Monte Carlo approach to compute the success probability of sending objects to the Moon

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Abstract

Increasing interest is growing within the space industry to design missions capable of sending unmanned objects (e.g. food supplies, instruments, experiment equipments, etc.) to our neighbouring planets. This has motivated the need for the development of low-cost technologies to make these missions ever more frequent and accessible. In this paper, an uncertainty study is conducted in response to the challenge of sending an unmanned object out to space as formulated in this year's Mathematical Competitive Games challenge (2016-2017). The object is launched from the Earth's surface towards the Moon and is given an initial thrust that lasts only for a few seconds. Given the intensity and duration of the thrust and all of the other parameters, such as the Earth's mass, air density, mass of the object, etc. The challengers ask *what is the probability of landing the object on the Moon surface given all the uncertainties in the parameters and the model?* A Monte Carlo approach is proposed to efficiently estimate the probability of the object landing on the Moon, complying with the assumptions provided by the challengers on the probability density functions. Different physical models and different underpinning assumptions are proposed to simulate the launch. It has been found that, once all uncertainties have been considered, about 37 % of the simulations lead to a successful landing on the Moon's surface. Conversely, the likelihood of successful landing on a specific area resulted fairly low. In addition, complementary sensitivity analyses are proposed which reveal the projectile motion is most affected by drag uncertainty and least affected by mass uncertainties.

Keywords: Uncertainty Quantification, Epistemic Uncertainty, Monte Carlo Simulation, Mathematical Competitive Game

Nomenclature

r_e the radius of the Earth.

r_m the radius of the Moon.

$\mathbf{x}_m(t)$ the Moon's position vector.

R_0 the radius of the Moon's orbit around Earth.

R projectile range.

θ_0 initial position of Moon with respect to the vertical.

ω the angular velocity of the Moon.

θ_t total angular displacement of Moon at time t.

M_e the expected mass of the Earth.

M_m the expected mass of the Moon.

m the mass of the projectile

$\mathbf{a}(t)$ projectile acceleration vector as a function of time.

$\mathbf{v}(t)$ projectile velocity vector as a function of time.

$\mathbf{x}(t)$ projectile position vector as a function of time.

t_m mission time of projectile.

$\mathbf{F}_e$ gravitational force vector of the Earth on the projectile.

$\mathbf{F}_m$ gravitational force vector of the Moon on the projectile.

$\mathbf{F}_D$ drag force vector on the projectile.

F_T thrust on the projectile.

α the velocity exponent in the expression for F_D.

ρ the density of air.

A the cross sectional area of the projectile.

C_D the air drag coefficient.

U_ρ uncertainty in the density of air.

U_α uncertainty in α.

U_e uncertainty in the mass of the Earth.

U_m uncertainty in the mass of the Moon.

U_θ uncertainty in the initial position of the Moon.

1. Introduction

The challenge presented refers to the novel "From the Earth to the Moon", by Jules Verne, 1865. *De la terre à la lune* tells the story of the Baltimore Gun Club, a society of weapons enthusiasts, and their attempts to build an enormous space gun and launch three people in a projectile with the goal of a Moon landing. Jules Verne even attempted some rough calculations (e.g. the requirements for the cannon) which considering the comparative lack of any data on the subject at the time, some of his figures are surprisingly close to reality, although physically not practicable. Part of the technical data and assumptions to be employed in solving the challenge are provided, please see [1] for the original problem description. A perfectly spherical projectile of radius 1 metre is subjected to an initial constant vertical thrust of $4 \times 10^6 N$ for 30 seconds. For simplicity, the Earth is assumed stationary, and the entire problem, therefore, is 2-dimensional in the ecliptic plane. The projectile motion is influenced by the gravitational pulls of the Moon and the Earth, as well as a drag force which magnitude depends on projectile velocity, position and altitude. The Moon is supposed to be a prefect sphere of radius 1738 [km] and mass 7.3×10^{22} [kg]. Its orbit has to be assumed circular with radius 384405 [km] and situated in the ecliptic plane, the speed around the Earth is assumed constant. The Earth is supposed to be a prefect sphere of radius 6378 [km] (although its average radius is 6371 [km]), mass is 6×10^{24} [kg] and it is assumed completely motionless. The launch is done at a time such that the projectile will hit the centre of the disc representing the Moon, if everything works to expectation. While there are no uncertainties in controllable quantities (e.g. the magnitude, direction, and duration of the thrust) the precise position of the Moon at the time of launch, masses of the Earth and Moon and other parameters are uncertain. The parameters characterised by uncertainties are:

1. The Moon position at the launch time.
2. The masses of the Earth and the Moon.
3. The air density at different altitudes.
4. The velocity exponent in the drag force expression.

2. Deterministic Model (Theory)

Newton's laws and equations of motion are used to describe the projectile position, velocity and acceleration as functions of time. The problem reduces

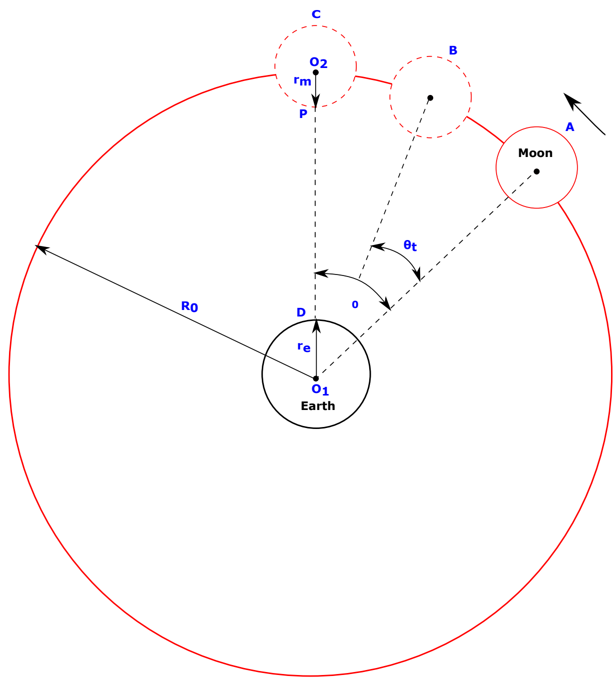

Figure 1: Simplified diagram of projectile-Moon system

to a set of (second-order non-linear) ordinary differential equation (ODE) in $\mathbf{x}(t)$. In this work, two deterministic models are described and adopted to solve the challenge. A first deterministic model (I) assumes that the projectile velocity vector is always towards the vertical (i.e. 1-D trajectory approach) and uses the MATLAB's ode 45 [3, 5] (Runge-Kutta 4^{th} order), to solve the set of ODE. The second deterministic model (II) relaxes the assumption on the velocity (i.e. 2-D trajectory approach) and uses a simple leapfrog solution sequence for the ODE. In both examples the atmospheric density, $\rho(x)$, is given by a lookup table, shown in Table 1. Between specified altitudes the density is assumed to be constant. At above 50000 m the density is zero. In both models a coordinate system is considered centered on a static, non-rotating Earth. In fact, such a coordinate system is non-inertial, and as such Newton's equations of motion are not guaranteed to hold, without introducing fictitious forces. Programming the equations of motion for this system in the frame of reference centered on the center of mass of the Earth-Moon system is not particularly difficult, and a test calculation in MATLAB shows that when our results are compared to this inertial coordinate system

Altitude $/m$	Temperature $/K$	Density $/kgm^{-3}$
0	288.150	1.22500
5000	255.650	0.736116
10000	223.150	0.412707
15000	216.650	0.193674
20000	216.650	0.0880349
25000	221.650	0.0394658
30000	226.650	0.0180119
35000	237.050	0.00821392
40000	251.050	0.00385101
45000	265.050	0.00188129
50000	270.650	0.000977525

Table 1: The density lookup table. In between altitudes the previous density is assumed to apply.

the difference in failure probability is minimal (∼1-3% difference in collision time).

2.1. Deterministic model I: 1-D trajectory approach

Shown in Fig. 1 is the simplified diagram of the projectile-Moon system, the Y-axis is oriented towards the vertical and the system is centred in the Earth geometric center. Ideally, one would expect $\mathbf{F}_m$ to act in the direction of the Moon, $\mathbf{v}$ to be tilted toward the Moon, and $\mathbf{F}_D$, parallel but opposite to $\mathbf{v}$. However, with the assumption that v is vertical, all forces are considered vertical. Thus, the projectile is forced to move on the vertical dotted line.

Applying Newton's second law of motion to the projectile,

$$\begin{aligned} m\frac{dv(t)}{dt} &= F_T - F_D + F_m - F_e \\ \frac{F_D}{m} &= \frac{\rho C_D \left(v(t)\right)^{\alpha}}{2m} \\ \frac{F_e}{m} &= \frac{M_e G}{x(t)^2} \\ \frac{F_m}{m} &= \frac{M_m G}{(R_0 - x(t))^2} \\ G &= 6.67408 \times 10^{-11} m^3 kg^{-1} s^{-2} \end{aligned} \tag{1}$$

where $x(t)$ and $v(t)$ are Y-components of the projectile position and speed vectors at time t, respectively. Substituting the relevant values into Equation 1,

$$
\begin{aligned}
\frac{dv(t)}{dt} &= 4 \times 10^3 - 7.85 \times 10^{-4} \rho \left(v(t)\right)^{\alpha} + f(x) \\
f(x) &= \left(\frac{4.8721}{\left(R_0 - x(t)\right)^2} - \frac{400.44}{\left(x(t)\right)^2} \right) \times 10^{12}
\end{aligned}
\tag{2}
$$

Equation 2 can be expressed in the form,

$$
\frac{d^2x}{dt^2} = C_0 - C\left(\rho\right) \times \left(\frac{dx}{dt}\right)^{\alpha} + f(x) \tag{3}
$$

Where $C_0 = 4 \times 10^3 k_1$ and $C\left(\rho\right) = 7.85 \times 10^{-4} \rho \cdot k_2$ with k_1 & k_2 being constants respectively defining the significance of the thrust and drag force to the projectile's motion. Owing to the imposed constraints on the projectile motion (see [1]),

$$
k_1 = \begin{cases} 1, & \text{if } t \leq 30 \ [s] \\ 0, & \text{otherwise} \end{cases} \qquad k_2 = \begin{cases} 1, & \text{if } x \leq 5 \times 10^4 + r_e \ [m] \\ 0, & \text{otherwise} \end{cases} \tag{4}
$$

Equation 3 is a non-linear second-order ordinary differential equation, which can be solved via an iterative numerical integration technique. Using the *ode45 solver* the equation is required to be decomposed into two first-order ordinary differential equations as follows,

$$
\begin{aligned}
\frac{dx}{dt} &= v \\
\frac{dv}{dt} &= C_0 - C\left(\rho\right) \times v^{\alpha} + f(x)
\end{aligned}
\tag{5}
$$

where v represents the velocity of the projectile. Given a range of time, $t_{span} = [t_{min}, t_{max}]$, and an initial displacement-velocity pair, (x_0, v_0), the solver returns $x(t)$ and $v(t)$. If t_{span} is not divided into time-steps, $x(t)$ and $v(t)$ are computed at steps between t_{min} and t_{max} determined by the solver.

2.1.1. 1-D trajectory: Deterministic results

Recall the objective here is to determine the projectile mission time, t_m. However, the parameters, C_0 and $C(\rho)$, of the projectile's equation of motion change at multiple instances during the mission. An iterative algorithm, therefore, is required to compute t_m. A fixed time-step would normally be used for this purpose, but this may result in unnecessary computational burden, owing to the varying external conditions the projectile's motion is subjected to. In this light, varying time-steps are used at different stages of the mission. The possible values of t_{span} are summarised thus;

$$t_{span} = \begin{cases} [0, 15], & \text{if } t \leq 30 \\ [0, 100], & \text{if } t > 30, x \leq 5 \times 10^4 + r_e \\ [0, 1000], & \text{otherwise} \end{cases} \tag{6}$$

The algorithm relies on the internal steps generated by the solver. This, however, poses some difficulty in obtaining the exact solution at specific stages of the mission, since the user has no control over which time-steps are considered by the solver. For instance, x and v are required at $t = 30$, but the solver may only provide values for say, $t = 28, 29.3, 30.1, 31$ and so on. In such a case, the two times, t_1 and t_2, closest to the desired time, $t_d \mid t_1 < t_d < t_2$, are used to define a new $t_{span} \mid t_{span} = [0, t_2 - t_1]$, divided to a time-step of 0.001. With $(x_0, v_0) = (x(t_1), v(t_1))$, the differential equation is solved again, to obtain $x(t_d)$ and $v(t_d)$. Similarly, if v and t are required at a point x_d metres from the surface of the Earth, t_1 and t_2 correspond to x_1 and $x_2 \mid x_1 < x_d < x_2$.

Let $\boldsymbol{H} = \{h_j\}^{11}$ and $\boldsymbol{\rho} = \{\rho_j\}^{11}$ respectively be the set of altitudes for the 11 atmospheric layers and its corresponding set of air densities (see Tab. 1). If $\boldsymbol{t}$ is the vector of times at which the projectile's motion is analysed, t_l, x_l, and v_l, the last elements of $\boldsymbol{t}$, $x(t)$, and $v(t)$, the algorithm, hereafter referred to as Algorithm 1, is summarised thus;

1. Set $(x_0, v_0) = (r_e, 0)$, $t_{span} = [0, 15]$, $k_1 = k_2 = 1$, $j = 1$, and $i = 0$. Where i is a flag that takes the value 1 if the projectile completes the mission.
2. Set $\rho = \rho_j$, update C_0 and $C(\rho)$ where necessary, solve Equation 5, and update $\boldsymbol{t}$, $x(t)$, and $v(t)$.
   (a) Terminate algorithm if any of the computed velocities is negative.

(b) Truncate $\boldsymbol{t}$, $x(t)$, and $v(t)$ at $x(t) = h_{j+1} + r_e$, discarding all values corresponding to $x(t) > h_{j+1} + r_e$.
(c) If $k_1 = 0$ and $t_l \geq 30$, truncate $\boldsymbol{t}$, $x(t)$, and $v(t)$ at $\boldsymbol{t} = 30$. Set $(x_0, v_0) = (x_l, v_l)$, $k_1 = 0$, $t_{span} = [0, 100]$, and go to step 2(e).
(d) Set $j = j + 1$ and $(x_0, v_0) = (x_l, v_l)$.
(e) Repeat step 2 until $j = 11$ and set $k_2 = 0$.

3. Solve Equation 5 and update $\boldsymbol{t}$, $x(t)$, and $v(t)$.
   (a) Terminate algorithm if any of the computed velocities is negative.
   (b) If $k_1 = 0$ and $t_l \geq 30$, truncate $\boldsymbol{t}$, $x(t)$, and $v(t)$ at $\boldsymbol{t} = 30$. Set $(x_0, v_0) = (x_l, v_l)$, $k_1 = 0$, and go to step 3(d).
   (c) If $x_l \geq R$, truncate $\boldsymbol{t}$, $x(t)$, and $v(t)$ at $x(t) = R$. Set $t_m = t_l$, $i = 1$, and go to step 4. Otherwise, proceed.
   (d) Set $(x_0, v_0) = (x_l, v_l)$ and $t_{span} = [0, 1000]$.
4. Output $\boldsymbol{t}$, $x(t)$, $v(t)$, t_m, and i.

Given t_m computed from the nominal values of the relevant projectile motion parameters, the expected initial position, θ_0, of the Moon is given by ωt_m. For $\alpha = 2.25$, $t_m = 1.0339 \times 10^4$ [s], $\omega = 2.6638 \times 10^{-6}$ [rad/s], and $\theta_0 = 0.0275$ [rad], based on a 27.321 day period of Moon revolution around the Earth. This phase of the analysis reveals that the projectile clears the 11 atmospheric layers in 22.95 [s]. During this period, its velocity vectors behave as plotted in Figs. 2 (top plot). The ripples in the velocity plot are a result of the non-continuous decrease in air density with increasing altitude. It could be seen from this plot that at the very early stages of the mission, the velocity is almost constant between adjacent layers. However, as the drag becomes less significant with altitude, the effects of the constant thrust become more prominent, and the projectile accelerates, even between adjacent layers. At an altitude of 5×10^4 metres, the projectile has already attained a velocity of 9876 [m/s], at which stage the only relevant forces are thrust and gravitation. Since the effect of these forces relative to the thrust is negligible, the projectile travels with constant acceleration, as shown in the centre plot in Fig. 2. By the time the constant thrust is removed at $t = 30$, the projectile has already travelled a distance of about 2.19×10^5 [m] and attained a velocity of 3.8×10^4 [m/s]. Beyond this point, the only forces acting on the projectile are the gravitational pulls of the Earth and the Moon. However, the former is much greater than the pull by the Moon, explaining the deceleration of the projectile. The projectile continues at this non-uniform deceleration until it is in the vicinity of the Moon's orbit. At this point, the

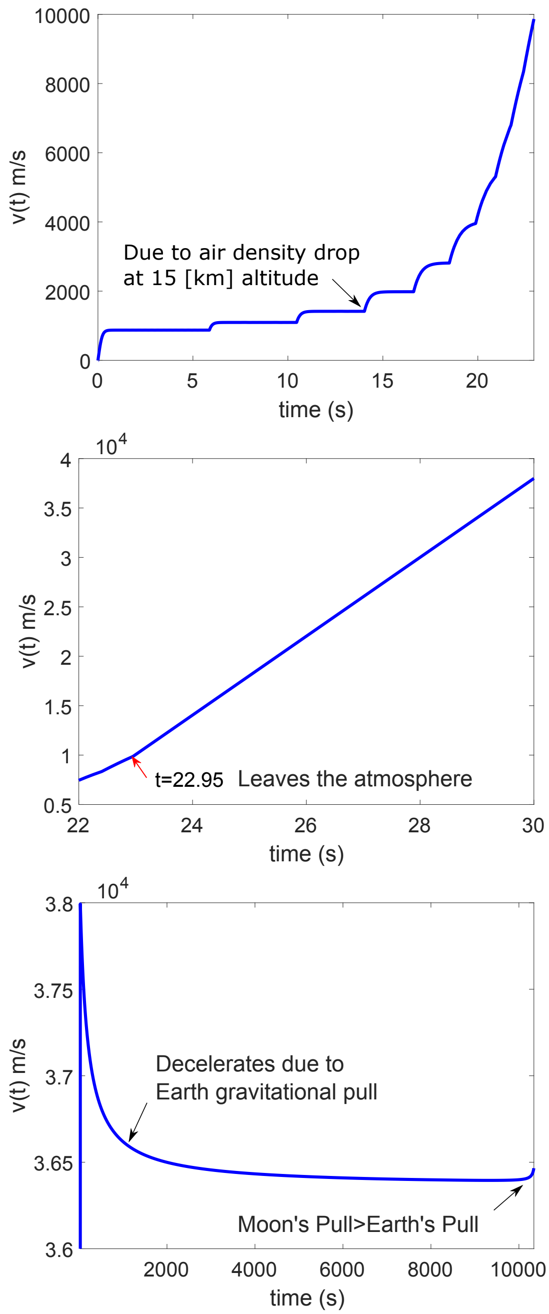

Figure 2: Projectile velocity behaviour inside the atmosphere (top plot) and outside the atmosphere (midle and bottom plots)

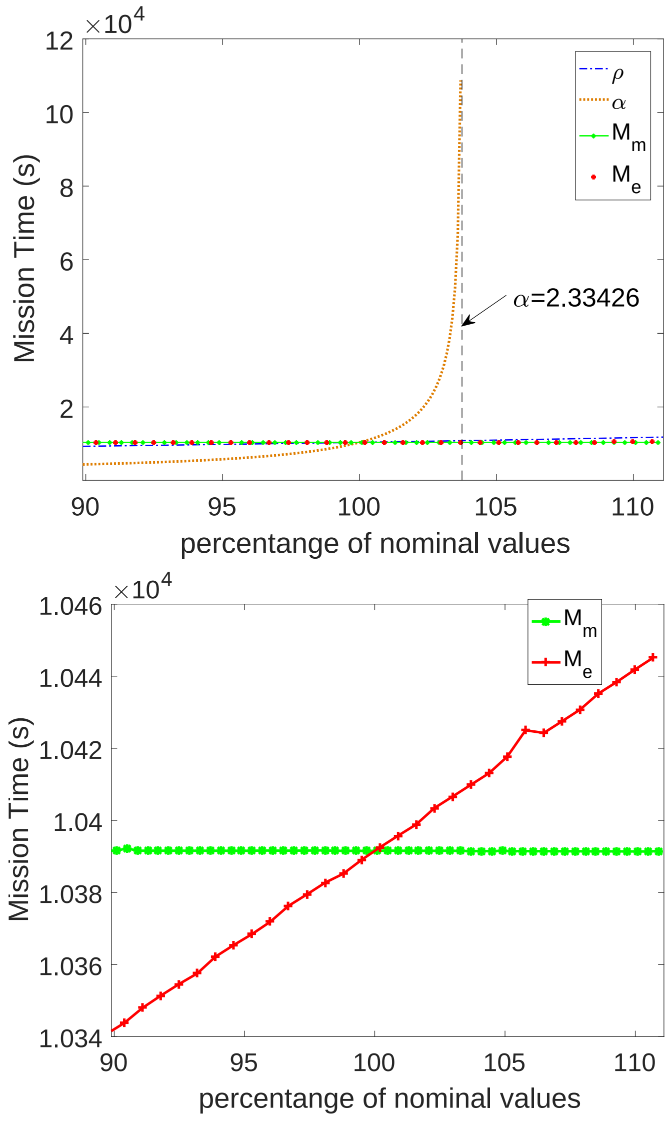

Figure 3: Projectile mission time sensitivity to motion equation parameters

pull by the Moon exceeds the pull by the Earth, and the projectile begins to accelerate non-uniformly, as depicted at about 10000 seconds in the bottom plot of Fig. 2.

To estimate the relative significance of the relevant parameters in the projectile's motion equation, a sensitivity analysis of its mission time to these parameters was carried out. As shown in Fig. 3, α affects the mission time the most, followed by ρ. The mission time increases exponentially as α approaches 2.33426, beyond which, the projectile never completes the mission before it starts falling back to Earth. Relative to drag, the gravitational forces seem almost insignificant and equal, though Fig. 3 (bottom panel) indicates the Earth's pull is much more relevant. In fact, the difference in mission time with the gravitational effect of the Moon ignored is only 0.4430 [s], and 458 [s] with the Earth's ignored.

2.2. Deterministic model II: 2-D trajectory approach

A second deterministic model is employed for the problem solution. The simplified diagram of the projectile-Moon system is similar to the one proposed in model I and depicted in Figure 4. The main difference is that F_m acts in the direction of the Moon, $\boldsymbol{v}$ can be tilted toward the Moon, and F_D, parallel but opposite to $\boldsymbol{v}$. The solution to the set of ODE is approximated by integration methods using a simple leapfrog integration method:

$$\begin{aligned}
m \cdot \boldsymbol{a}(t) &= \boldsymbol{F}_e(t) + \boldsymbol{F}_m(t) + F_T(t) + \boldsymbol{F}_D(t) \\
\boldsymbol{v}(t+\Delta_t) &= \frac{\boldsymbol{a}(t) + \boldsymbol{a}(t+\Delta_t)}{2} \cdot \Delta_t + \boldsymbol{v}(t) \\
\boldsymbol{x}(t+\Delta_t) &= \frac{1}{2}\boldsymbol{a}(t) \cdot \Delta_t^2 + \boldsymbol{v}(t) \cdot \Delta_t + \boldsymbol{x}(t) \\
\boldsymbol{F}_e(t) &= \frac{G \cdot m \cdot m_e}{|\boldsymbol{x}(t)|^3} \cdot \boldsymbol{x}(t) \\
\boldsymbol{F}_m(t) &= \frac{G \cdot m \cdot m_m}{|\boldsymbol{x}_{pm}(t)|^3} \cdot \boldsymbol{x}_{pm}(t) \\
\boldsymbol{F}_D(t) &= -\frac{1}{2} \cdot \boldsymbol{v}(t)^{\alpha} C_D \cdot \rho(\boldsymbol{x}(t)) \cdot A
\end{aligned} \tag{7}$$

where $\boldsymbol{x}_{pm} = \boldsymbol{x}(t) - \boldsymbol{x}_m(t)$ is the vector connecting projectile and Moon centre and Δ_t is the t^{th} time step, i.e. the difference between two consecutive

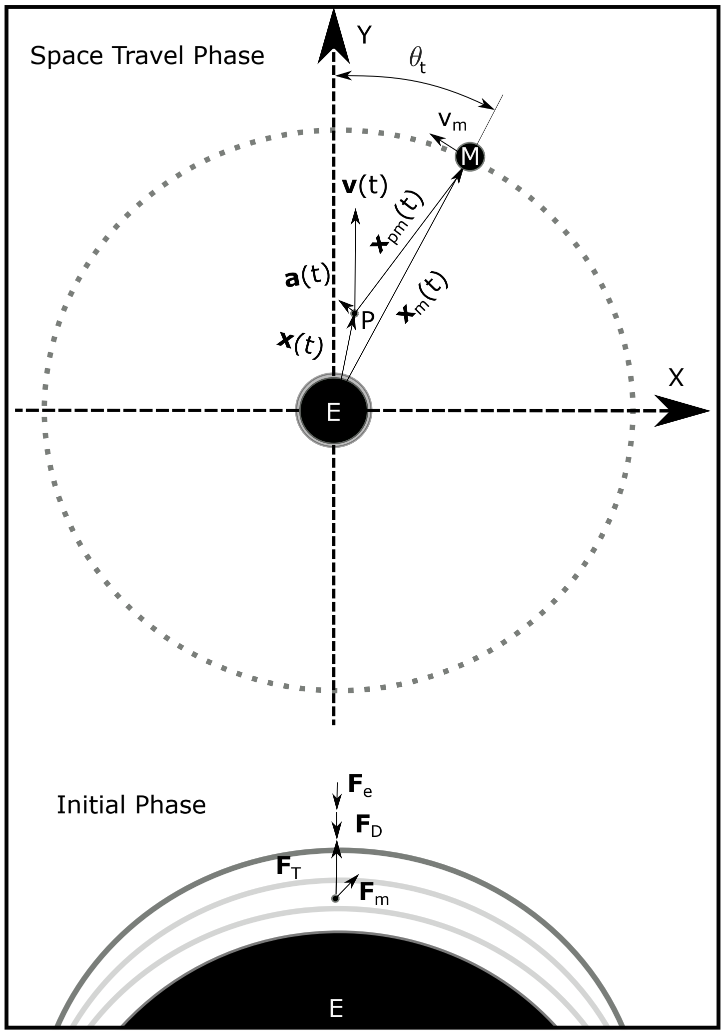

Figure 4: Position, velocity and force vectors used in the 2-D trajectory model.

instants for which the $\boldsymbol{x}$, $\boldsymbol{v}$ and $\boldsymbol{a}$ are evaluated. The Moon's position vector can be expressed in polar coordinates:

$$\boldsymbol{x}_m(t) = (R_0 \cdot sin(\theta_t)\ ,\ R_0 \cdot cos(\theta_t))$$

and the Moon's angular coordinate changes with time as:

$$\theta_{t+\Delta_t} = \theta_t - \frac{2\pi}{T_m}\Delta_t$$

where T_m is the period from Moon-rise to Moon-rise at the poles, measured to be 27.3216 days. The initial conditions on speed and position vectors are $\mathbf{x}(0) = (0, R_e)$, $\boldsymbol{v}(t) = (0, 0)$. Similarly to the previous approach, calculations are quickened by devising the time domain into 3 part, characterised by different Δ_t. The first phase is the 'take-off' (interval 0-50 [s] from the launch) a small time step $\Delta_t = 5$ [ms] is considered (i.e. to better capture rapid changes in $\boldsymbol{a}$ due to non-null and variable $\boldsymbol{F}_D(t)$). Second phase is the space travel from Earth to Moon between 50 seconds and 3500 seconds and $\Delta_t = 0.5$ [s]. In the third phase the projectile is approaching the Moon and it has been assumed $\Delta_t = 0.2$ [s]. The solution sequence is stopped for $t > 18000$ seconds or when projectile position is 1.2 times greater than the Moon's orbit $|\boldsymbol{x}(t)| \geq 1.2 \cdot R_0$.

2.2.1. 2-D trajectory: Deterministic results

Also for this second physical model, deterministic analysis has been performed fixing the uncertain quantities to their expected value (e.g. $\alpha = 2.25$, etc.). In this analysis the Moon's launch angle is fixed to be $\theta_0 = 0$ (i.e. Moon center on the Y axis at time 0). Figure 5 presents acceleration speed and position of the projectile for the simulation both in X-direction (left hand side) and Y-direction (right hand side). For the selected $\theta_0 = 0$, the projectile misses the target. However, it experiences a non-negligible pull from the Moon gravitational field (acceleration in the bottom left panel, after about 10000 seconds) which makes its trajectory deviate of about 10 [km] (Projectile position X panel on the top). The Y-component of the acceleration has been zoomed around the interval 0-50 seconds to improve graphical output. It can be observed that acceleration suddenly increases (step-like) and then rapidly decreases with an exponential-like trend. This is due to the imposed assumption of the density being constant at different altitudes. Once the projectile passes from a lower atmospheric layer to an upper layer, the density drops, the drag force drops and, in turn, the acceleration suddenly increases (because the trust is still pushing).

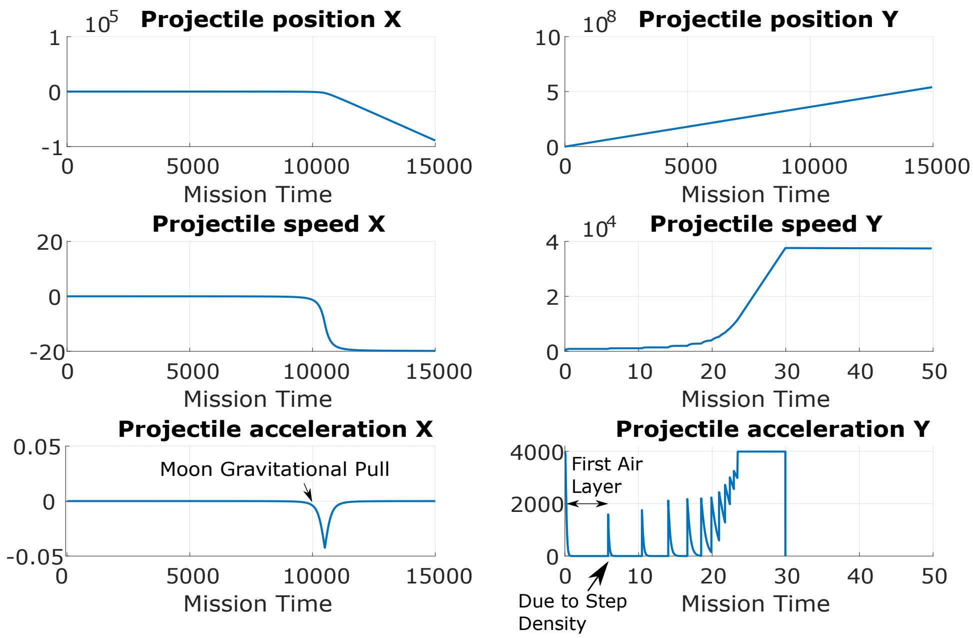

Figure 5: The deterministic result for the projectile position [m], velocity [m/s] and acceleration [m/s^2] vectors. Acceleration and velocity are zoomed in 0-50 seconds to improve graphical output.

In order to select a good expected Moon's angle θ_0 for the projectile launch, a Latin hyper-cube design of experiment (DOE) is used to explore the α-θ_0 space. A total of 5000 combinations of α (between 2 and 2.5) and Moon angle (between 0 and 0.05 rad) have been analysed while keeping other parameter fix to their expected values. Results are reported in Figure 6, blue stars markers represent successful simulations (i.e. projectile hits the Moon). It can be noticed that for α greater than 2.3 the projectile can not reach the Moon (it falls back on the Earth surface due to the strong drag force). For lower drags, it is possible to reach the Moon, although the launch window (i.e. set of θ_0 leading to success) is quite small and changes depending on the coefficient α. For instance if $\alpha = 2$ a θ_0 about 0.01 [rad] allows hitting the Moon, whilst for $\alpha = 2.25$ a firing angle within 0.02-0.03 [rad] allows completing the mission. This indeed makes the problem very challenging

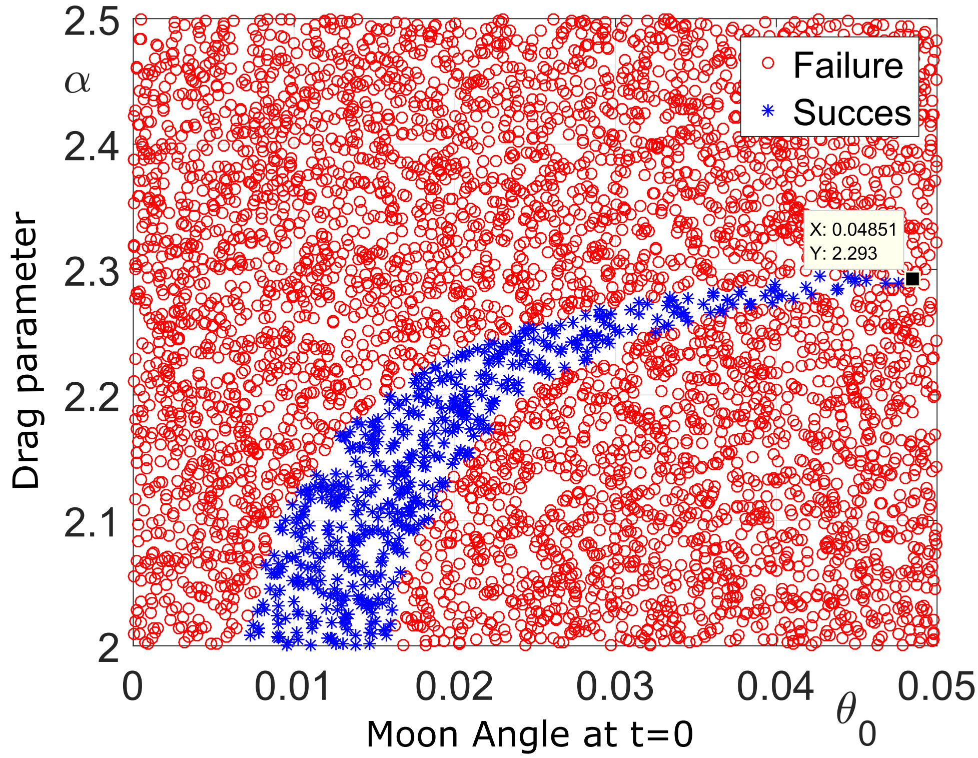

Figure 6: The 5000 Latin hyper-cube realisation for α and θ_0, clearly the launch window changes with α. The blue stars are success (projectile hits the Moon surface) while the red dots are failed simulations (projectile either fall back on the Earth surface of surpass the moon without landing).

especially considering that uncertainty is affecting the mission.

3. Uncertainty Quantification (UQ)

Sources of uncertainty have been classified in literature as aleatory and epistemic. The first is used to model random quantities, stochastic behaviours and inherent variability. Examples are natural variability the weather conditions or random fluctuation of quantities around a design value. The idea is that aleatory uncertainty can be quantified but can not be reduced by further data gathering. Conversely, epistemic uncertainty has been used to describe the lack of knowledge, missing information, imprecision in parameters and vagueness. It can be quantified and in particular situations reduced by further data gathering (although not always feasible due to time-cost constraints).

The challengers suggested in the problem statement that, usually, Gaussian distribution is assumed if the random quantities are reasonably well-known (for instance the mass of the Earth) and uniform laws for parameters on which the knowledge is rather poor (for instance the coefficient α). All choices should be clearly stated, since the result will depend upon the definition of these laws.

The two deterministic problem have been analyzed using similar approaches (Monte Carlo method) and slightly different probabilistic models describing the uncertainty sources. The results are compared and discussed and show good agreement.

3.1. Uncertainty Quantification: 1-D trajectory approach

The projectile's equation of motion (Eq. 1) uses the nominal values of the air density and the masses of the Moon and the Earth. With the introduction of uncertainties in these parameters, Equation 3 is re-written as,

$$\begin{aligned}\frac{d^2x}{dt^2} &= C_0 - C\left(\rho\right) \times \left(\frac{dx}{dt}\right)^{2.25U_\alpha} + f(x)\\ C\left(\rho\right) &= 7.85 \times 10^{-4} \rho k_2 U_\rho\\ f(x) &= \left(\frac{4.8721U_m}{\left(R_0 - x(t)\right)^2} - \frac{400.44U_e}{\left(x(t)\right)^2}\right) \times 10^{12}\end{aligned} \tag{8}$$

Similarly, the initial position of the Moon, θ, is given by $0.0275U_\theta$. The uncertainties, U_α, $U\rho$, U_m, U_e, and U_θ, are, therefore, random variables with a mean of 1. U_ρ has been assumed fixed for all 11 atmospheric layers. This assumption is based on the consideration that the same method was used to determine the air density at the various layers, in which case the uncertainties would be equal and correlated. U_θ and U_α are assumed a *Uniform distribution* with parameters $(0.99, 1.01)$ and $(0.8889, 1.1111)$ respectively, the remainder are assumed a *Normal distribution* with a 1% coefficient of variation.

The stochastic analysis is, therefore, reduced to checking for mission success for various samples of the pair, $(U_\alpha, U\rho, U_m, U_e, U_\theta)$. An indication function, s_n, taking the value 1 if the n^{th} mission is successful, and 0, otherwise

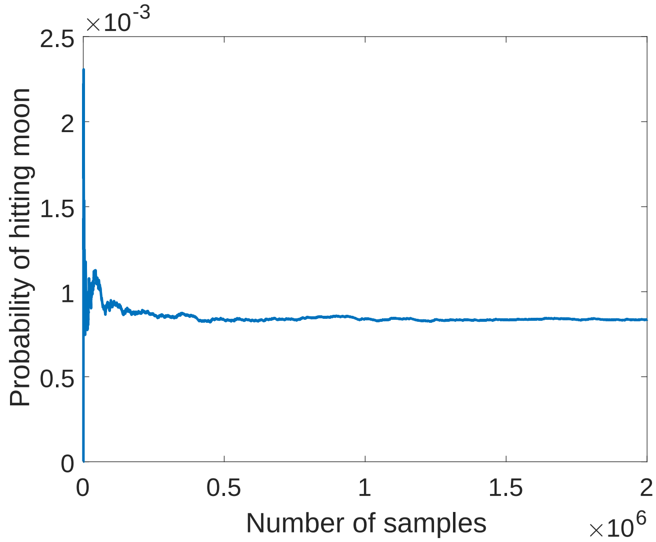

Figure 7: Variation of p with number of simulation samples, N

is used to record the simulation history. In summary,

$$s_n = \begin{cases} 1, & \text{if } i = 1 \quad \& \quad \omega t_m^{\{n\}} = 0.0275 U_\theta^{\{n\}} \\ 0, & \text{otherwise} \end{cases} \tag{9}$$

In Equation 9, the apex n denotes the n^{th} mission. For N missions, the probability, p, of success is given by,

$$p = \frac{1}{N} \sum_{n=1}^{N} s_n \tag{10}$$

The algorithm for the stochastic analysis is summarised thus,

1. Predefine the desired accuracy level (this could be the coefficient of variation) and set $n = 1$.
2. Set $s_n = 0$, generate the pair $(U_\alpha, U\rho, U_m, U_e, U_\theta)$, and solve the projectile motion using Algorithm 1.
3. Check for mission success if i, as obtained from step 2 equals 1 and update s_n accordingly. Otherwise, proceed.
4. Set $N = n$ and compute p.
5. Set $n = n + 1$.

6. Repeat steps 2 to 5 until the desired level of accuracy is attained and terminate algorithm.

For 2×10^6 samples, a success probability of 8.36×10^{-4} was obtained, with a coefficient of variation of 5.78%. The convergence of the simulation is highlighted by Fig. 7.

3.2. Uncertainty Quantification: 2-D trajectory approach

For the second model the uncertainty has also been characterized by assuming probability distribution functions for the uncertain quantities. Table 2 summarizes the assumptions made and ranges considered for the random variables. Discussions regarding the uncertainty on the Earth's mass and the Moon's mass are provided in Refs. [2] and [4]. These uncertainties are considered here as measurement errors/tolerances/imprecision (i.e. epistemic uncertainty). Uniform distributions were assumed for all of the uncertain parameters by applying Laplace's principle of indifference (i.e. maximum entropy). Notably, the uncertainty in the atmospheric density, ρ, in the second model differs to its treatment in the first model. In the second model each step in the function $\rho(x)$ has been considered to be a random variable. For this reason the second model considers more uncertain parameters than the first model.

	Distribution	Ranges	Ref.
m_e	Uniform	$(5.9722 \pm 0.0006)10^{24}$ [kg]	[2]
m_m	Uniform	$(7.3458 \pm 0.0007)10^{22}$ [kg]	[4]
α	Uniform	(2.25 ± 0.25)	[1]
θ_0	Uniform	(0.013,0.017) [rad]	[]
ρ	Uniform	$(\rho_i \cdot 0.95, \rho_i \cdot 1.05)\forall i$ [kg/m^3]	[]

Table 2: The probabilistic model used for the uncertainty quantification of the 2-D trajectory model.

3.3. 2-D trajectory approach

In order to check the success/failure of the mission, a simple indicator function for the success is introduced as follows:

$$I_{s1} = \begin{cases} 1 & \text{if } S_1 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

where S_1 is true if the following statement holds $\exists\ t:\ |\boldsymbol{x}_{pm}(t)| \leq r_m\ t = 0,..,T_{max}$. The condition $S1$ is met if the norm of the relative distance between projectile and Moon center is less or equal the moon radius. In other words, I_s is equal 1 if the projectile (represented by a point) intersect the projection of the Moon on the ecliptic plane (a disc of radius R_m). A second success criteria is defined to check if the projectile hits the Moon's centre, similarly to what proposed in the 1-D trajectory approach. The second indicator function is similar to the s_n (Eq. 9) and defined as follows:

$$I_{s2} = \begin{cases} 1 & \text{if } S_2 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

where the condition S_2 holds if $\exists\ t:\ |\boldsymbol{x}_{pm}(t)| \leq r_m\ \&\ x_{pm,x}(t) < \frac{x_{pm,y}(t)}{1e2}\ t = 0,..,T_{max}$ The success condition is met if the projectile falls very close to the 'center' of the Moon, specifically, if when the projectile hits the moon the X-component of the relative position vector projectile-Moon is 100 times smaller than its Y-component.

A direct Monte Carlo method is used to sample input uncertainty as described by Table 2 and forward samples to the 2-D solver. For each sample (one realization of the uncertain input), the set of equations previously described is solved. The output is saved and indicator function evaluated, the sampling stops once a predefined number of simulations ($N = 1e4$) is reached. The indicator function is used to estimate the probability of success (i.e. 1-P_f) by averaging the number of success over the total number of samples. A step-by-step procedure that summarizes the forward propagation approach is as follows:

1. Sample uncertain input variables accordingly to Table 2. The i^{th} sample vector will be $\mathbf{U}_i = [m_e, m_m, \alpha, \rho_1, ..., \rho_{11}, \theta_0]_i$,
2. Solve set of differential equations and obtain output, e.g. $|\boldsymbol{x}_{pm,i}(t)|$ for the given $\mathbf{U}_i$ and for $0 < t < T_{max}$.
3. Check if the travel is successful (i.e. $I_{s1,i} = 1$ and/or $I_{s2,i} = 1$) or failure (i.e. $I_{s1,i} = 0$ and $I_{s2,i} = 0$).
4. Store the indicator function, time to arrival and projectile trajectory $\boldsymbol{x}(t)$.
5. Repeat the procedure for all samples $i = 1, ..., N$.

The entire procedure relies upon the number of samples N, the more samples are generated the more accurate is the estimation of probability of success.

The probabilities of failure for the two success conditions are obtained as follows:

$$P_{f1} = 1 - P_{s1} = 1 - (\frac{\sum_{i=1}^{N} I_{s1,i}}{N} + \frac{\sum_{i=1}^{N} I_{s2,i}}{N})$$

$$P_{f2} = 1 - P_{s2} = 1 - \frac{\sum_{i=1}^{N} I_{s2,i}}{N}$$

The top and bottom plots of Fig. 8 show the two probability of failure for different number of Monte Carlo samples. It can be observed that P_{f1} converges nicely for increasing number of samples. Conversely, averaging of P_{f2} is not efficient due to the small number of successful simulations as confirmed by previous UQ analysis on the 1-D trajectory model. Accordingly to the probabilistic model presented in Table 2 P_{f1} is about 0.63 (0.37 the success probability) whilst P_{f1} is about 0.996 (and just 0.004 the probability of hitting the center).

Fig. 9 presents the points on the Moon where the simulated projectile landed and the two success criteria (S_1 and S_2). Fig. 10 present the same result as in Fig. 9 but with different prospective. The X-Y coordinates of the landed projectiles are transformed to polar coordinates w.r.t. the Moon's centre and the Kernel probability density estimated for the landing angle. Fig. 10 shows that, accordingly to the selected probabilistic model, is more likely to hit the 'right side of the Moon' (i.e. the angular sector between $\pi/2$ and $\pi/4$).

3.4. Sensitivity Analysis

Sensitivity analysis is the study of how the uncertainty in the output of a model can be apportioned to different sources of uncertainty in its inputs. For variance-based methods the sensitivity of the output to an input variable is measured by the amount of variance in the output caused by that input. The first order (e.g. additive) contribution of the input i to the variance of an output Y can be expressed as:

$$\frac{Var[E[Y|X_i = x_i]]}{Var[Y]}. \tag{13}$$

The numerator is the variance of the expectation of the output Y when fixing the input X_i to the value x_i and correspond to the Sobol's main effect. The

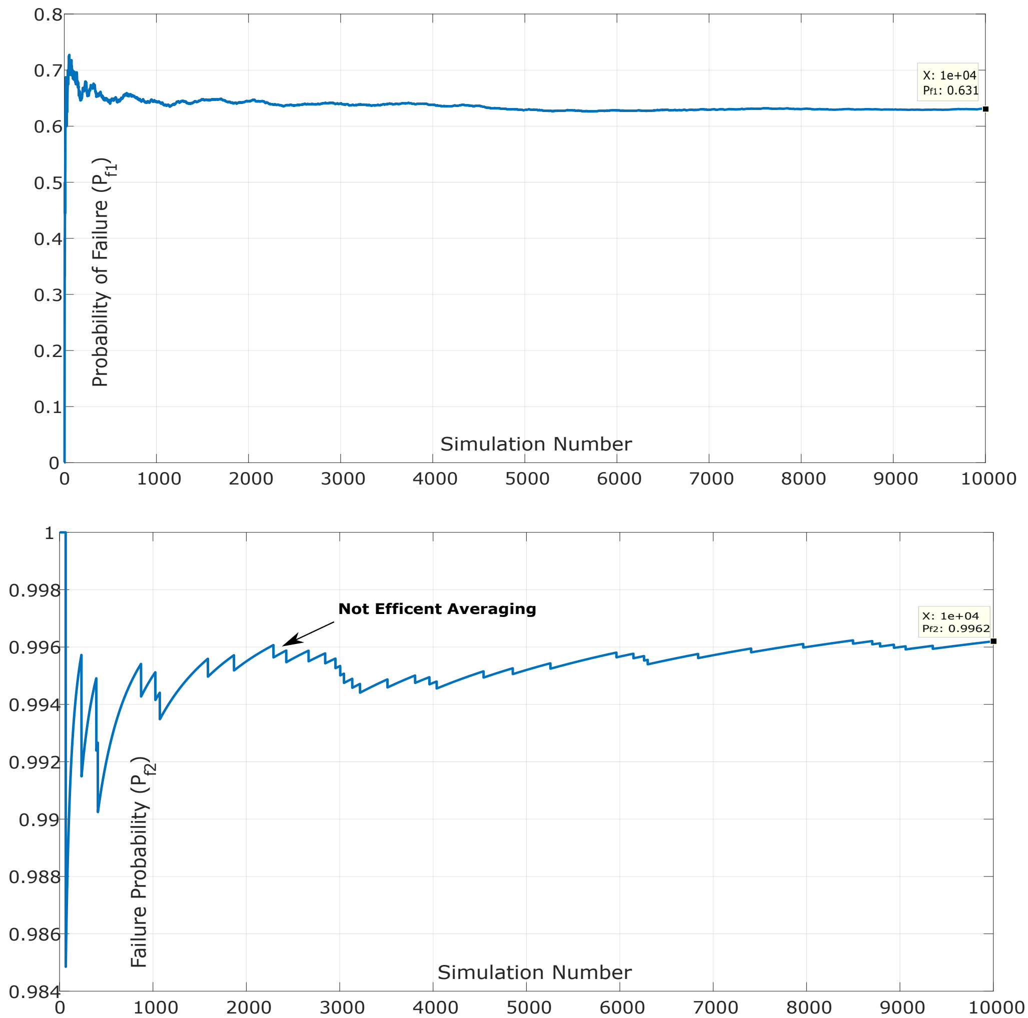

Figure 8: The convergence of the failure probability of hitting the Moon Surface (P_{f1}) and of hitting the Moon 'centre' (P_{f2}).

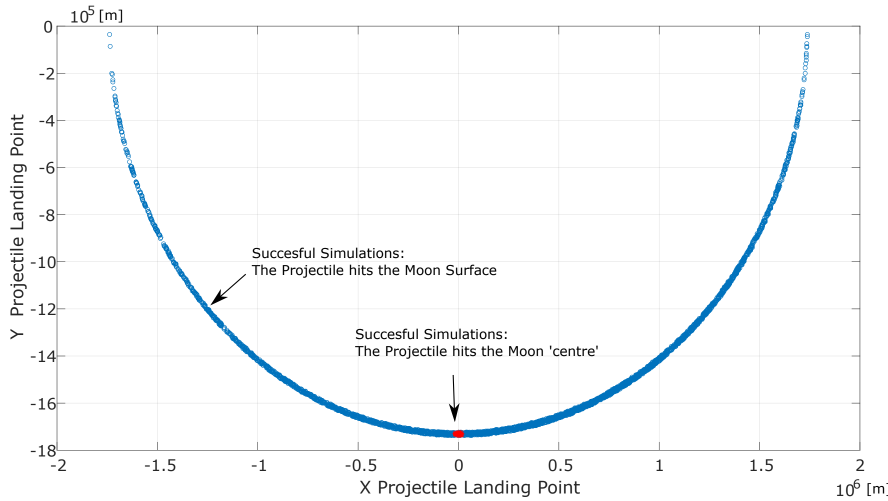

Figure 9: The projectile position when the simulation successfully stops. The blue dots are successful with respect to surface landing whilst the red dot are successful with respect to the Moon 'centre' landing (S_2 criteria).

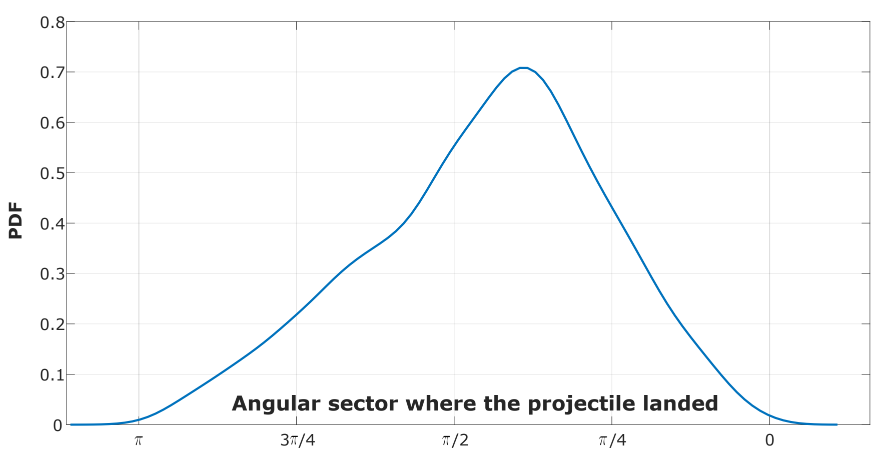

Figure 10: The probability distribution of the projectiles hitting the Moon expressed as Moon's angular sector. It can be noticed that, given the probabilistic model in Table 2, it is more likely to hit the 'right' side of the Moon surface (between $\pi/2$ and $\pi/4$).

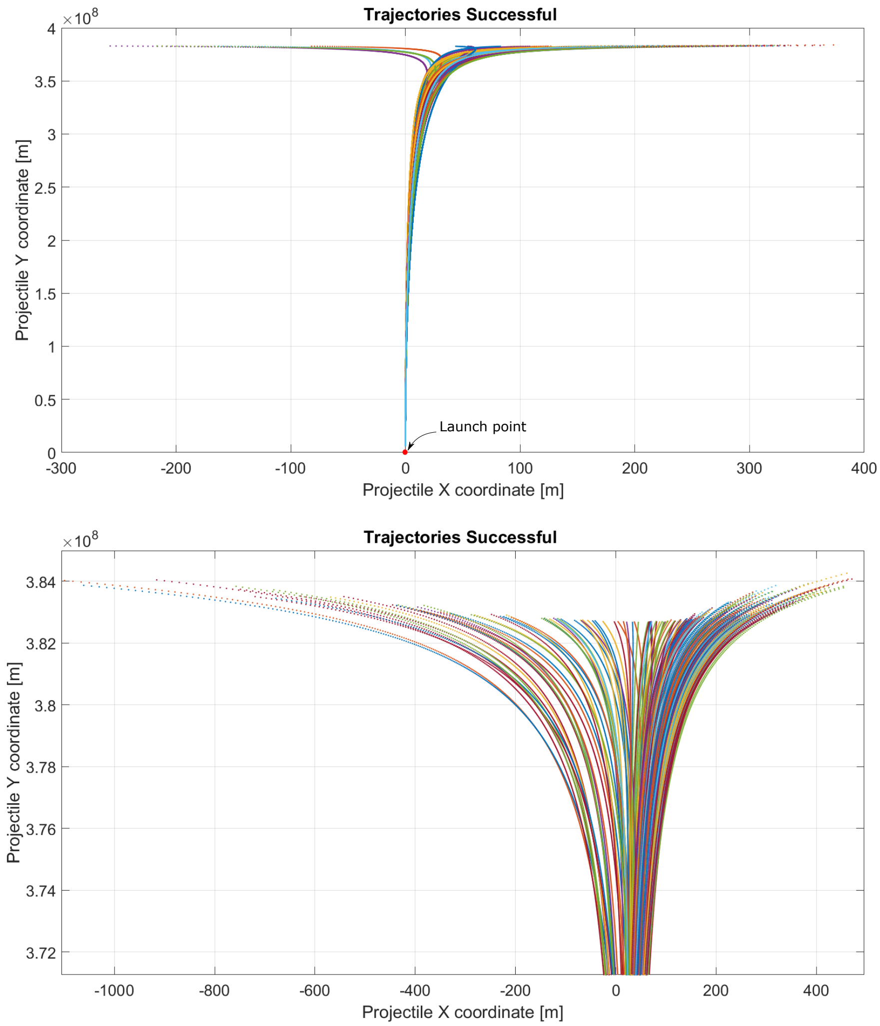

Figure 11: Two samples of 50 (top panel) and 250 (zoomed in bottom panel) projectile trajectories which result in successful simulations.

index has limitations (i.e. interaction effects are neglected). Nevertheless, the sensitivities scores can be efficiently obtained retaining the MC model output realizations for a negligible computational cost. The sensitivity indexes for the success indicator is displayed in Fig. 12. The result confirms that the model output resulted highly sensitive to α while the other inputs are less relevant.

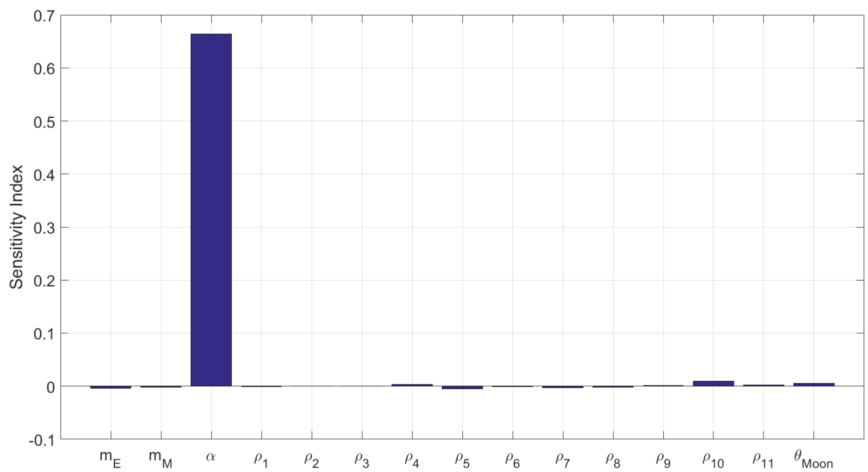

Figure 12: The main effect sensitivity score for the random variables accounted in the model. The α is clearly dominating the variability of the model output.

4. Discussion and Conclusion

Comparison between the 1-D and 2-D trajectory models shows good agreement between the results, especially in the first phase of the launch when the drag and thrust forces were dominant. The 2-D trajectory model shows that the projectile can experience a deviation greater than 1 km in the X-coordinate, which is attributable to the gravitational pull of the Moon. Although the 1-D and 2-D trajectory approaches where applied on slightly different probabilistic models, the obtained success probabilities (of hitting the Moon's 'centre') have the same order of magnitude (10^{-3}). This can be explained by considering the fact that the main key uncertainty driver for the problem is the parameter $z\alpha$, which has been identically modelled by both the 1-D and 2-D stochastic approaches. The probability of hitting the

Moon's surface was estimated to be about 0.37, which the authors belive was fairly high considering the extent of the uncertainties. Sensitivity analysis confirmed that the main driver for the uncertainty is the α exponent in the Drag force formula.

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