Competitive Mathematical Game 2011-2012

Choice of the best itinerary by a driver

Fédération Française des Jeux Mathématiques

and

Société de Calcul Mathématique SA

with the newspaper Auto Plus

I. Presentation of the games

The mathematical games, organized jointly by FFJM and SCM were started 3 years ago ; the previous ones were:

– In 2008-2009, conception of a transportation network (buses) in a city, in partnership with Veolia Transport ;

– In 2009-2010, conception of an electricity distribution network, in partnership with RTE.

These games deal with the solution of a problem of general interest, but simplified in order to allow a solution. Still, finding a solution usually requires several months of work. The candidates may compete individually, or as groups (typically, groups of students, writing a memoir).

Prizes fall into two categories:

Individual prizes:

For the winner: 500 Euros
For the second: 200 Euros
For the next three: 100 Euros each.

Group prizes:

For the winner: 500 Euros
For the second: 200 Euros
For the next three: 100 Euros each.

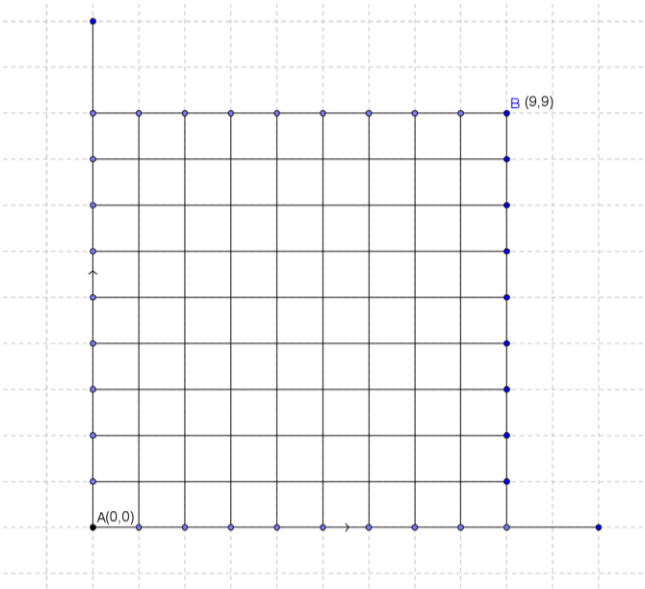
So the total amount of prizes is 2 000 Euros. The best solutions will be published on the web site of FFJM, on the web site of SCM, and on the web site of our partners.

The official announcement of the results will occur during the "Salon de la Culture et des Jeux Mathématiques", which is hold in Paris during the month of May. The prizes will be given at the same time. See the document "Règlement du Jeu-Concours", attached.

The deadline for sending the solutions is April 15th, 2012. The solutions should be written in English or French, under pdf format, and should be sent by email to : ffjm@wanadoo.fr.

II. The 2011-2012 prize

One has to chose the "best" itinerary in order to go from a city A to another city B, on a simplified map, made of ten vertical lines and ten horizontal lines. A is in the lower left corner and B is in the upper right corner, see the picture below. One can move only along the lines.



The elementary pieces are called "segments" ; they all have the same length, namely 100 km. So the distance from A to B is at least 1 800 km. The car is the same for everyone ; only the itinerary is to be computed.

As usual, the x axis has the orientation from left to right and the y axis from bottom to top. So the point A has coordinates $(0,0)$ and the point B $(9,9)$.

Each segment will be designated by the coordinates of its starting point and:

– the letter V (for "vertical") ; the segment has then the orientation from bottom to top;

– the letter H (for "horizontal") ; the segment has then the orientation from left to right.

For instance, the segment (3,4,V) goes from the point (3,4) to the point (3,5) ; the segment (5,7,H) goes from the point (5,7) to the point (6,7).

There are speed limits on each segment, and they differ from one segment to the other; these speed limits are given in the Excel file which is attached. These speed limits may be only 50, 90, 110, 130 km/h.

On the segments where the speed limit is 130 km/h (highways), there is a toll, which is arbitrarily fixed to the amount of 3 Euros for each segment.

Besides that, we have two types of cost:

- One is linked with the duration of the trip (hotel, restaurant, and so on). It will be given by the formula $c_1 = 5\,t$, where c_1 is a cost in euros and t is the time in hours;

- The second cost is linked with gas consumption for the car, which is given as a function of the speed, by a simplified formula:

$$conso = av + b$$

where $conso$ is the consumption in liters, v is the speed in km/h ; the coefficients a, b are respectively:

$$a = 0,0625\,,\ b = 1,875$$

The cost of gas will be fixed arbitrarily in a simplified manner to 1.3 Euro/liter.

A. Problem 1 (déterministic)

The car respects constantly all speed limits. Determine the itinerary (or itineraries) for which the total time is minimal and the itinerary (or itineraries) for which the cost is minimal.

B. Probleme 2 (probabilistic)

Some drivers do not care sufficiently about rules and they do not respect the speed limits.

The Government, in order to impose the respect of reglementations (and also perhaps in order to collect some fines), places some radars. There are two types of radars:

- Fixed radars, which are installed on known segments. There are 20 such fixed radars in total, and they are put on the following segments:

  (0,8,V) ; (1,2,V) ; (1,4,V) ; (2,0,V) ; (3,5,V) ; (5,3,V) ; (5,8,V) ; (7,2,V) ; (9,2,V) ; (9,5,V)
  (3,0,H) ; (8,1,H) ; (2,2,H) ; (5,3,H) ; (3,4,H) ; (0,5,H) ; (4,6,H) ; (4,7,H) ; (0,8,H) ; (7,9,H)

(this information can also be found in the Excel file)

- Mobile radars; their number is also 20. By convention, each mobile radar may be anywhere, on any segment, with equal probability. One can even find several mobile radars on the same segment, and one or more mobile radars on a segment which already contains a fixed radar. The Government really puts them at any place, at random.

A radar covers the totality of a segment, and we admit that the speed of the vehicle is constant on a whole segment.

On any segment, for the fixed and for the mobile radars, the detection rule is as follows:

- If the speed of the car is 10% above the speed limit, it will be detected with probability 0.7 and the fine is 100 Euros;

- If the speed of the car is 50% above the speed limit, it will be detected with probability 0.9 and the fine is 200 Euros.

Let T_0 be the minimal time found in paragraph A (legal time, observing all speed limits). A driver, who is in a hurry, would like to do the whole trip in a time $\leq 0{,}8\,T_0$. Then he may receive fines. The question is : what is the itinerary (itineraries) which minimize the expectation of the total cost of the trip, including fines, and what is the value of this expectation.

Please note : the question is about the a priori choice of an itinerary, taking into account the data given above. The driver wants, before he starts his journey, to find the most favourable itinerary ; the criterium which is chosen is the expectation of the cost, including fines. The question is not about the real cost of the trip, which is of course of probabilistic nature.

III. Comments

The exercise corresponds to a true preoccupation of the drivers (minimal time, minimal cost), but insists upon the fact that the available information is probabilistic.

The question is complex, because there is a large number of possible itineraries. If we restrict ourselves to the itineraries of minimal length (this length is 18 segments), there are $\binom{18}{9} =$ 48 620, such itineraries, but an itinerary with longer distance may result in a shorter time.

Many books, dealing with probabilities, have a satisfactory exposition, dealing with the axiomatic rules, but their applications to real life situations are not clear, and are even sometimes contradictory. Our reference book here will be:

Bernard Beauzamy : Méthodes Probabilistes pour l'étude des phénomènes réels. Ouvrage édité par la Société de Calcul Mathématique SA, ISBN 2-9521458-0-6, ISSN 1767-1175, mars 2004 (in French).