Société de Calcul Mathématique SA
Outils d'aide à la décision

GPS : Taking uncertainties into account

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Summary

The GPS (Global Positioning System) returns the position of an object, from the information provided by several satellites. We show how to compute an uncertainty about this position, using purely probabilistic methods. Our method does not assume the laws of errors to be Gaussian, and makes no use of the so called "Kalman filter", which is inappropriate in such a case.

I. Basic situation

The basic situation is as follows: we have K satellites, and each satellite S_k sends a signal, which is analyzed by a receiver R; the clocks of the satellites are supposed to be perfect and synchronized (this is the "absolute time") ; the clock of the receiver may differ from the absolute time by a value τ which may be positive or negative, and is unknown.

From the signal sent by each satellite, the receiver thus deduces the time $\frac{d_k}{c}+\tau$, where $d_k = d\left(R, S_k\right)$ is the distance from the receiver to the $k-$th satellite and c is the speed of light. In other words, we know the quantities $\delta_k = d_k + c\tau$, for $k = 1,\dots,K$.

We do not take into account here the so-called "ambiguities" (see [GPS]), namely the fact that the distance is defined only within a multiple of the wavelength ; in the present settings, the ambiguity is around 300 km, whereas our reasoning is essentially "local" (less than one km).

A. Computing a preliminary position

The equations $\delta_k = d_k + c\tau$ can be written, for $k = 1,\ldots,K$:

$$\left(\left(x_0 - x_k\right)^2 + \left(y_0 - y_k\right)^2 + \left(z_0 - z_k\right)^2 \right)^{1/2} + c\tau = \delta_k \tag{1}$$

which means:

$$\left(x_0 - x_k\right)^2 + \left(y_0 - y_k\right)^2 + \left(z_0 - z_k\right)^2 = \left(\delta_k - c\tau\right)^2 \tag{2}$$

In these equations, the unknown are x_0, y_0, z_0 (coordinates of the receiver) and τ (time shift of the receiver). Since we have 4 unknowns, we normally need 4 equations.

Let us write the first two equations, corresponding to the first two satellites.

$$\begin{cases} \left(x_0 - x_1\right)^2 + \left(y_0 - y_1\right)^2 + \left(z_0 - z_1\right)^2 = \left(\delta_1 - c\tau\right)^2 \\ \left(x_0 - x_2\right)^2 + \left(y_0 - y_2\right)^2 + \left(z_0 - z_2\right)^2 = \left(\delta_2 - c\tau\right)^2 \end{cases} \tag{3}$$

Taking the difference (3.1)-(3.2), we obtain:

$$2x_0\left(x_2 - x_1\right) + x_1^2 - x_2^2 + 2y_0\left(y_2 - y_1\right) + y_1^2 - y_2^2 + 2z_0\left(z_2 - z_1\right) + z_1^2 - z_2^2 = \delta_1^2 - \delta_2^2 - 2c\tau\left(\delta_1 - \delta_2\right)$$

that is:

$$2x_0\left(x_2 - x_1\right) + 2y_0\left(y_2 - y_1\right) + 2z_0\left(z_2 - z_1\right) + 2c\tau\left(\delta_1 - \delta_2\right) = x_2^2 - x_1^2 + y_2^2 - y_1^2 + z_2^2 - z_1^2 + \delta_1^2 - \delta_2^2 \tag{4}$$

This is a linear equation in the four unknowns x_0, y_0, z_0, τ, and the coefficients are known, as well as the right hand side of the equation.

We need at least four satellites in order to solve the equations.

- If we have 5 or more, we compute the linear equations 1-2, 1-3, 1-4, 1-5 and we solve a linear system. All unknowns are computed at the same time.

- If we have only four satellites, we compute the linear equations 1-2, 1-3, 1-4, solve these equations in terms of τ and substitute in equation (2) ; this way, we obtain a quadratic equation in τ, which we solve directly. One solution is eliminated for practical reasons (for instance the point would not be on the surface of the earth).

In the previous settings, the first satellite, used for all resolutions, is the one with strongest signal.

Of course, if we have more than 4 satellites, a difficulty comes from the fact that the corresponding equations will not be perfectly compatible, due to various errors. This difficulty is usually solved by means of an approximation procedure (taking an average position between the various possibilities) which is seldom correct. We describe below the correct approach, which is probabilistic.

Remark. - Another approach might be to take the differences of equations (1) for satellites 1 and 2 ; this way, we get the equation:

$$d\left(R,S_1\right)-d\left(R,S_2\right)=\text{constant}$$

which means that the point R belongs to a hyperboloid with foci S_1 and S_2. Using similar equations with S_1,S_3, and so on, we find that R is at the intersection of several hyperboloids, the foci of which are the satellites. An intersection of hyperboloids is, geometrically speaking, very easy to define (much easier than an intersection of spheres), but will require numerical procedures (it is non-linear), whereas the approach we describe here is entirely linear, except for the possible final solution of a quadratic equation (if the number of satellites is insufficient).

This geometrical approach, using hyperboloids, will not answer the question of the errors. If we have many hyperboloids, and if we take the errors into account, their intersection will not result in a single well-defined point.

II. Taking uncertainties into account

The question we solve now is: how to incorporate the information about the error laws, coming from each satellite.

A. *Deterministic Interval approach*

Assume first that each satellite sends a deterministic interval : the quantity δ_k is between two bounds $\delta_{k,\min}$ and $\delta_{k,\max}$. We have to look again at the resolution procedure seen above.

We have, for all k :

$$\delta_{k,\min}\le\left(\left(x_0-x_k\right)^2+\left(y_0-y_k\right)^2+\left(z_0-z_k\right)^2\right)^{1/2}+c\tau\le\delta_{k,\max}$$

which implies, for $k=1,2$:

$$\left(x_0-x_1\right)^2+\left(y_0-y_1\right)^2+\left(z_0-z_1\right)^2\le\left(\delta_{1,\max}-c\tau\right)^2 \qquad (2.1)$$

and:

$$\left((x_0 - x_2)^2 + (y_0 - y_2)^2 + (z_0 - z_2)^2\right)^{1/2} + c\tau \geq \delta_{2,\min}$$

which gives:

$$(x_0 - x_2)^2 + (y_0 - y_2)^2 + (z_0 - z_2)^2 \geq \left(\delta_{2,\min} - c\tau\right)^2 \tag{2.2}$$

So, the difference (2.1)-(2.2) gives:

$$\begin{aligned} 2x_0(x_2 - x_1) + 2y_0(y_2 - y_1) + 2z_0(z_2 - z_1) + 2c\tau\left(\delta_{1,\max} - \delta_{2,\min}\right) \leq \\ \leq x_2^2 - x_1^2 + y_2^2 - y_1^2 + z_2^2 - z_1^2 + \delta_{1,\max}^2 - \delta_{2,\min}^2 \end{aligned} \tag{2.3}$$

which is the equation of a half-space in a 4 dimensional space.

Similarly, we get:

$$\begin{aligned} 2x_0(x_2 - x_1) + 2y_0(y_2 - y_1) + 2z_0(z_2 - z_1) + 2c\tau\left(\delta_{1,\min} - \delta_{2,\max}\right) \geq \\ \geq x_2^2 - x_1^2 + y_2^2 - y_1^2 + z_2^2 - z_1^2 + \delta_{1,\min}^2 - \delta_{2,\max}^2 \end{aligned} \tag{2.4}$$

which is another half-space.

These half-spaces are delimited by hyperplanes, orthogonal respectively to the vectors

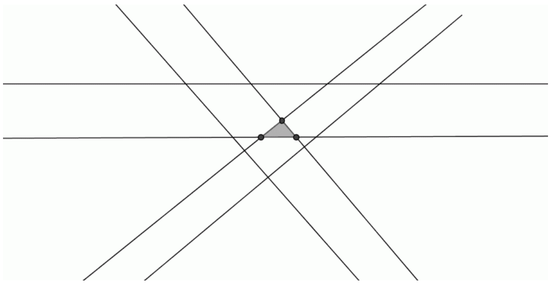
$$(x_2 - x_1), (y_2 - y_1), (z_2 - z_1), c\left(\delta_{1,\max} - \delta_{2,\min}\right)$$

and:

$$(x_2 - x_1), (y_2 - y_1), (z_2 - z_1), c\left(\delta_{1,\min} - \delta_{2,\max}\right)$$

These two hyperplanes are not parallel: the first three components of the orthogonal vectors are the same, but the fourth is not. Anyway, the intersection of the two half-spaces delimitates a portion of the 4-dimensional space, which we call a "slice".

Doing the same thing with the satellites S_1, S_3, S_1, S_4, and so on, we get an intersection of slices, which is a convex subset of the four-dimensional space. It will be bounded if the number of satellites is large enough; the more satellites we have, the smaller this intersection is.



We may finally take the intersection of this convex set with the set defined by equation (2.1), but this is not a convex set anymore. We need do so only if we have only 4 satellites available.

The set we finally obtain will contain the point R_0 computed previously, and, inside this set, all points are valid positions for the receiver. None of them is more likely than any other. We denote this set by Ω : set of admissible positions for the receiver.

B. Probabilistic approach

We have to distinguish between two factors:

- The time shift τ of the receiver may not be perfectly known, but is is assumed to be constant during the operation. It will not change from one hour to the next ;
- The distance information sent by each satellite may be affected by various elements, such as multipath errors, which may be different at each time of operation. They depend on the position of the receiver.

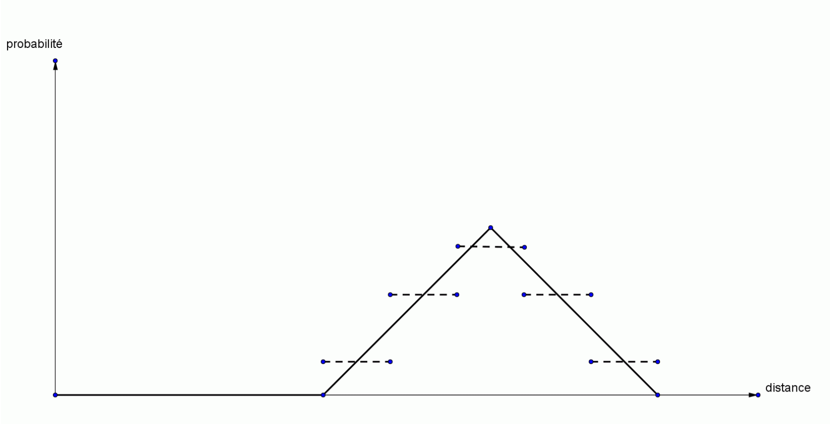
Both will be represented by probability laws, but of different nature. These probability laws will be assumed to be independent, which is quite realistic: the clock of the receiver has nothing to do with the estimates about the distance to the satellites.

1. The probability law about the time shift

Since we know nothing about it, we simply take a uniform law between two bounds $\tau_{\min}$ and $\tau_{\max}$. This density will be denoted by $\psi(\tau)$.

2. The probability law from each receiver

We assume that each satellite returns a law of probability, concerning its distance to the receiver: not just a single value, not an interval, but a complete probability law. The density of this law, of course, should be zero outside some interval $d_{\min}, d_{\max}$, so its shape may have the following form:



In this picture, we see two possibilities : either the law is given by a continuous function (here a triangle function, represented by a thick line), or, more often, it is given by a finite number of step functions (here, we have five such step functions, represented by dotted lines). In the latter case, we speak about a discrete law (opposed to continuous).

Here, we might for instance have the following situation: the triangle function starts at $d_0 = 25\,000\,km$ (because the satellites are usually at a distance around $20\,000\,km$ from the surface of the Earth) and finishes at $d = d_0 + 0.050\,km,$ meaning that the overall uncertainty is 50 m. The peak of the function is at $d = d_0 + 0.025\,km.$ Since the integral must be equal to 1, the height of the peak is $h = \frac{1}{5}$ (if the unit is 10 meters).

In practice, the law is usually given in a discrete form, that is under the form of a finite set of values.

In the above picture, we will have:

$$p_1 = \frac{1}{25} \text{ if } d_0 \le d < d_0 + 0.010\,km$$

$$p_2 = \frac{3}{25} \text{ if } d_0 + 0.010 \le d < d_0 + 0.020\,km$$

$$p_3 = \frac{9}{50} \text{ if } d_0 + 0.020 \le d < d_0 + 0.030\,km$$

$$p_4 = \frac{3}{25} \text{ if } d_0 + 0.030 \le d < d_0 + 0.040\,km$$

$$p_5 = \frac{1}{25} \text{ if } d_0 + 0.040 \le d < d_0 + 0.050\,km$$

We denote by f_k the probability law associated with the $k-$th satellite. It has the shape described above : either a triangle, or a few values.

We make one more assumption: we will assume that all errors are independent. This is not completely true, because part of the error may come from properties of the atmosphere near the receiver (local air pressure or moist), but specialists usually consider that this part is neglectible compared to the error coming from each satellite's clock. Also, the internal system of the receiver, which computes the distance to the satellite, using the signal it receives, may contribute to the error: this part also is ignored.

3. Construction of the admissible set

From each probability law f_k, we deduce the minimum $d_{k,\min} + c\tau_{\min}$ and the maximum $d_{k,\max} + c\tau_{\max}$ of the possible information δ_k sent to the receiver by the $k-$th satellite.

Using these bounds, we make the construction (intersection of sets) described in paragraph II.A above: what we obtain is the set of all possible positions of the receiver. As we said, we denote it by Ω.

4. Construction of the final density of probability

We use the method described in the book [MPPR], chapter XIII. We assume that the receiver is at some position (x, y, z) and that the time shift is at some value τ, and we compute the probability of such a disposition, which is therefore described as a point $M = M(x, y, z, \tau)$ in $\Omega \subset \mathbb{R}^4$. This is an application of Archimedes' Weighing Method (see [AMW]): we assume that the receiver is at some place, generate the information it would receive and compare it with the information it receives in reality.

We set:

$$\varphi(x, y, z, \tau) = \varphi(M) = c_0 \times \psi(\tau) \times f_1(x, y, z) \times \cdots \times f_K(x, y, z)$$

where $f_k(x, y, z)$, $k = 1, ..., K$, is the value of the density function related to the $k-$th satellite, computed at the point M, and c_0 is a normalization constant, so that $\int_\Omega \varphi(M)\, dxdydzd\tau = 1$. The independence assumption is used in this definition.

The function $\varphi(x, y, z, \tau)$ is a density of probability, which represents completely all probabilistic aspects. Outside the set Ω, this density is 0 (the point cannot be there) and the high values of φ represent the places where R is more likely to be. In particular, the expected position of the receiver will have the following coordinates:

$$E(R)_x = \int_\Omega x\varphi(x,y,z,\tau)\, dxdydzd\tau$$

$$E(R)_y = \int_\Omega y\varphi(x,y,z,\tau)\, dxdydzd\tau$$

$$E(R)_z = \int_\Omega z\varphi(x,y,z,\tau)\, dxdydzd\tau$$

Expected value of the time shift : $\int_\Omega \tau\varphi(x,y,z,\tau)\, dxdydzd\tau$

5. Practical construction of the density

In practice, one will not compute this density for any point M in Ω, but only for a finite number of points, disposed at a regular grid, both in space and time.

Let $x_{\min}$ be the lower x coordinate of all points in Ω and similarly $x_{\max}$, $y_{\min}, y_{\max}, z_{\min}, z_{\max}$. Let $\tau_{\min}, \tau_{\max}$ be the minimum and maximum values of the time shift, already defined.

We divide each interval $[x_{\min}, x_{\max}], [y_{\min}, y_{\max}], [z_{\min}, z_{\max}]$ into N equal sub-intervals and the interval $[\tau_{\min}, \tau_{\max}]$ into N' subintervals. We set:

$$M(n_1,n_2,n_3,n_4) = \left(x_{\min} + \frac{n_1}{N}(x_{\max} - x_{\min}), y_{\min} + \frac{n_2}{N}(y_{\max} - y_{\min}), z_{\min} + \frac{n_3}{N}(z_{\max} - z_{\min}), \tau_{\min} + \frac{n_4}{N'}(\tau_{\max} - \tau_{\min}) \right)$$

for $n_1, n_2, n_3 = 0, \ldots, N, \quad n_4 = 0, \ldots, N'$.

For instance, if the distances $x_{\max} - x_{\min}, y_{\max} - y_{\min}, z_{\max} - z_{\min}$ are of the order of 1 km, we take $N = 100$ and we will have a grid of ten meters in space. If $\tau_{\max} - \tau_{\min} = 0.01\,\text{s}$, we will need $N' = 3 \times 10^5$ points for our grid in time in order to achieve the same precision. Finally, we will have 3×10^{11} points to compute.

We compute the density function $\varphi(M)$ only at the points $M(n_1,n_2,n_3,n_4)$; everything seen before applies, and the expected position of the receiver is now given by the formulas:

$$E(R)_x = \frac{1}{N^3}\frac{1}{N'}\sum_{n_4=0}^{N'} \sum_{n_1,n_2,n_3=0}^{N} \frac{n_1}{N}\varphi(n_1,n_2,n_3,n_4)$$

$$E(R)_y = \frac{1}{N^3}\frac{1}{N'}\sum_{n_4=0}^{N'} \sum_{n_1,n_2,n_3=0}^{N} \frac{n_2}{N}\varphi(n_1,n_2,n_3,n_4)$$

$$E(R)_z = \frac{1}{N^3}\frac{1}{N'}\sum_{n_4=0}^{N'} \sum_{n_1,n_2,n_3=0}^{N} \frac{n_3}{N}\varphi(n_1,n_2,n_3,n_4)$$

Expected time shift : $\frac{1}{N^3}\frac{1}{N'}\sum_{n_4=0}^{N'}\sum_{n_1,n_2,n_3=0}^{N}\frac{n_4}{N'}\varphi(n_1,n_2,n_3,n_4)$

In practice, the sums do not run on all values of n_1,n_2,n_3,n_4, but only on the positions of M which are inside the admissible set Ω. This is decided from a system of linear inequations, of the type (2.3) and (2.4), so the number of points at which the computations need to be made is lower.

The precision about the time shift is more essential, and therefore the number of points for the discretization is higher in time than in space. In order to reduce the time for computation, one may start with a lower precision in time (for instance 10^2 instead of 3×10^5), perform the computation, deduce the most likely interval for τ and discretize again this subinterval only.

As we said earlier, the computation for the time shift needs to be done only once. When this time shift is computed, the expected value may be used constantly later, so the probability law on the time shift disappears.

Remark. – The whole approach described here applies to any single given time : at each t, the coordinates of the receiver are computed. Of course, a significant improvement will be obtained if we use the information that the receiver follows a continuous trajectory. Then, at each time t_0, we may use the predicted positions at all times $t < t_0$ and all these positions must fit into a continuous trajectory. Quite clearly, the estimate about the time shift will be more precise if it relies upon several positions, and not only one.

This is done again by purely probabilistic tools, as decribed in [MPPR], chap. 22 and 27. A concrete application was given in [MIS].

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