Archimedes Maps

and

Optimal location of monitoring points

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Summary

Where to put the monitoring points in a system is a very common problem (environment, industry, health, and so on). We show that Archimedes maps, that is, measure preserving transformations, provide the right tool for this problem. Conversely, we insist that random exploration, often preferred because of simplicity and low cost, should definitely be avoided.

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I. Presentation

Many of our contracts, both in industrial or environmental situations, deal with the following type of question: one wants to define "monitoring points" in a system (simple or complex) and one wants to know where to put them. Or, if the system has been already defined (for historical reasons), people want to know what its degree of validity is. Examples are:

– Where to put 10 stations for monitoring water quality on a river? This is a one-dimensional problem.

– For France, where to put 300 stations for air quality surveillance? This is a two-dimensional problem.

– For the Paris area, where should the firemen put their basis stations (these are the places where the vehicles stay, before they are called for fires or assistance to victims)? This is also a two-dimensional problem.
– Where to put temperature sensors in a nuclear reactor? This is a 3-dimensional problem.

– Assume we have a computational code, depending on $K = 40$ parameters. We want to identify say $N = 500$ situations which we will call "generic" or "characteristic", in the sense that we feel that with these N situations we can have a good idea of what the code produces in general. Where to put these situations? This is a K dimensional problem.

The general approach for these problems is: people choose these monitoring points at random. This approach is quite wrong. Indeed, when the resources are scarce and costly, as it is always the case in practice, one should think about the best way to use them, and one should not rely upon a random approach for that. For instance, if one wants to set relays on the highway between Paris and Lyon (for food or for gas), the best is to set them at equal intervals, certainly not to choose them at random.

We will follow Archimedes' approach (see [BB_Archimedes]) in two aspects : first, we will use regular subsets in order to dispose our monitoring system, and second, we will show that a monitoring system put on a simple space may be lifted to a more complex one (for instance, as he did from a disk to a sphere). This should apply as well to high dimensional settings.

II. Covering with balls

Mathematically speaking, the problem can be stated as follows : we have a set in a K dimensional Euclidean space, and we want to place N balls such that the reunion of these balls covers the set. Each ball, indeed, corresponds to the "radius of surveillance" of each sensor or resource.

For instance, for the firemen, this radius corresponds to the area which is under the jurisdiction of any of the basis stations.

This radius is sometimes given (we know that the device is capable of detection or action for instance to a distance of 10 km) or sometimes part of the problem, as it is the case for the firemen problem, but also for the computational code in the last example above.

Usually, the number of points is given, for reasons linked with the budget: we cannot afford more than X stations, and the question is: where to put them? If we have too few stations, with a range of activity which is too short, then obviously we cannot monitor the whole territory, but the problem remains: where should we put the equipment to use it at best?

The distance to be used is also not clear. For environmental purposes, one will use the usual Euclidean distance, which is the true geographical distance. But if one wants to study the neighborhood of a configuration in a computational code, it might better to use the maximum distance : two situations $(X_k)_{k=1,\dots,K}$, $(Y_k)_{k=1,\dots,K}$, will be close if for each k, $|X_k - Y_k| < \varepsilon$.

Such problems are related to the branch of mathematics called "Operations Research", but the tools brought by OR are often academic : it brings optimal solutions when all constraints and cost functions are available and well-defined, and this is never the case in practice.

III. Best strategy in practice

A. Type of covering

In practice, the best strategy for defining the surveillance points is as follows:

- Cover the territory by squares (or cubes, or hypercubes) of equal size, obtained from a regular covering. For instance, this means that the territory of France will be divided into squares of equal height and width (equal latitude and longitude).

- Take the center of each square and put a surveillance point at this place.

- Replace the square by a ball, with same center, enclosing the square.

In this approach, the size of each square is chosen according to the sensibility of each sensor. For instance, if a sensor has a range of 10 km, then the square will have a size of $10\sqrt{2}$ km (in dimension 2).

Proceeding that way, we are sure to cover the whole territory, because the squares already covered, and the balls are bigger. But of course, there is some overlap between the balls, so one may have the feeling that this solution is not optimal. This is indeed the case, but in low dimension the area which is lost is not large. Indeed, in dimension 2, if the square has size a, its area is a^2 and the disk has radius $\frac{a}{\sqrt{2}}$ and area $\frac{\pi a^2}{2}$, so we have lost a factor $\frac{\pi}{2}$ between the disk and the square.

In high dimensional situations, the loss is much bigger. Indeed, take a cube of side a in a K dimensional space. The radius of the enclosing ball is $R = \frac{a}{2}\sqrt{K}$, and the volume of this ball is:

$$vol(B_R) = \frac{R^K \pi^{K/2}}{\left(\frac{K}{2}\right)!} \quad \text{(for even } K\text{)}$$

and the ratio with the volume of the cube is:

$$ratio = \frac{K^{K/2}\pi^{K/2}}{2^K\left(\frac{K}{2}\right)!} \approx \left(\frac{\pi e}{2}\right)^{K/2}\frac{1}{\sqrt{\pi K}}$$

using Stirling's formula. We see that this ratio increases exponentially with the dimension. So, we do not have an optimal covering, starting with cubes and then converting them into bigger balls, but there is usually no other way to obtain the covering, except in simple situations. For instance, if our set is itself a ball, there are much better coverings, using smaller balls (see for instance [VERGER-GAUGRY]). But such a simple situation is seldom met in practice.

B. The number of points to be used

We observe that, in order to cover a cube in dimension K, one should use a number of smaller cubes which is obtained from a division on each axis, so it will necessarily be of the form x^K.

There is no other way to do it ; if we take a number of smaller cubes which is not of the form x^K, then it cannot cover the large cube. In order to see this, take for instance a cube of size 1 in dimension K, and take smaller cubes of size $1-\varepsilon$, for any $\varepsilon > 0$. No matter what the dimension is, you can never cover the first one by less than 2^K smaller ones. Indeed, the reunion of 2^K cubes of size $1-\varepsilon$ is a cube of size $2(1-\varepsilon)$, the points of which are defined by the inequalities :

$$0 \le x_k \le 1-\varepsilon \text{ or } 1-\varepsilon \le x_k \le 2(1-\varepsilon)$$

and all these inequalities are satisfied by some extreme point of the unit cube (which have coordinates 0 or 1) ; this means that if you remove any of the 2^K smaller cubes, one of the corners of the unit cube will not be covered anymore.

So, in practice, this means that if you want to cover some cube in dimension K, you should always use a number of surveillance points of the form x^K, for some x depending on the precision of the sensor. Using other numbers is useless, unless you accept the idea that some part of the territory is left without surveillance.

Let us now turn to random exploration. Especially in high dimensional situations, random exploration should not be done at random! It should obey very precise rules, as we now see.

IV. The dangers of random exploration

Many people consider wrongly that, in high dimensional situations, a completely random exploration is satisfactory. This belief comes from the fact that, in order to compute the integral of a function, one simply draws points at random, takes the average value of the function at these points, and this average converges to the mean value of the function, when the number of random points increases ; this convergence is independent of the dimension. Multiplying the average value of the function by the volume of the set, one gets the integral of the function over the set.

So, the computation of any integral, no matter how complicated it is, reduces to the determination of random points, with uniform law, in the domain where the integral takes place. This is of course a considerable simplification, though drawing points with uniform law in a domain may be quite complicated: this depends on the form of the domain.

The following facts should be emphasized:

- The above statement applies only to an integral, that is to the average of a function. It never applies to a max, a min, or any local information.

- The above statement is only asymptotic, when the number of random points tends to infinity. This convergence is very slow, and we have absolutely no control about it. In other words, for a particular problem, with given dimension and given number of points, we have no idea at all of the error, which is committed, approximating the integral by the average computed upon the random points.

- Different random runs will lead to different answers, making it difficult to compare the results: no normalization is possible this way.

We would like to illustrate these warnings with several explicit computations, coming from IRSN (Institut de Radioprotection et de Sûreté Nucléaire) preoccupations about the CATHARE code:

Fix in the sequel the dimension $K = 40$. We are working in the hypercube $[0,1]^K$, which means that all parameters have been normalized to vary in the interval 0 – 1. Consider the "central ball", that is the ball B with center $\left(\frac{1}{2},...,\frac{1}{2}\right)$ and radius 1/2 ; it touches the hypercube on all its faces and is the greatest ball included in the hypercube.

The volume of this ball is:

$$vol(B) = \left(\frac{1}{2}\right)^K \frac{\pi^{K/2}}{\left(\frac{K}{2}\right)!} = \frac{\pi^{20}}{2^{40} \times (20!)}$$

and an approximate value for this volume is:

$$vol(B) \approx 0.33 \times 10^{-20}$$

Assume you make N runs ; the probability that at least one of them penetrates in B is :

$$p_1 = 1 - (1 - vol(B))^N \approx N \times vol(B)$$

With $N = 100\ 000$ (which is already quite large!), the probability that you penetrate the ball at least once is:

$$p_1 \approx 0.33 \times 10^{-15}$$

which means that, in practice, you have no chance at all that any of your runs will penetrate the largest ball in the hypercube.

If now we take the ball with same center, but with radius 1 (this is a ball which is not contained in the hypercube), the same computation gives a probability to enter the ball, in 100 000 trials:

$$p_1 \approx 0.00036$$

In other words, in practice, all the trials will be at distance more than 1 from the center of the hypercube.

Take now a smaller hypercube, say of size $c = 0.7$, inside the unit hypercube. Its volume is c^K and the probability to penetrate it at least once, among $N = 10\ 000$ runs, is :

$$p_2 = 1 - \left(1 - c^K\right)^N \approx 0.0063$$

So, here again, though the smaller hypercube seems to be very large inside the unit one, the chances to penetrate it at least once are very small.

Let us now show that the random computation of a multiple integral, by means of Monte-Carlo methods, in high dimensional spaces, may lead to very poor results.

We consider the function:

$$f(\lambda_1, ..., \lambda_K) = c\lambda_1^2 ... \lambda_K^2$$

on the set:

$$S = \left\{ (\lambda_1, ..., \lambda_K) \, ; \, \lambda_k \geq 0, \sum_k \lambda_k = 1 \right\}$$

and c is a normalization constant, such that $\int_S f = 1$.

These functions are probability densities; with a slightly different terminology, they are called "Dirichlet densities", and they are studied in our book "Nouvelles méthodes probabilistes pour l'étude des risques" [NMP].

Take here again $K = 40$ and $N = 10\,000$.

In order to produce uniformly distributed points in S, one proceeds as follows (see [NMP] and Luc Devroye [Devroye]):

- Generate K random variables X_k with uniform law on 0 – 1 ;

- Take $Y_k = Log\left(\frac{1}{X_k}\right)$: they follow an exponential law.

- Compute $U = \sum_{k=1}^{K} Y_k$;

- Set $Z_k = \frac{Y_k}{U}$: they give uniformly distributed points in S.

We repeat this process N times.

Now, the measure of S is $(K-1)!$, so the quantity :

$$(K-1)!\frac{1}{N}\sum_n f(\lambda_1^{(n)},...,\lambda_K^{(n)})$$

should be close to the integral of the function. But for $N = 10\,000$, the result is in the range 0.1.

This comes from the fact that the function is too small at most places and has one sharp "bump". This bump is too sharp to be detected by only 10 000 points in a 40-dimensional space. With 100 000 points, the estimate is roughly 0.45 for this function. Again, it becomes extremely bad for the function $f(\lambda_1,...,\lambda_K) = c\left(\lambda_1...\lambda_K\right)^3$.

Our recommendations are:

- Use random methods only at the very beginning of a problem, when one knows nothing: sending points at random may reveal interesting situations.

- As soon as specific situations are in evidence, use deterministic methods, in order to investigate them.

– In any case, use random methods only to compute averages, never min or max.

V. Archimedes Maps

A. Definition

In a recent seminar about Archimedes' work [BB_Archimedes], we illustrated the fact that his techniques lead to the definition of measure preserving maps from a complex situation to a simpler situation ; an example was given from a piece of sphere to a piece of disk [BB_Archimedes2].

A "measure preserving map" is a map such that two sets of equal measure in the original space have same measure in the destination space. Precisely, let E_1 and E_2 be two sets equipped with some measures m_1 and m_2 respectively ; we define an Archimedes map as follows :

Definition. - If f is a bijective transformation from E_1 onto E_2, it is an Archimedes map if $m_2(f(A_1)) = m_2(f(A_2))$, for any two sets with $m_1(A_1) = m_1(A_2)$.

The measures m_1 and m_2 can be normalized differently : what we require is that two sets of equal measures have images of equal measures. We do not require $m_2(f(A)) = m_1(A)$.

B. Archimedes maps as the solution to the surveillance problem

In our first paragraph, we explained that covering a set by regular squares of same size led to a simple solution of the surveillance problem, best in practice. But this solution still has some drawbacks:

– Some squares overlap the territory, so we use a sensor for almost nothing.

– This covering refers to areas only; it cannot be used if more complex measures are to be used.

For instance, assume that we want to draw a map of France, divided into sectors, where each sector has the same population, or the same amount of pollution, and so on: the regular covering will be inappropriate.

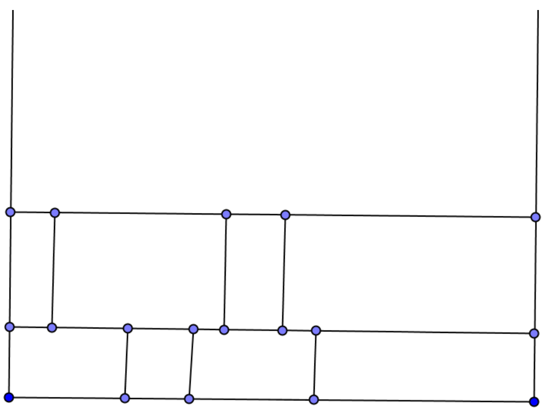
Let us take the example of a 2 D territory, France (F), and the population problem. We want to divide the country into small pieces of equal population.

Assume we have constructed an Archimedes map from the unit cube C to F, for the usual area measure. Divide the unit cube into 10^6 small squares of identical dimensions ; for this, we divide each side of the unit square into $N = 1000$ parts. Using an Archimedes map, transfer these N^2 small squares $C_{i,j}$, $i, j = 1, \ldots, N$, to the same number of sets $S_{i,j}$ in F : they are disjoint (except for the boundaries), they have the same area, and they cover F .

Now, let $s_{i,j}$ be the population of the set $S_{i,j}$; bringing back these numbers to C, we obtain a probability law on C : the probability of $C_{i,j}$ is $p_{i,j} = \dfrac{s_{i,j}}{\sum_{i',j'=1}^{N} s_{i',j'}}$.

From any probability law on C, we can deduce a grouping in zones of equal probability. This can be done in an exact way, when the probability law is continuous, in an approximate way when the probability is discrete, as it is the case here.

For this, we proceed as follows; the value 0.1 is taken as an example. First, we choose an index i_1 such that $\sum_{i=1}^{i_1} \sum_{j=1}^{N} p_{i,j} \approx 0.1$ (this is an horizontal strip of probability 0.1), and then an index i_2 such that $\sum_{i=i_1+1}^{i_2} \sum_{j=1}^{N} p_{i,j} \approx 0.1$ (this is a second horizontal strip of probability 0.1), and so on. So, we have 10 horizontal strips of probability 0.1. Then, inside each strip, we use the same process : find fist and index j_1 such that $\sum_{i=1}^{i_1} \sum_{j=1}^{j_1} p_{i,j} \approx 0.01$, then an index j_2 such that $\sum_{i=1}^{i_1} \sum_{j=j_1+1}^{j_2} p_{i,j} \approx 0.01$, and so on. What we obtain is of the following type:



We have horizontal strips, not all of same height, and each strip is made of rectangles, all of same height in each strip, but not of same width. Each rectangle is made of a certain number of the original small squares.

Let now R_n be an enumeration of these rectangles; they all have same probability, with respect to the probability law associated to the population. Let us send these rectangles to F by means of our Archimedes map, and let R'_n be their images. These sets cover F , are disjoint (except for boundaries in common), and all have the same population. We have solved our problem; cover France by means of zones of equal population.

Let us take a second example of the same nature. We recently had a contract with the Firemen Brigade in Paris : they are interested in grouping their assistance vehicles into zones of "equal load", meaning that all zones should approximately have the same demand (these demands concern mostly assistance to victims). At present, there are more than 400 small zones, of unequal load, and the question is to define a grouping which would be homogeneous. We solved this question in an empirical manner, but Archimedes maps would provide the tool for a precise solution.

C. More properties for the maps

When a grouping of zones is performed, usually people want two more properties, which do not follow readily from the definition we gave about an Archimedes map:

1. **These zones should be in one piece**

This comes from practical reasons, since a zone is to be attributed to some surveillance system, so one would not be happy if the zone consisted in many small pieces scattered on the whole territory.

What this means mathematically speaking is unclear. We might decide that the zones must be connected, but they do not need to be connected by arcs (which means that they may contain holes).

In the case of the Archimedes maps we construct below, this will be automatic : if two squares in C have a boundary in common, so will have their images in F.

2. **No zone should be too long or too large.**

If a zone was very thin and long, such as a narrow band near the sea, the surveillance would require many sensors. Therefore, a bound must be given on the possible diameter of any zone in F , depending on the diameter of the original zone in C . The simplest condition ensuring this is of the form:

$$diam(f(Z)) \leq const.\, diam(Z)$$

This condition will also be satisfied on the examples we give below, because they are continuous, with continuous inverse, on compact sets of $\mathbb{R}^2$ or $\mathbb{R}^3$.

D. *Another application of Archimedes maps*

Let us mention finally another possible application of Archimedes maps, besides the construction of zones of equal weight. We may use them to construct points with uniform distribution in the destination space, or part of it. For this, in our example above, we construct points with uniform distribution in the unit square C (which is immediate) and send them, using Archimedes map, to the destination space F; the measure preserving property will ensure that the images will follow a uniform law also.

In this latter case, properties of connectedness and size of diameter are not required.

VI. Measure preserving maps between simple sets

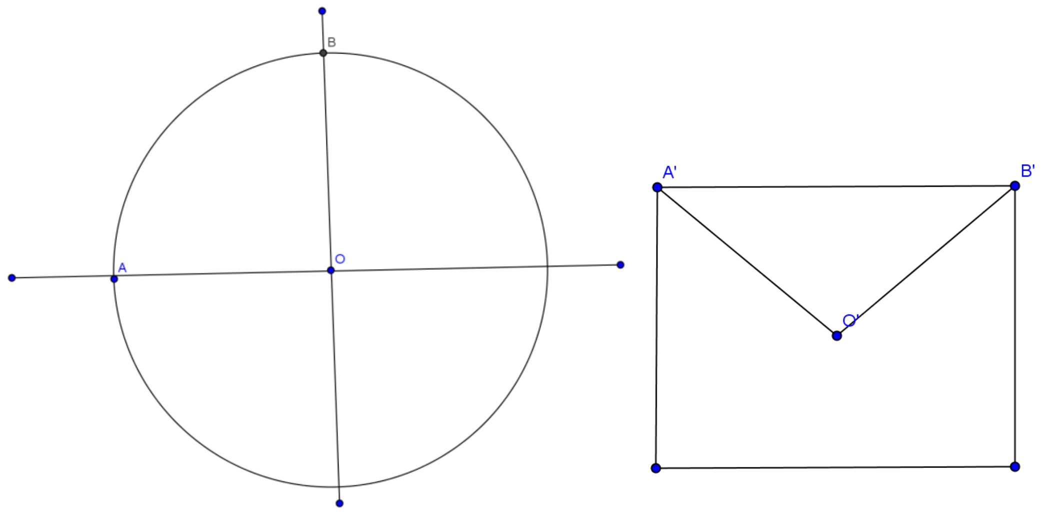
In his work "On the Sphere and the Cylinder" (see [BB_Archimedes]), Archimedes shows that there is a measure preserving map between the half sphere and the disk. Such maps between two sets are, in practice, extremely useful as we just saw.

The simplest set, for these purposes, is of course the cube (or hypercube, in higher dimension), because all meshes are easy to define, and so is the implementation of random points. So, we want to construct Archimedes maps from or onto the cube, whenever possible.

A. *Two dimensional structures*

Proposition 1. – *There is an Archimedes map, in dimension 2, from the disk to the cube.*

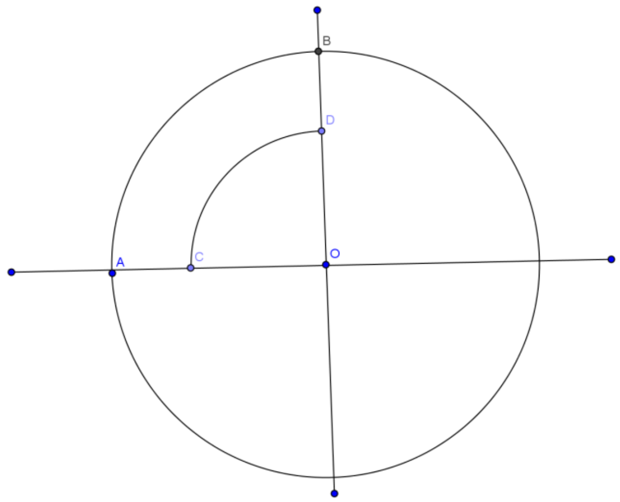
Proof. We follow Archimedes method.

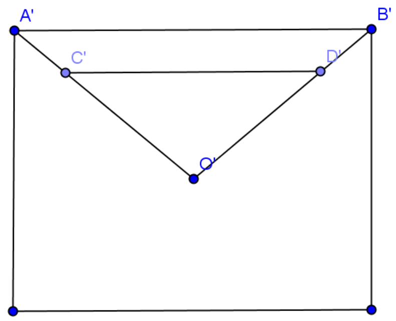


We will show that the sector AOB (quarter of the disk) can be transformed into the triangle A'O'B' (quarter of the square), by an Archimedes map.

We normalize the following way : the disk has radius $AO = 1$ and the square has width $A'B' = 1$.

First, take any point C between A and O, and consider the circle of center O, radius OC. Let CD be the corresponding quarter.





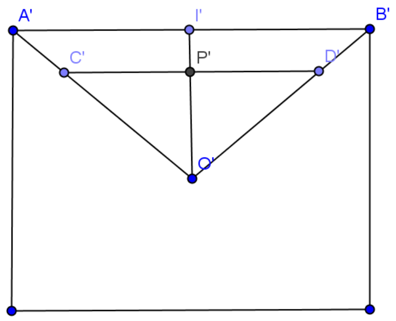
Then the arc CD will be transformed into a straight line C'D'. In order to satisfy the measure preserving property, we want:

$$\frac{area(\text{sector}(ABCD))}{area(disk)} = \frac{area(\text{trapeze}(A'B'C'D'))}{area(square)} \quad (1)$$

Let $OC = c$. We have:

$$\frac{area(\text{sector}(ABCD))}{area(disk)} = \frac{1}{4}\left(1 - c^2\right) \quad (2)$$

For the trapeze, let I' be the middle of A'B' and P' the middle of C'D'; let $h = O'P'$.



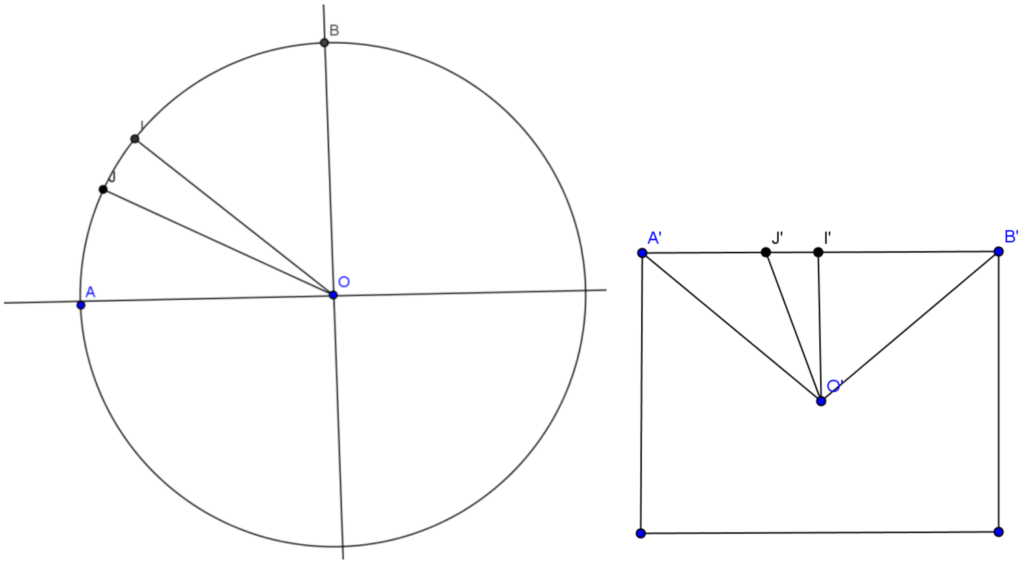
Then the area of the trapeze is:

$$
\begin{aligned}
area(trapeze) &= \frac{A'B' + C'D'}{2} \times I'P' \\
&= \left(A'I' + C'P'\right) \times \left(O'I' - O'P'\right) \\
&= \left(\frac{1}{2} + h\right)\left(\frac{1}{2} - h\right) \\
&= \frac{1}{4} - h^2
\end{aligned}
$$

Since the area of the square is 1, equations (1) and (2) lead to:

$$h = \frac{c}{2} \qquad (3)$$

Now, let us consider an angular sector of type OIJ, where I is (on the circle) the middle between A and B, and J is any point on the arc AB. Then the angular sector OIJ is transformed into a triangle $O'I'J'$:



and we want as before:

$$\frac{area(\text{sector}(OIJ))}{area(disk)} = \frac{area(\text{triangle}(O'I'J'))}{area(square)} \qquad (4)$$

Let ϑ be the angle OIJ. Then:

$$\frac{area(\text{sector}(OIJ))}{area(disk)} = \frac{\vartheta}{2\pi}$$

Let $x = I'J'$. Then:

$$area(triangle(O'I'J')) = \frac{x}{4}$$

So, condition (4) is equivalent to:

$$x = \frac{2\vartheta}{\pi} \qquad (5)$$

Combining (3) and (5), we have defined explicitly our Archimedes map, from the disk onto the square : a point M in the disk, with $OM = c$ and $angle(OI, OM) = \vartheta$ is transformed to a point M' with $O'P' = h = \frac{c}{2}$, where P' is the orthogonal projection of M' onto $O'I'$, and $M'P' = \frac{2\vartheta}{\pi}$.

(Here, in his own words, Archimedes would have written "area of the unit disk" instead of π.)

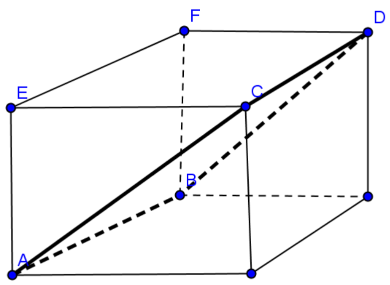
The transformation we have constructed is explicit (it can be programmed in a computer) and it is continuous.

B. Three dimensional structures

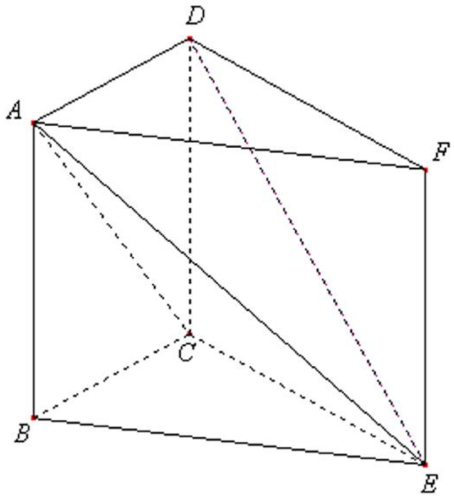
The same construction can be carried out in three dimensions:

Proposition 2. – *In dimension 3, there is an Archimedes map from the unit ball to the unit cube.*

Proof. The cube has 8 summits. So we first slice it into two, using a diagonal plane, namely $ABCD$ in the picture below : this way we obtain two identical prisms, with triangular base. Each of them has 6 summits.



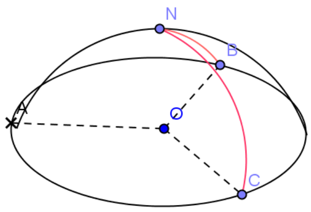
Now, a prism with triangular section can be decomposed into 3 equal pyramids with triangular base (Euclid, XII, Prop. 7), as the picture below shows:



(picture reproduced from http://www.math.uqam.ca/~tanguay_d/Pdf%20des%20articles/Pyramides.pdf)

Now, each pyramid has 4 summits, and our cube is decomposed into 6 such pyramids.

On the ball, we proceed the same: first we divide the ball into two half balls. Then each cup is divided into 3 equal pieces, with equal angles at the pole N :



Here the upper half-sphere $ANBC$ is divided into 3 equal slices with summit N, namely ANB, CNB, BNA. Each slice has 4 summits, so we can proceed.

On each slice, the Archimedes map is built as we did in the previous paragraph. Let us consider the slice $NAOB$; it has 4 summits, which are sent to the summits of the corresponding pyramid $N'A'O'B'$. Then, for the interior points, one proceeds as before: first playing with the radius, and then with two angles in spherical coordinates.

Playing with the radius is an homothety of center O ; it transfers to an homothety of center O'.

If we introduce a point C on the arc AB, it will transfer to a point C' on the segment $A'B'$, and we must have the property :

$$\frac{volume(slice(ANCO))}{volume(ball)} = \frac{volume(pyramid(A'N'C'O'))}{volume(cube)}$$

And if we introduce a point D on the arc AN, it transfers to a point D' on the segment $A'N'$, with the property :

$$\frac{volume(slice(DNBO))}{volume(ball)} = \frac{volume(pyramid(D'N'B'O'))}{volume(cube)}.$$

The general idea is this: a cube has summits, and a ball does not. So, one should divide the cube, in order to reduce the summer of summits, and divide the ball, in order to increase the number of summits. For both, this should be done in a regular way: all pieces must have the same volume. When this is done, one is able to construct the Archimedes map easily, on each piece, as we did before.

We think that the construction above can be carried in higher dimensions.

C. General constructions in two dimensions

In two dimensions, anyhow, the above propositions can be generalized:

Proposition 1. – *In $\mathbb{R}^2$, there is an Archimedes map from any convex compact set with non-empty interior onto the unit disk. This map is bicontinuous.*

Proof.

Obviously, the assumption "nonempty interior" is necessary: there cannot be an Achimedes map from a segment onto the unit disk.

Let K be a convex compact set with non-empty interior, let ∂K be its boundary. Let O be any point in the interior of K. Let Ω be the closed unit disk, and let O' be its center. The point O will be transformed into O'.

Let A be any point of the boundary ∂K and A' be any point of the boundary $\partial\Omega$ (the unit circle). The point A will be sent to A'.

Let λ, $0 \le \lambda \le 1$. Let λK be the set obtained from K by an homothety of center O and coefficient λ. Let $\partial(\lambda K)$ be the boundary of this set. The set $\partial(\lambda K)$ will be transformed into the circle C_μ, with center O' and radius μ such that :

$$\frac{area(\lambda K)}{area(K)} = \frac{area\ disk(\Omega_\mu)}{area\ disk(\Omega)} \tag{1}$$

where Ω_μ is the disk with center O' and radius μ.

Let now B be any point of the boundary ∂K ; it will be transformed into a point B' of C_μ such that :

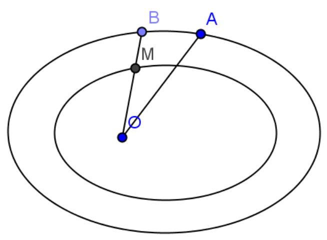
$$\frac{\text{area sector}(AOB)}{\text{area}(K)} = \frac{\text{area sector}(A'O'B')}{\text{area}(\Omega)} \tag{2}$$

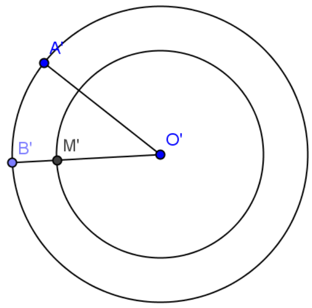
Note that properties (1) and (2) are not pointwise definitions.

Now, let M be any point in K. Let λ, $0 \le \lambda \le 1$ be such that $M \in \partial(\lambda K)$ and B be the intersection of ∂K with the half-line drawn from O, with same orientation as OM.

Then the image of M is the point M' which is at the intersection of the circle C_μ (μ defined by (1)) and the segment $O'B'$, where B' is defined by (2). Both properties ensure that the area is preserved: see the Lemma below.

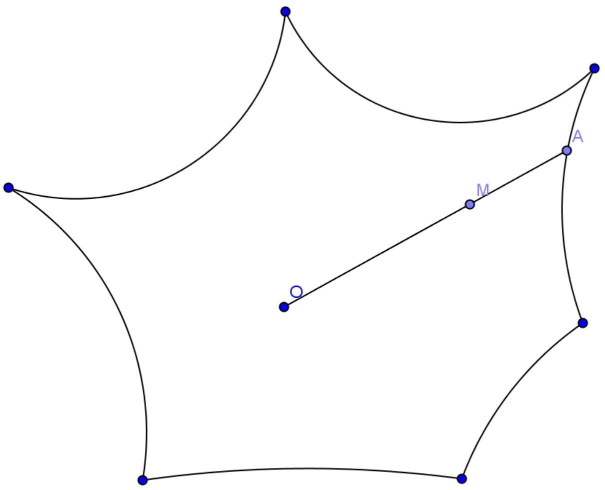
We observe that the Archimedes map thus constructed is continuous and respects segments starting at O : the image of any segment OB (B on the boundary of K) is the segment $O'B'$.





Remark

The previous proposition does not really require the fact that K is convex. In fact, a weaker assumption suffices, with the same proof. This weaker assumption is that K is "star shaped", meaning that it has non empty interior, and there is a point O in this interior, such that for any point A in K, the segment OA is also in K.

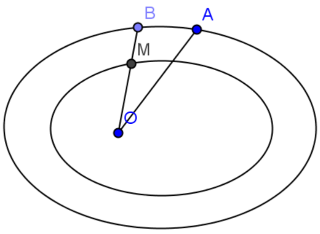


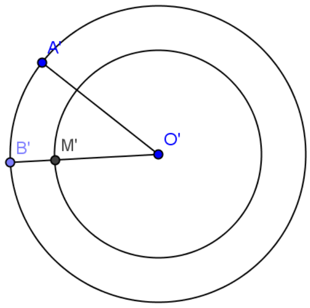
Lemma 2. – *Constructing Archimedes maps*

Let K be a convex compact subset of $\mathbb{R}^2$ with non-empty interior. Let Ω be a disk in $\mathbb{R}^2$. Let O be a point in the interior of K, O' be the center of Ω, A an arbitrary point on the boundary ∂K, A' be an arbitrary point on the boundary $\partial\Omega$. Assume that, for any point B on ∂K, the point B' on $\partial\Omega$ satisfies the property :

$$\frac{\text{area sector}(AOB)}{\text{area}(K)} = \frac{\text{area sector}(A'O'B')}{\text{area}(\Omega)} \tag{1}$$

Let M be any point in K and let B be the associated point on the boundary ∂K :





Let λ, $0 \le \lambda \le 1$ such that $\overrightarrow{OM} = \lambda\,\overrightarrow{OB}$. Define M' by $\overrightarrow{OM'} = \lambda\,\overrightarrow{OB'}$, where B' is defined by (2). Then the application $M \to M'$ is an Archimedes map from K onto Ω.

Proof of Lemma 2.

Let B_1, B_2 be any two points on the boundary ∂K; let λ, μ any numbers with $0 \le \lambda, \mu \le 1$, and let M_1, M_2, N_1, N_2 be the intersections of OB_1, OB_2 with $\partial(\lambda K), \partial(\mu K)$ respectively (see picture). In order to ensure the preservation of the measure, all we have to show is that:

$$\frac{area(M_1M_2N_1N_2)}{area(K)} = \frac{area(M_1'M_2'N_1'N_2')}{area(\Omega)} \tag{2}$$

Indeed, the sigma-algebra generated (by countable unions and intersections) by such "rectangles" is the same as the sigma-algebra generated by all Borel subsets of K.

But property (1) follows immediately from the same property for sectors:

$$\frac{area(\text{sector}(M_1OM_2)}{area(K)} = \frac{area(\text{sector}(M_1'O'M_2'))}{area(\Omega)} \tag{3}$$

But:

$$area(\text{sector}(M_1OM_2)) = \lambda^2\ area(\text{sector}(B_1OB_2))$$

and:

$$area(\text{sector}(M_1'OM_2')) = \lambda^2 \, area(\text{sector}(B_1'OB_2'))$$

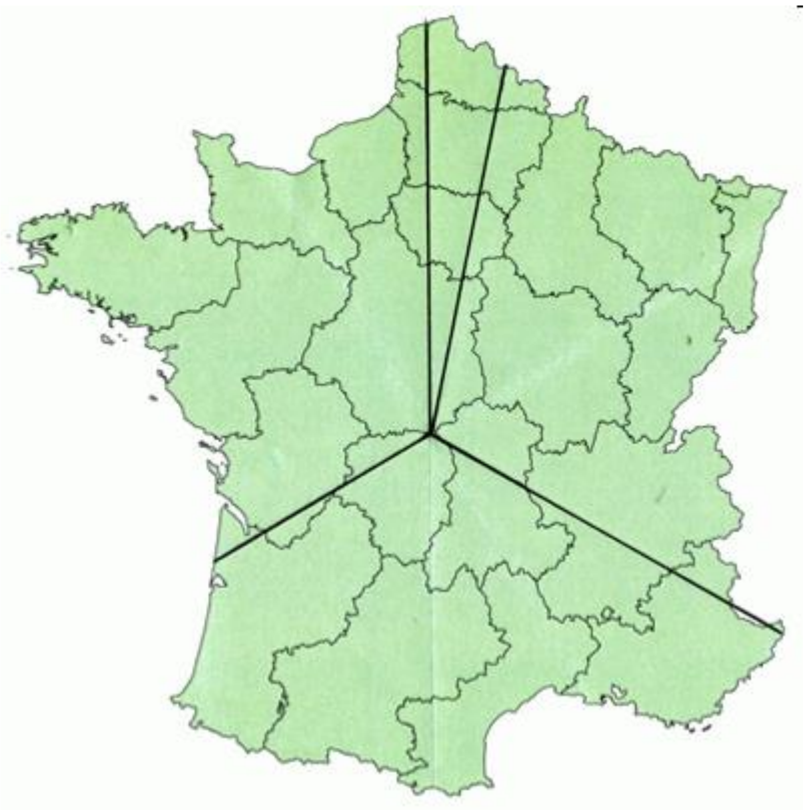
Finally,

$$\frac{area(\text{sector}(B_1OB_2))}{area(K)} = \frac{area(\text{sector}(B_1'OB_2'))}{area(\Omega)}$$

follows obviously from (1): this proves the Lemma.

D. An example of Archimedes map

This example is due to Stéphanie Premel, during her internship at SCM, summer 2010. It shows an Archimedes map of France, divided into four zones of equal population. It is obtained from a central point, sending a hexagon to a disk, and back. Of course, we might have more than four sectors, and each sector might, in turn be divided into smaller pieces.



Archimedes map of France: division into four zones of equal population (Stéphanie Premel)

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