Construction of Archimedes transforms

Answer to a problem raised by the *Société de Calcul Mathématique* and the *Caisse Centrale de Réassurance*

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Abstract

This note answers the question raised by the *Société de Calcul Mathématique, S.A.* and the *Caisse Centrale de Réassurance* about the existence of mappings between a compact subset K of $\mathbb{R}^n$ and the unit cube, such that parts of equal volumes are mapped onto parts of equal volumes. Such mappings are called *Archimedes transforms*. The answer is affirmative. An explicit construction is given, together with a computation algorithm. The same construction moreover permits to "archimedeanly" map any continuous mass distribution of finite total mass onto the uniform distribution of the unit cube. Continuity and injectivity of the constructed mapping are also discussed.

Contents

1 Introduction

In [2], the *Société de Calcul Mathématique, S.A.* (SCM) and the *Caisse Centrale de Réassurance* (CCR) ask whether, given a compact subset K of $\mathbb{R}^n$, there exists a transform $f : K \to [0,1]^n$ such that whenever A and B are measurable subsets of K with equal volume, $f(A)$ and $f(B)$ have equal volume as well. Such a mapping is called an *Archimedes transform* in [2]. The aim is to easily divide K into parts of equal volume.

An other goal is to divide K into parts containing the same amount of some resource K would be endowed of. Mathematically speaking, this means that, given a measure μ on K, one wishes to divide K into parts of equal mass with respect to μ. A manner to do so, given an Archimedes transform onto the unit cube, is explained in [1]. But of course, if one could map K onto the unit cube, such that parts of equal μ-masses have images with equal volumes, the problem would be treated more easily.

It is explicitly stated in [2] that the required f needs not preserve measures of subsets. It however suffices that f multiplies them by a constant—and in fact one easily sees that it is also necessary, see Section 4.2. After applying a convenient homothety to K, such that its image has unit volume, one thus looks for a transform that preserves measure.

Such transforms have already been constructed in the mathematical literature. C. Villani's book [5] gives an overview of them in its first chapter (in particular an abstract, non-constructive, existence theorem). One of them may be applied to the problem at hand. It is particularly appealing because it may be computed quite simply, and it may be extended to the case when the source data are more general measures than the uniform one on a compact subset. The construction has been proposed independently by H. Knothe [3] in a geometric setting and M. Rosenblatt [4] for probabilistic matters. The idea may also be found in [1] in a somewhat hidden manner.

There are however some drawbacks to this transform: when the source compact K is not convex, it may fail to be continuous, and fail to be injective over the boundary of K. In the case when the source is a general measure μ, the construction works when this measure has a density with respect to the Lebesgue measure. Injectivity and continuity may fail as previously, depending on the convexity of the support of μ, but continuity may also fail if the density of μ is not a continuous function over its support. Nonetheless, these disadvantages should have little weight compared with the simplicity of computation.

Let us state right now the computation algorithm which is, as far as I am aware, the main novel contribution in this text. It certainly seems very mysterious at this point; this is why, in the sequel, the conceptual construction is presented first.

Theorem. *Let μ be a measure on a compact set $K \subset [a_1,b_1]\times\cdots\times[a_n,b_n]$, absolutely continuous with respect to the Lebesgue measure: $\mu = \rho(x_1,\dots,x_n)\mathrm{d}x_1\cdots\mathrm{d}x_n$, with positive, finite total mass. The case when ρ is the characteristic function 1_K of K is a special case.*

1. *Set*
$$R_1(x_1,\dots,x_n) := \int_{a_n}^{x_n} \rho(x_1,\dots,x_{n-1},s)\mathrm{d}s.$$

2. *For $1 \le \ell \le n-1$, iteratively compute*
$$R_{\ell+1}(x_1,\dots,x_{n-\ell}) := \int_{a_{n-\ell}}^{x_{n-\ell}} R_\ell(x_1,\dots,x_{n-\ell-1},s,b_{n-\ell+1})\mathrm{d}s.$$

The transformation $(x_1, \ldots, x_n) \mapsto (y_1, \ldots, y_n)$ *defined by*

$$y_k = f_k(x_1, \ldots, x_k) := \frac{R_{n-k+1}(x_1, \ldots, x_{k-1}, x_k)}{R_{n-k+1}(x_1, \ldots, x_{k-1}, b_k)},$$

is an Archimedes transform from K *endowed with* μ *to the unit cube* $[0,1]^n$ *endowed with the usual volume, i.e. it maps subsets with equal* μ*-mass to subsets with equal usual volume.*

The result is valid even if the support of μ *is not bounded: just replace* a_k *and* b_k *above by* $\pm\infty$ *respectively. One however still requires that the total mass be positive and finite.*

We also use our method to compute explicitly an Archimedes transform from the 3-dimensional unit ball to the unit cube: see formula (4) in section 2.1.

This note is organized as follows. The Archimedes transform is constructed in section 2. We first give an example in 2.1, hoping to give an idea of the general construction. The conceptual algorithm is then presented in section 2.2, and section 2.3 is devoted to the proof of its validity. Section 2.4 then gives the computational algorithm, proving the above theorem. In the second part, section 3, we discuss continuity and injectivity of the constructed transform. Finally, the third part, section 4, contains some remarks, relating the 2-dimensional solution of [1] to the construction given here (paragraph 4.1), and the proof that Archimedes transforms necessarily multiply volumes of subsets by a constant (section 4.2).

2 Construction of the Archimedes transform

We begin in section 2.1 with an example, that of the 3-dimensional ball, hoping to give a hint about the general construction. The general conceptual algorithm is then presented in section 2.2. It is the clearest way to present the idea, but it is obvious that actual computations should run it backwards. The practical algorithm is then presented in section 2.4. Before that, we prove in section 2.3 that our construction indeed solves the problem.

2.1 An example: the 3-dimensional ball

To begin with, let us explain the construction of the Archimedes transform onto the unit cube in the particular case of the 3-dimensional ball. For consistency of notations with the rest of the text, we shall take x_1, x_2 and x_3 as spatial coordinates in the ball: let us denote the unit ball by $B := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\}$. Each point of B will be mapped to a point of the cube $[0,1]^3$ whose coordinates will be written y_1, y_2, y_3.

As explained in the introduction, we construct a transform $f : (x_1, x_2, x_3) \mapsto (y_1, y_2, y_3)$ that multiplies the volumes of subsets by a constant: i.e. for any $A \subset B$, $\mathrm{vol}(f(A)) = \frac{\mathrm{vol}(A)}{\mathrm{vol}(B)} = \frac{3}{4\pi}\mathrm{vol}(A)$. The construction uses 3 steps (and will use n steps in dimension n):

1. first y_1 will be chosen depending only on x_1: $y_1 = f_1(x_1)$,
2. then y_2 depending on x_1 and x_2: $y_2 = f_2(x_1, x_2)$,
3. and finally $y_3 = f_3(x_1, x_2, x_3)$.

It may not be obvious that the constructed f is an Archimedes transform. In this particular case, we will only check it, postponing the general proof until the general algorithm is given.

First step:

Asking y_1 to depend only on x_1 means that each slice $x_1 = \text{cst}$ of the ball is mapped onto a slice $y_1 = \text{cst} = f_1(x_1)$ of the cube. If we require f_1 to be increasing, the part of the ball located below a plane $x_1 = c$ is then mapped to the part of the cube located below the plane $y_1 = f_1(c)$ (light green parts in figure 1 below).

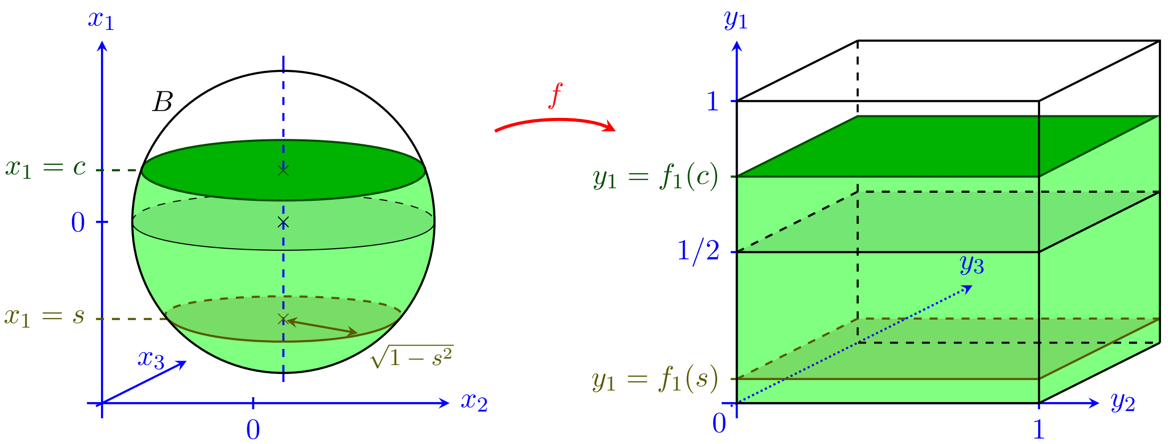


Fig. 1 : The horizontal slice $x_1 = c$ of the ball is mapped onto the horizontal slice of the cube such that the volumes of the light green parts are proportional by a factor $3/4\pi$.

Thus their volumes need to be proportional by a factor $3/4\pi$. The volume of the light green part (in fig. 1) of the cube is just $f_1(c)$. The volume of the light green part (in fig. 1) of the ball is the integral of the areas of the disks at $x_1 = s$, whose respective radii are $\sqrt{1-s^2}$, for $-1 \leq s \leq c$: it is $\int_{-1}^{c} \pi(\sqrt{1-s^2})^2 \mathrm{d}s$. Therefore f_1 is defined by

$$f_1(c) = \frac{3}{4\pi} \int_{-1}^{c} \pi(1-s^2)^2 \mathrm{d}s = \frac{-c^3 + 3c + 2}{4}. \tag{1}$$

Second step:

Let us fix $x_1 = c_1$ as a parameter; we have just chosen to send points of the ball that have $x_1 = c_1$ as their first coordinate to points of the cube having $y_1 = f_1(c_1)$ as their first coordinate. We now define the mapping $x_2 \mapsto y_2 = f_2(c_1, x_2)$.

The idea is to imitate the first step above as if we were building an Archimedes transform from the slice $x_1 = c_1$ of the ball, which is a disk of radius $r := \sqrt{1-c_1^2}$, onto the slice $y_1 = f_1(c_1)$ of the cube, which is a unit square. To do so, we map each segment $x_2 = c_2$ of the disk onto the segment $y_2 = f_2(c_1, c_2)$ of the square, so that the domains located below them, respectively in the disk and in the cube, have areas proportional by a factor $1/\pi r^2$.

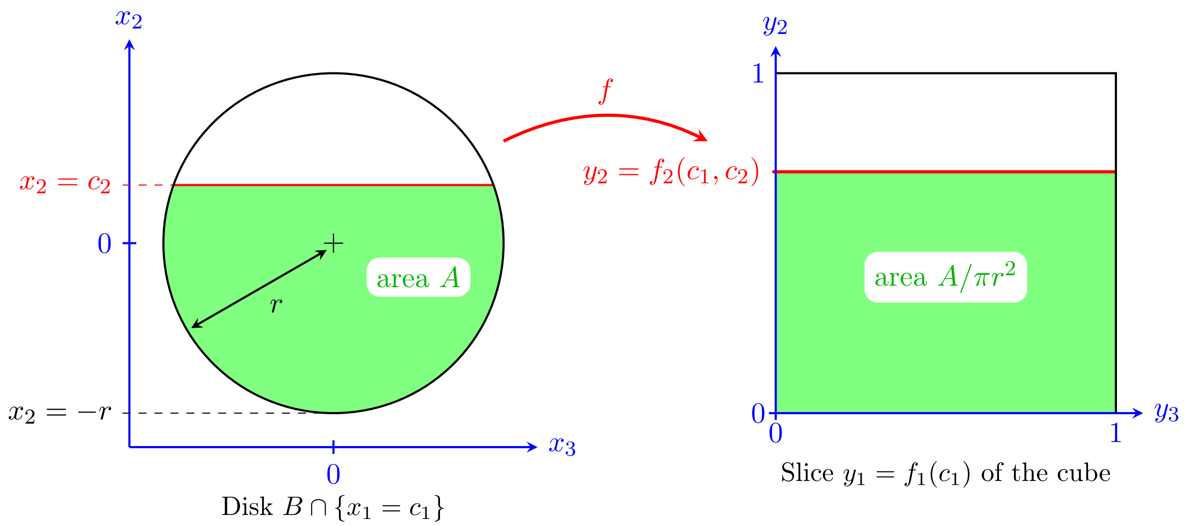


Fig. 2 : Mapping from the slice $x_1 = c_1$ of the ball onto the slice $y_1 = f_1(c_1)$ of the cube. The segment $x_2 = c_2$ of the disk is sent to the segment $y_2 = f_2(c_1, c_2)$ of the square, so that the areas of the green parts are proportional by a factor $1/\pi r^2$.

The area of the green part of the square (in fig. 2) is $f_2(c_1, c_2)$, and that of the green part of the disk is $2\int_{-r}^{c_2} \sqrt{r^2 - s^2}\mathrm{d}s$. Thus we set

$$\begin{aligned} f_2(c_1, c_2) &= \frac{2}{\pi r^2} \int_{-r}^{c_2} \sqrt{r^2 - s^2}\mathrm{d}s \\ &= \frac{1}{2} + \frac{1}{\pi}\left[\arcsin\left(\frac{c_2}{r}\right) + \frac{c_2}{r}\sqrt{1 - \frac{c_2^2}{r^2}}\right] \end{aligned}$$

i.e.

$$f_2(c_1, c_2) = \frac{1}{2} + \frac{1}{\pi}\left[\arcsin\left(\frac{c_2}{\sqrt{1 - c_1^2}}\right) + \frac{c_2\sqrt{1 - c_1^2 - c_2^2}}{1 - c_1^2}\right]. \tag{2}$$

Third step:

Once y_1 and y_2 are chosen, it remains to determine y_3. That is, we need to define the mapping from the segment $x_1 = c_1$, $x_2 = c_2$ of the ball to the segment $y_1 = f_1(c_1)$, $y_2 = f_2(c_1, c_2)$ of the cube. Let us use the most obvious way: a dilation-translation, i.e. an affine function.

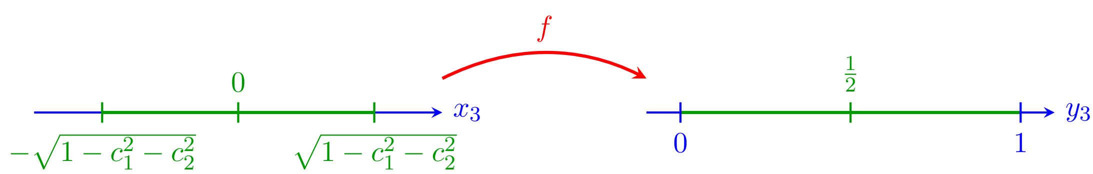


Fig. 3 : Mapping from the segment $x_1 = c_1$, $x_2 = c_2$ of the ball to the segment $y_1 = f_1(c_1)$, $y_2 = f_2(c_1, c_2)$ of the cube.

Thus we set

$$f_3(c_1, c_2, x_3) = \frac{1}{2} + \frac{x_3}{2\sqrt{1 - c_1^2 - c_2^2}}. \tag{3}$$

Conclusion:

Summing up formulas (1), (2) and (3), we have defined

$$(4)\qquad f(x_1,x_2,x_3) := \begin{pmatrix} f_1(x_1) & = \frac{1}{4}(-x_1^3+3x_1+2) \\ f_2(x_1,x_2) & = \frac{1}{2}+\frac{1}{\pi}\left[\arcsin\left(\frac{x_2}{\sqrt{1-x_1^2}}\right)+\frac{x_2\sqrt{1-x_1^2-x_2^2}}{1-x_1^2}\right] \\ f_3(x_1,x_2,x_3) & = \frac{1}{2}+\frac{x_3}{2\sqrt{1-x_1^2-x_2^2}} \end{pmatrix}.$$

Notice that these formulas may not be well defined when (x_1,x_2,x_3) is outside the ball.

Let us check that this f is truly an Archimedes transform. Noticing that f_1 does not depend on x_2 or x_3, and f_2 does not depend on x_3, we see that the Jacobian matrix of f has the form

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & 0 & 0 \\ * & \frac{\partial f_2}{\partial x_2} & 0 \\ * & * & \frac{\partial f_3}{\partial x_3} \end{pmatrix}.$$

We have

$$\frac{\partial f_1}{\partial x_1} = \frac{3}{4}(1-x_1^2), \qquad \frac{\partial f_2}{\partial x_2} = \frac{2}{\pi}\frac{\sqrt{1-x_1^2-x_2^2}}{1-x_1^2}, \qquad \frac{\partial f_3}{\partial x_3} = \frac{1}{2\sqrt{1-x_1^2-x_2^2}}.$$

The determinant of the Jacobian matrix is $\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2}\frac{\partial f_3}{\partial x_3} = \frac{3}{4\pi}$. Thus f multiplies volumes of subsets of the ball by the constant $\frac{3}{4\pi}$: therefore it is an Archimedes transform.

2.2 General construction

We now tackle the case of a general measure: K is a compact subset of $\mathbb{R}^n$ with non-empty interior, endowed with a measure μ absolutely continuous with respect to the Lebesgue measure: $\mu = \rho(x)\mathrm{d}x$, with ρ supported in K. The initial problem raised in [2] will be the special case when μ is the usual volume measure restricted to K, i.e. $\rho = 1_K$.

We need to assume that $\mu(K) > 0$. Up to scaling μ by a constant, we may—and will—assume it has total mass 1. The problem will be solved if we construct a mapping

$$f : (x_1,\dots,x_n) \longmapsto (y_1,\dots,y_n) \in [0,1]^n$$

such that for any (measurable) subset A, $\mathrm{vol}\big(f(A)\big) = \mu(A)$. We do so by an inductive process on the dimension.

First step:

The compact K lies between two hyperplanes $\{x_1 = a_1\}$ and $\{x_1 = b_1\}$. We first map each horizontal slice $K_c := \{x_1 = c\} \cap K$ onto a horizontal slice $\{y_1 = f_1(c)\}$ of the cube $[0,1]^n$ in the following way:

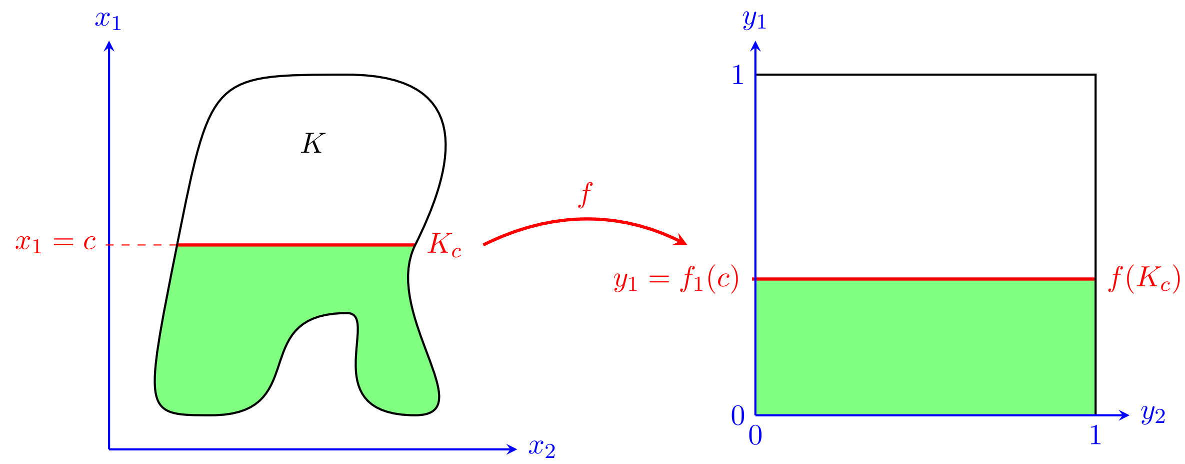


Fig. 4 : The horizontal slice $x_1 = c$ of K is mapped onto the horizontal slice of the cube such that the green domains have equal measure. The $(n-1)$-dimensional mapping from K_c to its image is determined by induction on the dimension.

The volume of the bottom $\{y_1 \leq f_1(c)\}$ of the cube, which equals y_1, has to be equal to the mass of the bottom $\{a_1 \leq x_1 \leq c\}$ of K. That is, we set:

$$\begin{aligned} f_1(c) &:= \mu\big(\{a_1 \leq x_1 \leq c\} \cap K\big) \\ &= \int_{x_1=a_1}^{x_1=c} \left[\int_{\text{slice } x_1=\text{cst of } K} \rho(x_1, \dots, x_n) \mathrm{d}x_2 \cdots \mathrm{d}x_n \right] \mathrm{d}x_1. \end{aligned} \tag{5}$$

Each of the slices K_{x_1} is a $(n-1)$-dimensional compact, that we endow with the total-mass-1 measure (depending on x_1)

$$\mu'_{x_1} := \frac{\rho(x_1, x_2 \dots, x_n)}{Z'_{x_1}} \mathrm{d}x_2 \cdots \mathrm{d}x_n,$$

where $Z'_{x_1} := \int_{K_{x_1}} \rho(x_1, x_2 \dots, x_n) \mathrm{d}x_2 \cdots \mathrm{d}x_n$ is a normalization "constant" (actually depending on x_1) such that $\mu'_{x_1}(K_{x_1}) = 1$.

Remark. Notice that when μ is the (normalized) volume measure of K, i.e. when ρ is the characteristic function 1_K of K, then μ'_{x_1} is the $(n-1)$-dimensional volume measure of the slice K_{x_1}, normalized by the "constant" Z'_{x_1} so as to have total measure 1. Thus Z'_{x_1} is the usual $(n-1)$-dimensional volume of K_{x_1}.

Since μ'_{x_1} is absolutely continuous with respect to the $(n-1)$-dimensional Lebesgue measure, we may proceed inductively to map "archimedeanly" K_{x_1} and μ'_{x_1} to the $(n-1)$-dimensional cube $\{f_1(x_1)\} \times [0,1]^{n-1}$.

Second step:

To illustrate, let us enter the details of the second step. We study points of K having $x_1 = c_1$ as their first coordinate. We have already determined to send them to points having $y_1 = f_1(c_1)$. We now choose how to transform the coordinate x_2, applying the first step to K_{c_1} and μ'_{c_1}.

The compact K lies between two hyperplanes $\{x_2 = a_2\}$ and $\{x_2 = b_2\}$. Denote by K_{c_1,c_2} the $(n-2)$-dimensional slice $\{x_1 = c_1,\ x_2 = c_2\}$ of K. We map it onto the $(n-2)$-dimensional cube $\{f_1(c_1)\} \times \{f_2(c_1,c_2)\} \times [0,1]^{n-2}$, where f_2 is defined by:

$$\begin{aligned} f_2(c_1,c_2) &:= \mu'_{c_1}\left(\{a_2 \leq x_2 \leq c_2\} \cap K_{c_1}\right) \\ &= \frac{1}{Z'_{c_1}} \int_{x_2=a_2}^{x_2=c_2} \left[\int_{K_{c_1,x_2}} \rho(c_1,x_2,\ldots,x_n)\mathrm{d}x_3\cdots\mathrm{d}x_n\right]\mathrm{d}x_2 \end{aligned} \tag{6}$$

To continue the induction, we endow K_{x_1,x_2} with the measure

$$\mu''_{x_1,x_2} := \frac{\rho(x_1,x_2,\ldots,x_n)}{Z''_{x_1,x_2}}\mathrm{d}x_3\cdots\mathrm{d}x_n, \quad \text{where } Z''_{x_1,x_2} := \int_{K_{x_1,x_2}} \rho(x_1,x_2,\ldots,x_n)\mathrm{d}x_3\cdots\mathrm{d}x_n,$$

which has total mass 1. Here again, when μ is the volume measure of K, μ''_{x_1,x_2} is the $(n-2)$-dimensional volume measure of the slice K_{x_1,x_2}, normalized by Z''_{x_1,x_2}; thus Z''_{x_1,x_2} is the usual volume of K_{x_1,x_2}.

$\mathbf{k^{th}}$ step:

The compact K lies between the hyperplanes $\{x_k = a_k\}$ and $\{x_k = b_k\}$, other new notations should be obvious regarding steps 1 and 2. Similarly to these steps, we map the $(n-k)$-dimensional slice $K_{c_1,\ldots,c_k}$ onto the slice $\big\{y_1 = f_1(c_1), \ldots, y_k = f_k(c_1,\ldots,c_k)\big\}$ of the cube, where

$$\begin{aligned} f_k(c_1,\ldots,c_k) &:= \mu^{(k-1)}_{c_1,\ldots,c_{k-1}}\left(\{a_k \leq x_k \leq c_k\} \cap K_{c_1,\ldots,c_{k-1}}\right) \\ &= \frac{1}{Z^{(k-1)}_{c_1,\ldots,c_{k-1}}} \int_{x_k=a_k}^{x_k=c_k} \left[\int_{K_{c_1,\ldots,c_{k-1},x_k}} \rho(c_1,\ldots,c_{k-1},x_k,\ldots,x_n)\mathrm{d}x_{k+1}\cdots\mathrm{d}x_n\right]\mathrm{d}x_k. \end{aligned} \tag{7}$$

To pursue the induction, we endow $K_{x_1,\ldots,x_k}$ with the total-mass-1 measure

$$\mu^{(k)}_{x_1,\ldots,x_k} := \frac{\rho(x_1,\ldots,x_n)}{Z^{(k)}_{x_1,\cdots,x_k}}\mathrm{d}x_{k+1}\cdots\mathrm{d}x_n, \qquad \text{where } Z^{(k)}_{x_1,\ldots,x_k} := \int_{K_{x_1,\ldots,x_k}} \rho(x_1,x_2,\ldots,x_n)\mathrm{d}x_{k+1}\cdots\mathrm{d}x_n. \tag{8}$$

Remark. When μ is the volume measure of K, $\mu^{(k)}_{x_1,\ldots,x_k}$ is only the $(n-k)$-dimensional normalized volume measure of the slice $K_{x_1,\ldots,x_k}$. The normalization "constant" $Z^{(k)}_{x_1,\ldots,x_k}$ is thus the usual $(n-k)$-dimensional volume of $K_{x_1,\ldots,x_k}$.

Last step:

Of course we could end at the $(n+1)^{\text{th}}$ step, where the problem is in dimension 0, therefore not very difficult. We choose to give the details of the last actual step, the n^{th} one, which is very simple too. Indeed one just has

$$f_n(x_1,\cdots,x_n) := \frac{1}{Z^{(n-1)}_{x_1,\ldots,x_{n-1}}} \int_{a_n}^{x_n} \rho(x_1,\ldots,x_{n-1},t)\mathrm{d}t \tag{9}$$

with $Z^{(n-1)}_{x_1,\ldots,x_{n-1}} := \int_{a_n}^{b_n} \rho(x_1,\ldots,x_{n-1},t)\mathrm{d}t$. Here again a_n and b_n are such that K lies between the hyperplanes $x_n = a_n$ and $x_n = b_n$.

Remark. When μ is the volume measure of K, $K_{x_1,\ldots,x_{n-1}}$ is a union of disjoint, vertical intervals (if K is convex there is only one), and $\mu^{(n-1)}_{x_1,\ldots,x_{n-1}}$ is the length measure, normalized by

$Z^{(n-1)}_{x_1,\dots,x_{n-1}}$ so as to have total measure 1. Geometrically, the meaning of (9) is the following: first set the total length to 1 by a convenient, global dilation; then translate separately the disjoint segments to make them fit into the vertical segment $\{f_1(x_1)\}\times\cdots\times\{f_{n-1}(x_1,\dots,x_{n-1})\}\times[0,1]$.

Conclusion:

Eventually, we end up with the transformation

$$f:(x_1,\dots,x_n)\longmapsto\begin{cases} y_1=f_1(x_1) & \text{given by (5)}\\ y_2=f_2(x_1,x_2) & \text{given by (6)}\\ \vdots & \vdots\\ y_k=f_k(x_1,\dots,x_k) & \text{given by (7)}\\ \vdots & \vdots\\ y_n=f_n(x_1,\dots,x_n) & \text{given by (9)}\end{cases}\tag{10}$$

The measures $\mu^{(k)}_{\dots}$ and normalization "constants" $Z^{(k)}_{\dots}$ appearing in formulas (5), (6), (7) and (9) are given by (8).

Remarks.

- In the case of general measures μ, we have assumed that they have compact support, included into K, for simplicity. It is clearly not necessary: one just needs to replace above the bounds for K (i.e. a_k and b_k) with $\pm\infty$ respectively.
- In the case of a general μ, the ρ used here is uniquely defined only modulo equality almost everywhere. Changing it would modify the constructed f on a subset of measure 0. One needs to be careful when ρ is not continuous, in particular at the boundary of K when $\rho=1_K$.
- There is a difficulty when ρ vanishes along some $(n-k)$-plane $x_1=c_1,\ \dots,\ x_k=c_k$: indeed we then have $Z^{(k)}_{c_1,\dots,c_k}=0$, $Z^{(k+1)}_{c_1,\dots,c_k,x_{k+1}}=0,\ \dots,\ Z^{(n)}_{c_1,\dots,c_k,x_{k+1},\dots,x_n}=0$, so the normalizations of the low-dimensional measures $\mu^{(j)}_{c_1,\dots}$, $j\ge k$, are not possible. Thus the images by f of the points having $c_1,\ \dots,\ c_k$, as their k first coordinates can't be computed. This however happens only for a set of points of measure 0, and can be remedied by modifying ρ on this subset. In the sequel of this text, we shall assume that it is done.

2.3 Proof of correctness

Proposition 1 (Rosenblatt [4], Knothe [3])**.** *The mapping f defined by* (10) *is an Archimedes transform, i.e. when A and B are measurable subsets of K with $\mu(A)=\mu(B)$, then* $\mathrm{vol}(f(A))=\mathrm{vol}(f(B))$.

Proof. After normalizing μ if needed, so that $\mu(K)=1$, we prove that for any measurable $A\subset K$, $\mathrm{vol}\big(f(A)\big)=\mu(A)$, by induction on the dimension. The result is trivial in dimension 0, but we shall begin with dimension 1 in order to keep the argument intuitive.

We will denote by vol_n the usual n-dimensional volume. We still consider for simplicity that $K\subset[a_1,b_1]\times\cdots\times[a_n,b_n]$, though if the support of μ is not bounded one may take $\pm\infty$ for a_k and b_k.

Initialization: In dimension 1, the first step of the algorithm of section 2.2 is the only one: f is defined by

$$y_1=f(x_1)\text{ such that }y_1=\mathrm{vol}_1\big([0,y_1]\big)=\mu\big(K\cap(-\infty,x_1]\big).$$

Therefore f preserves the measure of the subsets of the form $K \cap (-\infty, x_1]$. Since they generate the σ-algebra of borelian subsets of K, we are done.

Induction: Let A be a measurable subset of $K \subset \mathbb{R}^n$, and let A_{x_1} denote the $(n-1)$-dimensional slice $x_1 = \text{cst}$ of A: $A_{x_1} := A \cap K_{x_1}$. Recall the first step in the construction of f: K_{x_1} is mapped into a $(n-1)$-dimensional cube using the $(n-1)$-dimensional construction associated to the measure μ'_{x_1} defined there. By induction we thus have

$$\mathrm{vol}_{n-1}\big(f(A_{x_1})\big) = \mu'_{x_1}(A_{x_1}) \tag{11}$$

where the image $f(A_{x_1})$ is, by construction, the slice $y_1 = f_1(x_1)$ of $f(A)$, which we shall denote by $\big[f(A)\big]_{f_1(x_1)}$.

Now by the very definition of μ'_{x_1} and Z'_{x_1}, see (8),

$$\mu(A) = \int_{a_1}^{b_1} \left[\int_{A_{x_1}} \rho(x_1, \dots, x_n) \mathrm{d}x_2 \cdots \mathrm{d}x_n \right] \mathrm{d}x_1 = \int_{a_1}^{b_1} \mu'_{x_1}(A_{x_1}) Z'_{x_1} \mathrm{d}x_1.$$

Plugging equation (11) here, we obtain

$$\mu(A) = \int_{a_1}^{b_1} \mathrm{vol}_{n-1}\left(\big[f(A)\big]_{f_1(x_1)} \right) Z'_{x_1} \mathrm{d}x_1.$$

Recalling the definitions of f_1 and Z'_{x_1}, we have $f_1(x_1) = \int_{a_1}^{x_1} Z'_s \mathrm{d}s$. The change of variable $y_1 = f_1(x_1)$, $\mathrm{d}y_1 = Z'_{x_1} \mathrm{d}x_1$, is correct as soon as $x_1 \mapsto Z'_{x_1}$ is L^1 and non-negative, which is true. In the last equation above, it gives:

$$\mu(A) = \int_0^1 \mathrm{vol}_{n-1}\left(\big[f(A)\big]_{y_1} \right) \mathrm{d}y_1 = \mathrm{vol}_n\big(f(A)\big).$$

This means that, assuming the mapping (10) in dimension $n-1$ is “archimedean”, it is also in dimension n.

Conclusion: In any dimension, the mapping (10) is an Archimedes transform. □

2.4 Computation algorithm

The first step of the algorithm of section 2.2 involves n-dimensional integrations, which happen to be usual integrals of the $(n-1)$-dimensional integrals computed for step 2. Thus, clearly, computations should begin at the last step and run the algorithm backwards. Indeed, we see from (8) that the normalization “constants” $Z^{(k)}_{\dots}$ appearing in formulas (5), (6), (7) and (9) are such that:

$$Z^{(k-1)}_{x_1,\dots,x_{k-1}} = \int_{s=a_k}^{s=b_k} Z^{(k)}_{x_1,\dots,x_{k-1},s} \mathrm{d}s, \tag{12}$$

$$f_k(x_1, \dots, x_k) = \frac{1}{Z^{(k-1)}_{x_1,\dots,x_{k-1}}} \int_{s=a_k}^{s=x_k} Z^{(k)}_{x_1,\dots,x_{k-1},s} \mathrm{d}s. \tag{13}$$

with, say, $Z^{(n)}_{x_1,\dots,x_n} := \rho(x_1, \dots, x_n)$ to initialize the recursion (12). Here again, a_k and b_k are such that K (or the support of μ) is included into the parallelepiped $[a_1, b_1] \times \cdots \times [a_n, b_n]$. In case μ has unbounded support one may take $a_k = -\infty$ and $b_k = +\infty$. We also assume again that ρ is supported in K, multiplying it by the characteristic function 1_K if needed.

We inductively compute the functions

$$R_{n-k+1}(x_1,\dots,x_k) := \int_{s=a_k}^{s=x_k} Z^{(k)}_{x_1,\dots,x_{k-1},s}\mathrm{d}s, \tag{14}$$

for k ranging downwards from n to 1. Because of (12), one then has

$$Z^{(k-1)}_{x_1,\dots,x_{k-1}} = R_{n-k+1}(x_1,\dots,x_{k-1},b_k).$$

Thus, plugging that equation into (14) with k lowered by 1:

$$R_{n-k+2}(x_1,\dots,x_{k-1}) = \int_{s=a_{k-1}}^{s=x_{k-1}} R_{n-k+1}(x_1,\dots,x_{k-2},s,b_k)\mathrm{d}s,$$

which allows the inductive computation. Equation (13) now gives

$$f_k(x_1,\dots,x_k) = \frac{R_{n-k+1}(x_1,\dots,x_k)}{R_{n-k+1}(x_1,\dots,x_{k-1},b_k)}.$$

Summing up, we get the theorem stated in the introduction:

Proposition 2. *Compute inductively*

$$R_1(x_1,\dots,x_n) := \int_{a_n}^{x_n} \rho(x_1,\dots,x_{n-1},s)\mathrm{d}s.$$

$$R_{\ell+1}(x_1,\dots,x_{n-\ell}) := \int_{a_{n-\ell}}^{x_{n-\ell}} R_\ell(x_1,\dots,x_{n-\ell-1},s,b_{n-\ell+1})\mathrm{d}s.$$

The Archimedes transform (10) *is defined by*

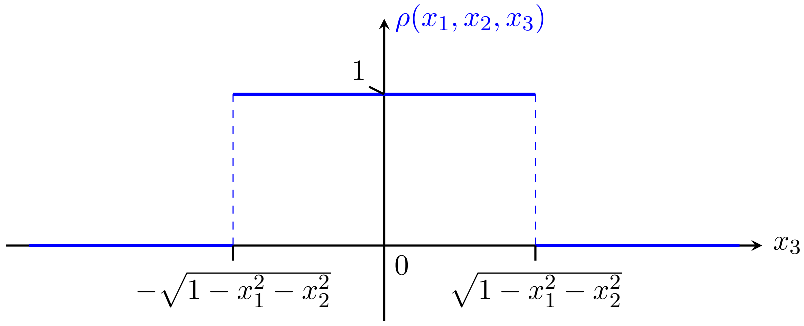
$$y_k = f_k(x_1,\dots,x_k) = \frac{R_{n-k+1}(x_1,\dots,x_{k-1},x_k)}{R_{n-k+1}(x_1,\dots,x_{k-1},b_k)}.$$

Remarks.

- This algorithm in fact does not need to first normalize μ: the normalization is only needed for the computation of y_1, and is achieved by the division by $R_n(b_1)$, which in full generality equals $\mu(K)$, at this step.
- We notice that when μ is the volume measure $1_K(x)\mathrm{d}x$ of a polyhedron K, R_1 is a piecewise affine function—the length of the vertical segments included in K. By induction, each R_ℓ is then piecewise polynomial, which allows explicit integration. This is also true when K may be triangulated into simplices over which ρ is polynomial.
- There does not seem to exist such a simple algorithm for the reverse transformation. It can however be computed numerically, noticing that f is, in a certain sense, "triangular":
  - Since $y_1 = f_1(x_1)$ with f_1 non-decreasing, one first finds $x_1 = f_1^{-1}(y_1)$ numerically. Of course f_1^{-1} can be discontinuous: that needs to be the case if K—or the support of μ—is made up of two pieces separated by a band $p \le x_1 \le q$.
  - Once x_1 is fixed, $y_2 = f_2(x_1,x_2)$ with f_2 non-decreasing with respect to x_2. So, given x_1 just computed at the previous step, one may find numerically $y_2 = f_2(x_1,\cdot)^{-1}(x_2)$. Here again discontinuities may occur for some x_1.
  - *et cætera.*

2.4.1 Application to the 3-dimensional ball

To illustrate this algorithm, let us verify that we recover formula (4) in the case of the 3-dimensional unit ball. Here we have $\rho(x_1, x_2, x_3) = 1_{x_1^2+x_2^2+x_3^2 \leq 1}$ (with obvious notations):

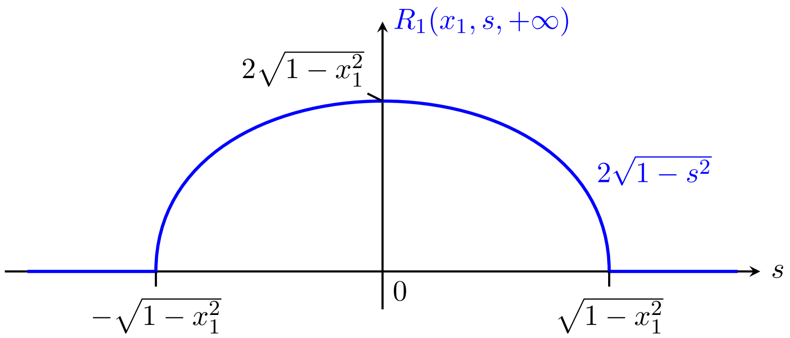


Therefore $R_1(x_1, x_2, x_3) = \int_{-\infty}^{x_3} \rho(x_1, x_2, s)\mathrm{d}s$ satisfies

$$(15) \qquad R_1(x_1, x_2, x_3) = \begin{cases} 0 & \text{if } x_1^2 + x_2^2 > 1, \\ 0 & \text{if } x_3 < -\sqrt{1-x_1^2-x_2^2}, \\ x_3 + \sqrt{1-x_1^2-x_2^2} & \text{if } -\sqrt{1-x_1^2-x_2^2} \leq x_3 \leq \sqrt{1-x_1^2-x_2^2}, \\ 2\sqrt{1-x_1^2-x_2^2} & \text{if } x_3 \geq \sqrt{1-x_1^2-x_2^2}. \end{cases}$$

It follows that

$$R_1(x_1, s, +\infty) = \begin{cases} 0 & \text{if } x_1^2 + s^2 > 1, \\ 2\sqrt{1-x_1^2-s^2} & \text{else:} \end{cases}$$



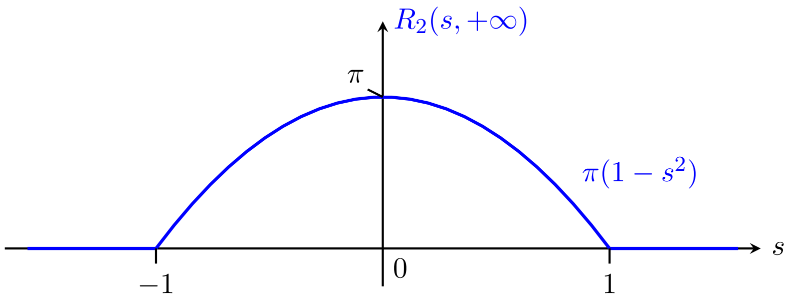
From that we compute $R_2(x_1, x_2) = \int_{-\infty}^{x_2} R_1(x_1, s, +\infty)\mathrm{d}s$: if $|x_2| \leq \sqrt{1-x_1^2}$ one has

$$(16) \qquad \begin{aligned} R_2(x_1, x_2) &= 2\int_{-\sqrt{1-x_1^2}}^{x_2} \sqrt{1-x_1^2-s^2}\mathrm{d}s \\ &= \frac{\pi(1-x_1^2)}{2} + (1-x_1^2)\arcsin\left(\frac{x_2}{\sqrt{1-x_1^2}}\right) + x_2\sqrt{1-x_1^2-x_2^2}, \end{aligned}$$

and else

$$(17) \qquad R_2(x_1, x_2) = \begin{cases} 0 & \text{if } |x_1| > 1 \text{ or } x_2 < -\sqrt{1-x_1^2}, \\ \pi(1-x_1^2) & \text{if } x_2 > \sqrt{1-x_1^2}. \end{cases}$$

Thus we compute that $R_2(s, +\infty) = 0$ if $|s| > 1$, and $R_2(s, +\infty) = \pi(1-s^2)$ else:



Therefore we have

$$R_3(x_1) = \int_{-\infty}^{x_1} R_2(s, +\infty) \mathrm{d}s = \begin{cases} 0 & \text{if } x_1 < 1, \\ \pi(x_3 + 1 - \frac{1+x_1^3}{3}) & \text{if } -1 \leq x_1 \leq 1, \\ \frac{4}{3}\pi & \text{if } x_1 > 1. \end{cases} \tag{18}$$

We may now readily check, when (x_1, x_2, x_3) is a point of the unit ball, that:

$$f_1(x_1) = \frac{R_3(x_1)}{R_3(+\infty)}\text{: formula (1) is coherent with (18).}$$

$$f_2(x_1, x_2) = \frac{R_2(x_1, x_2)}{R_2(x_1, +\infty)}\text{: formula (2) is coherent with (16) and (17).}$$

$$f_3(x_1, x_2, x_3) = \frac{R_1(x_1, x_2, x_3)}{R_1(x_1, x_2, +\infty)}\text{: formula (3) is coherent with (15).}$$

3 Continuity and injectivity

The Archimedes map constructed in section 2 may be discontinuous and may not be injective, even in simple cases. We illustrate this fact by some examples in section 3.1. We show however in 3.2 that injectivity holds in restriction to the interior of the compact K, or respectively the interior of the support of μ. We also show in section 3.3 that if K is convex—respectively when the density ρ of μ has convex support in the interior of which it is continuous—our Archimedes transform is continuous, and actually bicontinuous, from the interior of K onto the interior of the cube.

3.1 Examples of bad behavior

How injectivity and continuity may fail in the case of a compact K endowed with its volume measure:

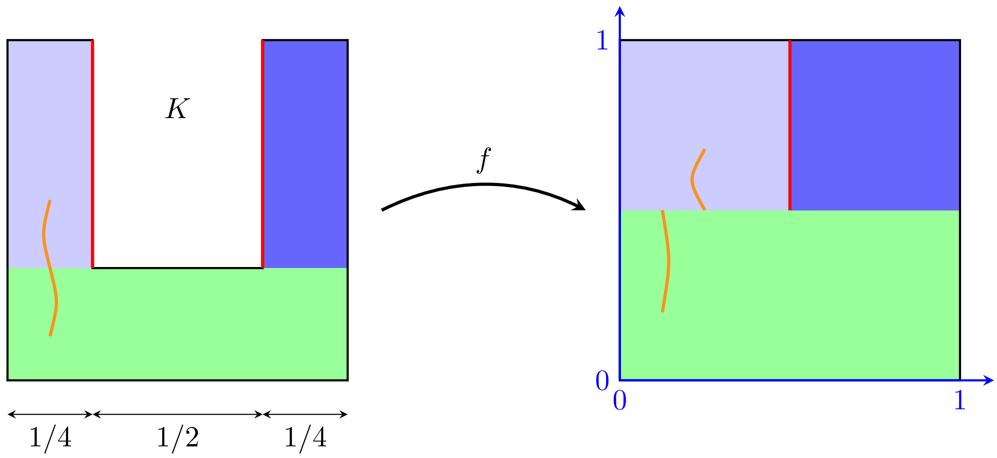


Fig. 5 :

- The green rectangle is stretched vertically but not horizontally. The blue ones *are* stretched horizontally by a factor 2. Therefore continuity fails at the blue-green interfaces. For example, we have drawn in orange a continuous curve in K, and its image.
- The red parts of the boundary of K are merged by f, therefore injectivity fails.

How continuity may fail in the case of a convex K endowed with a measure having a non-continuous density:

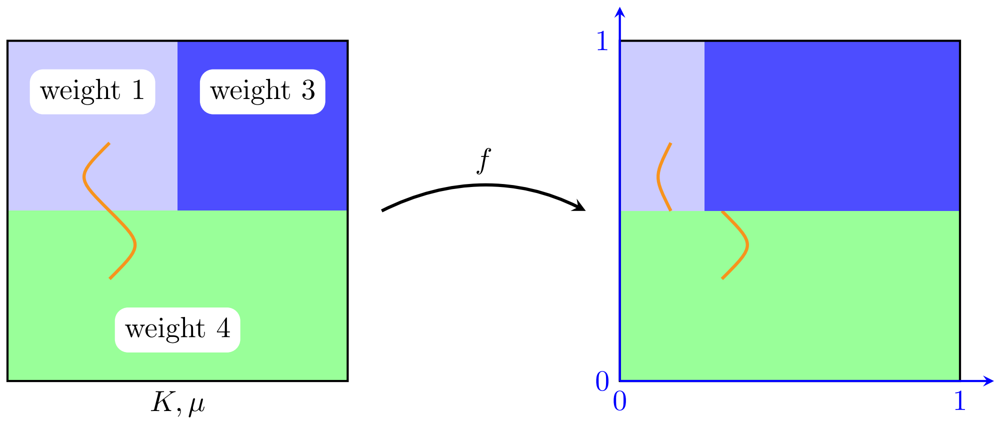


Fig. 6 : The green rectangle is not deformed. The light blue one is horizontally squeezed by a factor 2. The dark blue one is horizontally stretched by a factor 3/2. Therefore, just as before, continuity fails at the green-blue interface. Again, we have drawn a continuous curve in orange and its non-continuous image.

3.2 Injectivity

Proposition 3. *The transform* (10) *is injective in restriction to points admitting a neighborhood where the density ρ does not vanish.*

Remark. We insist that we require ρ to vanish *nowhere*, and not only almost nowhere, in a neighborhood of the considered points. As already remarked, if ρ vanishes along some k-plane directed by some of the coordinate axes, failures of injectivity may occur along this k-plane. However, μ determines ρ only up to a subset of measure 0, so we assume that ρ has been chosen so that this case doesn't happen.

Proof. Let $p = (p_1, \dots, p_n) \in K$ such that there exists an open, non-empty parallelepiped $(s_1,t_1)\times\cdots\times(s_n,t_n) \subset K$ in which p lies and where ρ does not vanish. Let $p' = (p'_1, \dots, p'_n) \in K$.

We consider first the case where $p'_1 \neq p_1$, say $p'_1 > p_1$. The band $p_1 \leq x_1 \leq p'_1$ intersects the parallelepiped $[s_1,t_1]\times\cdots\times[s_n,t_n]$; the intersection is the parallelepiped $[p_1, \min(t_1,p'_1)] \times [s_2,t_2]\times\cdots\times[s_n,t_n]$ which has positive μ-measure m, since ρ does not vanish there. Thus

$$f_1(p'_1) = \mu(K \cap \{x_1 \leq p'_1\}) = \mu(K \cap \{x_1 \leq p_1\}) + \mu(K \cap \{p'_1 \leq x_1 \leq p_1\}) \\ \geq f_1(p_1) + m > f_1(p_1).$$

Therefore p and p' have distinct images by f.

Now we assume that $p_1 = p'_1$ and $p_2 \neq p'_2$: then $f(p)$ and $f(p')$ will be separated during the second step of the algorithm of section 2.2 just as above. If $p_2 = p'_2$ and $p_3 \neq p'_3$ they will be separated at step 3, and so on by induction on the dimension. Notice that it is for the induction that the positivity of ρ everywhere is needed: to insure at each step that the measure $\mu^{(k)}_{p_1,\dots,p_k}$ is positive in a neighborhood of p in the $(n-k)$-dimensional slice $K_{p_1,\dots,p_k}$. □

3.3 Continuity

Proposition 4. *Let here K be the support of μ, i.e. the set of points all of whose neighborhoods have positive measure. Assume that*

- *K is convex,*
- *the density ρ of μ is continuous and positive in the interior of K.*

Then the Archimedes transform (10) *is a homeomorphism from the interior of K to the interior of the cube* $(0,1)^n$.

Proof. It is a simple fact from the theory of convex bodies that the families of lower-dimensional slices $K_{x_1,\dots,x_k}$, $x_1 \in \mathbb{R}$, …, $x_k \in \mathbb{R}$, are continuous families of convex subsets, except maybe at the boundary of K. If ρ is continuous in the interior of K, the various integrals

$$\int_{K_{x_1,\dots,x_k}} \rho(x_1,\dots,x_n)\mathrm{d}x_{k+1}\cdots\mathrm{d}x_n$$

then depend continuously on $(x_1,\dots,x_k)$. From the formulas in section 2.2 we thus see that f is continuous in the interior of K.

We notice now that f may be inverted as follow: the slice $y_1 = c$ of the cube is mapped onto the slice $x_1 = g_1(c)$ such that

$$\mu\left(K \cap \{x_1 \leq g_1(c)\}\right) = y_1,$$

and the way this mapping is achieved is determined by induction on the dimension. It is exactly the algorithm of section 2.2 where the role of K and the cube have been exchanged.[1] We thus build $g = f^{-1}$ on the interior of the cube.

Now, because the cube is convex and its volume measure has continuous density on its interior, the previous result on continuity holds: g is continuous in the interior of the cube. Therefore f is a homeomorphism. □

[1] In fact the algorithm perfectly applies to Archimedes transform for any measured convex (K,μ) to any other one (L,ν): it is the construction of Knothe [3].

4 Remarks

4.1 Choice of coordinates

The construction of section 2 uses euclidean coordinates. But of course the construction works as well if $x_1, \ldots, x_n$ is any system of curvilinear coordinates that covers K, and if $y_1, \ldots y_n$ is any system of curvilinear coordinates over the cube—or any other reference domain, such as the unit ball. The process is the following, assuming that x_1 ranges from a_1 to b_1, and y_1 from a'_1 to b'_1: we map the slice $x_1 = c$ to $y_1 = f_1(c)$, such that the bands (or sectors, or whatever) $\{a_1 \leq x_1 \leq c\}$ and $\{a'_1 \leq y_1 \leq f_1(c)\}$ have equal mass. Then we proceed by induction on the dimension similarly to section 2.2.

For example, the construction of [1] of an Archimedes transform from a 2-dimensional star-shaped compact K to the unit disk may be seen as an application of the algorithm given in section 2.2, where x_1 and x_2 are modified polar coordinates adapted to K, and y_1 and y_2 the usual polar coordinates on the disk.

4.2 Preservation of the volume by an Archimedes transform

Let $f : K_1 \to K_2$ be a measurable transform. For f to be an Archimedes transform, it suffices that f multiplies volumes by a constant $\lambda > 0$. As mentioned in the introduction, it is in fact necessary, as proven here in a few lines. We shall assume that K_1 and K_2 have positive volumes, respectively $V_1 > 0$ and $V_2 > 0$.

Assume indeed that f is an Archimedes transform. Then there exists a function $\varphi : [0, V_1] \to [0, V_2]$ such that whenever A has volume v, $f(A)$ has volume $\varphi(v)$. Because of the addition properties of the Lebesgue measure, for any infinite series $\sum x_n$ with non-negative terms and whose sum is lower than or equal to V_1,

$$\varphi\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} \varphi(x_n).$$

Therefore φ is an additive function (within its domain) and is continuous from the left. It is now a classical exercise that there exists a constant $\lambda > 0$ such that $\forall x,\ \varphi(x) = \lambda x$: one first sees that for any $k \in \mathbb{N}$ and $x \leq V_1/k$, $\varphi(kx) = k\varphi(x)$. This immediately extends to $k \in \mathbb{Q}_+$, and then to $k \in \mathbb{R}_+$ by continuity. Thus φ may be extended to a linear function on $\mathbb{R}$, which is the claimed result.

5 Conclusion

Section 2.2 presents a construction of an Archimedes transform from any compact subset K of $\mathbb{R}^n$ of positive volume to the unit cube, i.e. a mapping transforming subsets of equal volumes into subsets of equal volumes. The construction readily extends to Archimedes mappings relatively to any continuous measure μ with finite (and positive) total mass.

This procedure is particularly appealing because of a simple algorithm for computations, as given by Proposition 2 (and by the Theorem in the introduction). In the case where the data K and μ are a piecewise polynomial measure on a compact polyhedron, calculations may be done explicitly.

Injectivity and continuity are discussed in Section 3. In particularly it is shown by Proposition 4 that when K is convex and μ has a density which is positive and continuous in the interior of K, then the constructed Archimedes transform is injective on the interior of K and bicontinuous. Continuity with respect to the data K and μ is not treated, because, since in general the transform is not continuous, and even only defined up to a subset of measure 0, the precise meaning of "continuous dependance with respect to K and μ" is not very clear. The formulas of Proposition 2 however allow to think that, given any reasonable meaning to that expression, it should be true. This allows to work with approximate data. A precise estimation could be useful, but is beyond the purpose of this text.

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