Société de Calcul Mathématique, S. A.
Algorithmes et Optimisation

High Dimensional Archimedes Transformations

Prizes offered by SCM

With the support and participation of the

"Caisse Centrale de Réassurance", Paris, France

I. Physical description of the problem

We have a sandbag, or a bag full of water, or of any incompressible fluid. This bag may have any shape. Can we transform it into a cube, or into a ball ?

For a cube, the answer is obviously yes : use an hydraulic press, or any similar device, with six plates, and press the bag until it becomes a cube. For a ball, there is no simple mechanical answer, but we know that we can transform a cube into a ball by means of an Archimedes Transformation (see [2]).

II. Mathematical description of the problem

An "Archimedes transformation" (or measure preserving transformation) is a transformation such that two sets of equal measure in the original space have same measure in the destination space. Precisely, let E_1 and E_2 be two sets equipped with some measures m_1 and m_2 respectively ; we define an Archimedes transformation as follows :

Definition. - If f is a bijective transformation from E_1 onto E_2, it is an Archimedes transformation if $m_2(f(A_1)) = m_2(f(A_2))$, for any two sets with $m_1(A_1) = m_1(A_2)$.

The measures m_1 and m_2 can be normalized differently : what we require is that two sets of equal measures have images of equal measures. We do not require $m_2(f(A)) = m_1(A)$.

In the sequel, we deal with the usual Lebesgue measure in the Euclidean space $\mathbb{R}^3$.

The problem is the following : let K be any (convex) compact subset of $\mathbb{R}^3$ (possibly with reasonably regular boundary) ; is there an Archimedes transformation from K onto the unit cube ?

The assumptions "convex" and "regular boundary" might not be necessary.

The physical problem and the mathematical problem are equivalent : if you take two subsets A_1 and A_2 of the sandbag, with same volume (same Lebesgue measure), they contain the same number of molecules (same number of grains of sand), and so will have their images after the compression by the hydraulic press. So the hydraulic press realizes an Archimedes transformation.

Recall (see [1]) that Archimedes showed the existence of such a transformation from a spherical cup onto a disk.

III. What else do we want ?

We would like this Archimedes transformation to be constructive (not just an existence property), even if this construction is done using several steps. We would be glad if some other properties could be established, such as the continuity of the transformation (global, or local). Finally, we would be interested in the solution of this problem for any dimension $n \geq 3$ (we have already done it for dimension 2).

IV. The prizes

SCM and CCR will offer two prizes of 1,000 Euros each for the best solutions, to be sent before June 30th, 2011, to the email address : scm.sa@orange.fr (answers in English, please).

Partial solutions, or incomplete solutions, will also be considered and may share part of the prize. In fact, we are interested by any available information on this problem.

V. Why do we need this ?

For us, the direct need comes from a contract with the "Caisse Centrale de Réassurance" about extreme events. We have to compute complicated integrals, and this cannot be done analytically : we must use Monte-Carlo methods, on strange sets in $\mathbb{R}^n$: typically, $x_i \geq 0, \sum_{i=1}^{n} x_i = 1$, and some specific orderings on the x_i. So we must produce uniformly distributed points on such sets. We did it using a method due to Peter Robinson, using Luc Devroye's book (see the references in [2]), but it would be immediate if we had an Archimedes transformation between this set and the hypercube in $\mathbb{R}^n$: uniformly distributed points in the hypercube are easy to construct, and then one takes them back to the set.

We define an Archimedes map of a 2d territory, with respect to some parameter, as a decomposition of this territory into zones where the parameter takes the same value. For instance, an Archimedes map of France, with respect to population, would be a decomposition into zones with equal population. Such maps are useful in order to define the ressources : for instance, each zone should have the same number of schools. We can also think of Archimedes maps for electrical consumption, medical needs, and so on : see [2] for other examples.

Similarly, we can define a 3d-Archimedes map, with respect to some criterion, as a decomposition of the soil into pieces where the criterion is constant : for instance, we might have a 3d map of the subsoil of a country, into 10 zones, where each zone contains the same amount of petrol reserves, or the same amount of pollution, or the same amount of radioactivity. The idea is the same as in 2d : each zone deserves the same amount of resources.

Here again, the existence of an Archimedes Transformation f from the hypercube $[0,1]^n$ to a set E makes the Archimedes maps in E very easy to construct : one builds the map in the hypercube first, and takes them to E by means of f.

VI. References

[1] Bernard Beauzamy : A l'occasion du 2222ème anniversaire de la mort d'Archimède, séminaire exceptionnel : Un théorème d'Archimède et sa démonstration d'origine, 5 mai 2010, SCM SA. http://scmsa.eu/archives/CLQ_SCM_2010_05_Archimede.pdf

[2] Bernard Beauzamy : Archimedes Maps and Optimal location of monitoring points, september 2010, SCM SA. http://scmsa.eu/archives/ART_BB_Archimedes_maps_2010_09.pdf