



## Simple Random Walks in the Plane: An Energy-Based Approach

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### Abstract

The behavior of the sums  $\sum_{n=1}^N X_n$ , where  $X_n$  are independent random variables, taking the values  $\pm 1$  with probability  $1/2$ , has been extensively studied ; the most noticeable result is Khinchin's Law of the Iterated Logarithm (1924). Such results are of probabilistic nature, and very little is known about their quantitative versions.

Here, we introduce a completely new approach. The probabilistic problem is replaced by the propagation of an "energy" in the plane, with possible absorption by a "barrier". The question becomes entirely deterministic. The use of Chebycheff's polynomials allows us to reduce the problem to an operator theory setting: iterates of a matrix, with eigenvectors and eigenvalues of trigonometric type, and Hilbert space decomposition. A continuous barrier  $b(x)$  is replaced by a succession of "tunnels", of increasing diameter, in which the energy propagates with possible absorption. During each period, we are able to determine precisely what the energy profile is. Finally, we show that the game terminates eventually (that is, the total energy tends to zero) if and only if the barrier satisfies  $\int_1^{+\infty} \frac{dx}{b^2(x)} = +\infty$ .

Keywords: *Sums of random variables, energy, operator theory*

## I. Presentation of the problem

We consider a simple random walk in the plane : a sequence of random variables  $X_n$  with values  $\pm 1$ , probability  $1/2$  in each case. Let  $S_N = \sum_{n=1}^N X_n$  be the sum of the first  $N$  variables. This random walk can be viewed as a game between two players  $A$  and  $B$ ; at the  $n^{\text{th}}$  step, the first player receives 1 Euro from the second player if  $X_n = +1$  and conversely if  $X_n = -1$ . So the sum  $S_N$  represents the increase of fortune of  $A$  compared to  $B$ , with  $S_0 = 0$ . We introduce a "barrier", that is a function  $b(n)$ , positive, increasing, and decide that the game stops at step  $n$  if  $S_n = \pm b(n)$ . The question is : what is the probability that the game continues after  $N$  steps, and, if so, what is the probability distribution of  $S_n$ ? Of course, the answers to these questions depends on the properties of the barrier.

Among the many existing results on this topic, let us mention in particular:

- Feller's "Gambler's ruin" ; see [Feller]. The problem may be stated as follows : given an initial fortune and a barrier, what is the probability to reach the barrier without having first reached the barrier  $y = 0$  (which means ruin) ? The gambler's ruin does not care about a specific time, whereas we compute the probability on each vertical for each specific time. We thank Doron Zeilberger for useful discussions about this comparison.
- Khintchin's Law of the Iterated Logarithm (1924); see [Khintchin]: almost surely, when  $n \rightarrow +\infty$ :

$$\limsup \frac{S_n}{\sqrt{2n \text{Log}(\text{Log}(n))}} = +1 \text{ and } \liminf \frac{S_n}{\sqrt{2n \text{Log}(\text{Log}(n))}} = -1$$

We will here present a new approach to such problems, which is "energy based" and not probabilistic in nature. This will allow us to develop a unified framework, and to obtain quantitative estimates which were not known previously.

Indeed, the probabilistic appearance of Khinchin's laws is misleading. Looking at such a statement, one may have the impression that, for a given player, there are some unknown forces which will, sooner or later, bring his fortune close to Khintchin's curves (a Khintchin curve is of the form  $y = \pm \sqrt{2x \text{Log}(\text{Log}(x))}$ , of course). This is completely wrong ; at any time, the game is only governed by the  $\pm 1$  rule, with equal probability.

What Khintchin's laws say, and, more generally, what any result about random walks say, is that there are more paths with some properties than paths with other properties. They are not individual results about each path; they are results about the number of paths with a given characteristic. Such results are in fact of combinatorial nature. For instance, at time  $n$ , the proportion of paths which never touched the curve  $y = \sqrt{n}$  tends to 0 when  $n \rightarrow +\infty$ .

Our approach relies upon a concept derived from "energy absorption". Our aim is to obtain quantitative estimates, of the following form:

*Given a curve  $y = b(n)$ , what is the proportion of paths which never touched the curves  $\pm b(n)$  before the instant  $n$  ?*

## Part I - Basic settings

### I. Elementary facts about the random walk

At any time  $n$ , we have  $|S_n| \leq n$ . The values of  $S_n$  are even if  $n$  is even, and are odd if  $n$  is odd. In what follows, we will always restrict ourselves to the case where  $n$  is even. This means that the elementary game consists in two repetitions,  $X_1 + X_2$ , with :

$$P(X_1 + X_2 = -2) = \frac{1}{4}, P(X_1 + X_2 = 0) = \frac{1}{2}, P(X_1 + X_2 = 2) = \frac{1}{4} \quad (1)$$

The following Lemma is well-known (see for instance [1]) ; it simply reflects the combinatorics:

**Lemma 1.** - *Let  $A_{2n,2k}$  be the point of coordinates  $(2n, 2k)$ , with  $k = -n, \dots, n$ . The number of paths from 0 to  $A_{2n,2k}$  is:*

$$N(2n, 2k) = \binom{2n}{n+k}$$

We write  $N(A) = N(0 \rightarrow A)$  for the total number of paths, starting at 0, finishing at  $A$  and, more generally,  $N(A \rightarrow B)$  for the number of paths starting at  $A$ , finishing at  $B$ .

### II. Introducing the energy

Instead of looking at various paths, and computing their probability, we consider that, at time 0, a unit of energy is put at the origin. This unit will then divide itself in two halves, at time  $n=1$ , one at the point  $(1,1)$  and one at the point  $(1,-1)$ . More generally, every time a division point is met, the available energy divides equally into the two possible paths. At any step, in this configuration, the sum is always 1.

In this basic setting, since the energy 1 is put at 0 and since there is a total of  $2^{2n}$  possible paths  $N(A_{2n,2k})$  at time  $2n$ , each point  $A_{2n,2k}$  receives an amount of energy equal to:

$$P(S_{2n} = 2k) = e(A_{2n,2k}) = \frac{1}{2^{2n}} \binom{2n}{n+k} \quad (2)$$

We see that the repartition of energy is given by a binomial law: there is more energy at the central points and very little energy at the extreme points ( $A_{2n,2n}$  and  $A_{2n,-2n}$ ). The amount of energy at a point is the probability to reach this point by the random walk.

We consider only the even values of  $n, k$  ; let  $f(n, k) = e(A_{2n,2k})$  be the energy put at the point of coordinates  $2n, 2k$ . It satisfies for any  $k$  :

$$f(n, k) = \frac{1}{4} f(n-1, k-1) + \frac{1}{2} f(n-1, k) + \frac{1}{4} f(n-1, k+1) \quad (3)$$

We can restrict ourselves to  $k \geq 0$ , using the symmetry of the process. Since  $f(n, 1) = f(n, -1)$ , we have :

$$f(n, 0) = \frac{1}{2} f(n-1, 0) + \frac{1}{2} f(n-1, 1) \quad (4)$$

A first general result is given by:

**Proposition 2.** - For all  $k \geq 0$ ,  $f(n, k) \rightarrow 0$  when  $n \rightarrow +\infty$ .

### Proof of Proposition 2

We consider the Banach space  $l_1$  of sequences  $x_k$  such that  $\sum_{k=0}^{+\infty} |x_k| < +\infty$ . The linear operator  $T$  which sends a sequence  $x_k$  to the sequence  $y_k$  defined by  $y_0 = \frac{1}{2}x_0 + \frac{1}{2}x_1$  and, for  $k \geq 1$   $y_k = \frac{1}{4}x_{k-1} + \frac{1}{2}x_k + \frac{1}{4}x_{k+1}$ , is an isometry on the subset of positive sequences (the sequence  $x_k$  and the sequence  $y_k$  have same sum). One checks immediately that if the  $x_k$  are decreasing, so are the  $y_k$ . We start with the sequence  $X_0 = (1, 0, \dots)$  and consider its iterates under the operator  $T$  ; let  $X_n = T^n X_0$ . Let  $X_n(0)$  be the first coordinate of the sequence  $X_n$ . It is clear on the definition that the sequence  $X_n(0)$  is decreasing, and the same applies to each coordinate  $X_n(j)$ . Since all are positive, each sequence  $X_n(j)$  must have a limit, denoted by  $l_j$ , when  $n \rightarrow +\infty$ . But we have  $X_n(0) = \frac{1}{2}X_{n-1}(0) + \frac{1}{2}X_{n-1}(1)$ , which implies  $l_0 = \frac{1}{2}l_0 + \frac{1}{2}l_1$  and therefore  $l_0 = l_1$ . The same applies to each coordinate:  $l_0 = l_1 = \dots = l_j$ . The limiting value must be the same

for all coordinates. The condition  $\sum_j X_n(j) = 1$ , for any  $n$ , implies  $\sum_j l_j \leq 1$ , so the  $l_j$  must all be 0. This proves Proposition 2.

In this preliminary approach, the total amount of energy remains the same at each time step.

### III. Introducing the barrier

Now, we introduce a curve,  $y = b(x)$ , defined for  $x > 0$ , continuous and differentiable, increasing. Indeed, for practical purposes, it will be more convenient to work on a curve than on a sequence  $b(n)$ . We investigate the probability that the random walk, up to time  $N$ , remains constantly between the curve and its symmetric, which means  $|S(n)| < b(n)$ ,  $n = 1, \dots, N$ . Our representation, in order to investigate this phenomenon, will be the fact that the barrier  $b$  absorbs the energy. This means that, for any path which touches the barrier, the corresponding energy disappears.

**Proposition 1.** - *Let  $y = b(x)$  be any barrier. The total energy left, at time  $N$ , is equal to the total probability to reach any of the points  $A_{N,j}$  below the curve, that is  $j < |b(n)|$ , without ever touching the curve at any time before ( $n \leq N$ ).*

#### Proof of Proposition 1

This is a mere rephrasing of the disparition of energy. Any time a path touches the barrier, it is annihilated, so what remains is the set of paths which never touched the barrier.

If a time  $N$  is fixed, and a barrier  $b$  is fixed, we will call admissible a path with never touches it (at any time  $n \leq N$ ). For any point  $A$  in the plane, let  $N_{ad}(A)$  be the number of admissible paths which reach  $A$ , and  $p_{ad}(A) = \frac{N_{ad}(A)}{2^N}$  the probability to reach  $A$  by an admissible path.

Proposition 1 states that:

$$\sum_{j=-N}^N e(A_{N,j}) = \sum_{j < |b(N)|} p_{ad}(A_{N,j})$$

## Part II : The case of a constant barrier

### I. Preliminary results

We compute the number of admissible paths when the barrier is a simple horizontal line segment. There are some differences, depending if  $y$  is odd or even; we will restrict ourselves to the odd case (still, the time  $n$  is even).

**Lemma I.1.** - *Let  $y = 2\xi + 1$  ( $\xi \geq 0$ ) be an horizontal line segment. Let  $A_{2n,2k}$ , with coordinates  $(2n, 2k)$ , be any point that the random walk may reach, with  $k \leq \xi$ . The number of paths, starting at 0, finishing at  $A_{2n,2k}$ , which touch the horizontal segment at a time before  $2n$  is  $N(A_{2n,2\xi+2-2k})$ , where  $A_{2n,2\xi+2-2k}$  is the symmetric of  $A_{2n,2k}$  with respect to the line segment.*

#### Proof of Lemma I.1

This property is well-known (see for instance [1]), under the name of "reflexion principle".

In the sequel, we denote by  $W_{2n}$  the "vertical" at time  $2n$ . This is the set of points  $A_{2n,2k}$ ,  $k = -n, \dots, n$ . We also denote by  $E_{2n}$  the total energy on this vertical :  $E_{2n} = \sum_{k=-n}^n e(A_{2n,2k})$ .

**Proposition I.2.** - *Assume that our barrier is the line segment  $y = 2\xi + 1$ . The energy left at time  $2n$  is:*

$$E_{2n} = 1 - \frac{2}{2^{2n}} \sum_{j=\xi+1}^n \binom{2n}{n+j} = \frac{1}{2^{2n}} \sum_{j=-\xi}^{\xi} \binom{2n}{n+j}$$

#### Proof of Proposition I.2

The barrier  $y = 2\xi + 1$  has two effects :

- No point  $A_{2n,2k}$  above this segment receives any energy at all ; there is a drop of total energy equal to the probability to reach this point;
- For every point strictly below this segment, there is a drop of energy equal to the probability to reach its symmetric.

Since both terms are equal, the total drop of energy (that is the total energy "swallowed" by the segment), instead of reaching  $W_{2n}$ , is  $\frac{2}{2^{2n}} \sum_{j=\xi+1}^n \binom{2n}{n+j}$ . This proves Proposition I.2.

It is clear from Proposition 2 that  $E_{2n} \rightarrow 0$  when  $n \rightarrow +\infty$ . We make this statement quantitative:

**Proposition I.3.** - For all  $\xi$  and  $n$ , we have the estimate:

$$E_{2n} \leq \frac{2\xi}{\sqrt{\pi n}} + \sqrt{\frac{2}{n}}$$

### Proof of Proposition I.3

We have, using the approximation of the binomial law by the normal law:

$$\frac{1}{2^{2n}} \sum_{j=-\xi}^{\xi} \binom{2n}{n+j} \approx \int_{-\xi}^{\xi} \exp\left(-\frac{t^2}{2\sigma^2}\right) \frac{dt}{\sigma\sqrt{2\pi}} \text{ with } \sigma^2 = 2n$$

In order to make this approximation precise, we use Berry-Esseen Theorem [Berry-Esseen], which may be stated as follows:

For all  $x$  and all  $n$  :

$$\left| P\left(\frac{S_n}{\sqrt{n}} \leq x\right) - \int_{-\infty}^x \exp\left(\frac{-t^2}{2}\right) \frac{dt}{\sqrt{2\pi}} \right| \leq \frac{1}{\sqrt{n}}$$

With  $\sigma = \sqrt{2n}$  and  $\xi = \frac{x\sqrt{2n}}{2}$ , it becomes:

$$\left| P(S_{2n} \leq 2\xi) - \int_{-\infty}^{2\xi} \exp\left(\frac{-t^2}{2\sigma^2}\right) \frac{dt}{\sigma\sqrt{2\pi}} \right| \leq \frac{1}{\sqrt{2n}}$$

that is:

$$\left| \frac{1}{2^{2n}} \sum_{k \leq \xi} \binom{2n}{n+k} - \int_{-\infty}^{2\xi} \exp\left(\frac{-t^2}{2\sigma^2}\right) \frac{dt}{\sigma\sqrt{2\pi}} \right| \leq \frac{1}{\sqrt{2n}} \quad (1)$$

and also with  $-\xi$  :

$$\left| \frac{1}{2^{2n}} \sum_{k \leq -\xi} \binom{2n}{n+k} - \int_{-\infty}^{-2\xi} \exp\left(\frac{-t^2}{2\sigma^2}\right) \frac{dt}{\sigma\sqrt{2\pi}} \right| \leq \frac{1}{\sqrt{2n}} \quad (2)$$

Taking the difference, we obtain:

$$\left| \frac{1}{2^{2n}} \sum_{k=-\xi}^{\xi} \binom{2n}{n+k} - \int_{-2\xi}^{2\xi} \exp\left(\frac{-t^2}{2\sigma^2}\right) \frac{dt}{\sigma\sqrt{2\pi}} \right| \leq \sqrt{\frac{2}{n}} \quad (3)$$

From (3), we deduce:

$$E_{2n} \leq \int_{-2\xi}^{2\xi} \exp\left(\frac{-t^2}{2\sigma^2}\right) \frac{dt}{\sigma\sqrt{2\pi}} + \sqrt{\frac{2}{n}} \leq \frac{2\xi}{\sqrt{\pi n}} + \sqrt{\frac{2}{n}}$$

which proves Proposition I.3.

For  $\xi = 0$ , we find the estimate  $E_{2n} \leq \sqrt{\frac{2}{n}}$ , whereas a direct application of Stirling's formula gives  $E_{2n} \leq \frac{1}{\sqrt{\pi n}}$ , so the estimate in Proposition I.3 is not best possible.

## II. Gaussian interpretation

Formula (3) gives immediately, with  $\sigma^2 = 2n$

$$\left| \frac{1}{2^{2n}} \sum_{k=\xi_1}^{\xi_2} \binom{2n}{n+k} - \int_{2\xi_1}^{2\xi_2} \exp\left(\frac{-t^2}{2\sigma^2}\right) \frac{dt}{\sigma\sqrt{2\pi}} \right| \leq \sqrt{\frac{2}{n}} \quad (1)$$

This formula has an interpretation, namely that the energy, on any vertical  $W_{2n}$ , between the levels  $2\xi_1$  and  $2\xi_2$ , may be viewed as a gaussian integral between these two levels, with a Gaussian law, the variance of which is the distance between 0 and the vertical (this distance is  $2n$ ). The error in this approximation is smaller than  $\sqrt{\frac{2}{n}}$ .

This setting is much easier to handle, since Gaussian integrals are simpler to manipulate than binomial sums. Let us give a complete reinterpretation of the previous paragraph: energy absorption in case of a barrier at  $\xi$ . In this continuous setting, there is no need to differentiate between the odd and even cases, which is also a simplification.

The symmetric of a point  $A_{n,t}$  with respect to the barrier  $y = \xi$  is  $A_{n,2\xi-t}$ . Therefore, the density of energy sent by 0 to the point  $A_{n,t}$ , taking into account the annihilation by the barrier, is :



$$f_n(t) = \frac{1}{\sigma\sqrt{2\pi}} \left( \exp\left(-\frac{t^2}{2\sigma^2}\right) - \exp\left(-\frac{(2\xi-t)^2}{2\sigma^2}\right) \right), \text{ for } t \leq \xi, 0 \text{ if } t > \xi \quad (2)$$

with  $\sigma = \sqrt{n}$ . So this is simply the difference of two gaussian functions with same variance.

From (2), we easily obtain the profile of energy, on the vertical  $W_n$ , that is the position of the point of maximal energy:

**Proposition II.1.** - *The point of maximal energy on the vertical  $W_n$  has coordinate:*

$$t_n \approx \frac{3\xi}{2} - \frac{1}{2}\sqrt{\xi^2 + 4n}$$

### Proof of Proposition II.1

Indeed, the condition  $f'_n = 0$  is equivalent to:

$$\frac{t}{t-2\xi} = \exp\left(\frac{-2\xi(\xi-t)}{n}\right) \quad (3)$$

The function  $h(t) = \frac{t}{t-2\xi}$  is decreasing, has the limit 1 at  $-\infty$  and takes the value  $-1$  for  $t = \xi$ .

The function  $g(t) = \exp\left(\frac{-2\xi(\xi-t)}{n}\right)$  is increasing, has limit 0 when  $t \rightarrow -\infty$ , and takes the

value 1 for  $t = \xi$ . Therefore, a unique solution  $t_n$  of equation (3) exists. When  $n \rightarrow +\infty$ , we have

the rough estimate  $\frac{t}{t-2\xi} \sim 1 - \frac{2\xi(\xi-t)}{n}$ , that is  $t_n \approx \frac{3\xi}{2} - \frac{1}{2}\sqrt{\xi^2 + 4n}$ , which implies that

$t_n \rightarrow -\infty$  and proves Proposition II.3.

## III. Notation

In what follows, we will need the following concepts:

- The total energy on the vertical :  $E(W_{2n}) = \sum_{k=-n}^n e(2n, 2k)$  ;
- The quadratic energy :  $E_2(W_{2n}) = \left( \sum_{k=-n}^n (e(2n, 2k))^2 \right)^{1/2}$ .

For any vector  $V = (v_1, \dots, v_N)$ , we will use the following classical norms:

$$|V|_1 = \sum_{j=1}^N |v_j|$$

$$|V|_2 = \sqrt{\sum_{j=1}^N v_j^2}$$

#### IV. The case $\xi = 1$

In this simple case, we have :

**Proposition IV.1.** - For  $\xi = 1$  (barrier at  $\pm 3$ ), the energy is :

$$e(2n, 2) = e(2n, -2) = \frac{1}{4} \left(\frac{3}{4}\right)^{n-1}, \quad e(2n, 0) = \frac{1}{2} \left(\frac{3}{4}\right)^{n-1}$$

The total energy at time  $2n$  is  $E(2n) = \left(\frac{3}{4}\right)^{n-1}$  and the quadratic energy is  $E_2(2n) = \left(\frac{3}{4}\right)^{n-1} \sqrt{\frac{3}{8}}$ .

##### Proof of Proposition IV.1

At the time  $n = 2$ , the energy distribution is:  $e(2, 2) = \frac{1}{4}$ ,  $e(2, 0) = \frac{1}{2}$ ,  $e(2, -2) = \frac{1}{4}$  (the barrier plays no role). We set  $f(n, i) = e(2n, 2i)$ , so we get  $f(1, 1) = \frac{1}{4}$ ,  $f(1, 0) = \frac{1}{2}$ , and we have the recurrence relations:

$$f(n+1, 0) = \frac{1}{2}(f(n, 0) + f(n, 1)), \quad f(n+1, 1) = \frac{1}{4}(f(n, 0) + f(n, 1))$$

Set  $x_n = \frac{1}{2}(f(n, 0) + f(n, 1))$ . Then  $x_1 = \frac{1}{2}(f(1, 0) + f(1, 1)) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4}\right) = \frac{3}{8}$ .

We obtain the equations:

$$f(n+1, 0) = x_n, \quad f(n+1, 1) = \frac{1}{2}x_n, \quad \text{and} \quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{2}x_n\right) = \frac{3}{4}x_n, \quad \text{which gives} \quad x_n = \frac{1}{2}\left(\frac{3}{4}\right)^n.$$

So the energy profile at time  $2n$  is:

$$e(2n, 0) = f(n, 0) = x_{n-1} = \frac{1}{2}\left(\frac{3}{4}\right)^{n-1}, \quad e(2n, 1) = e(2n, -1) = f(n, 1) = \frac{1}{2}x_{n-1} = \frac{1}{4}\left(\frac{3}{4}\right)^{n-1}$$

The total energy at the instant  $2n$  is the sum of all terms, that is  $E(2n) = \left(\frac{3}{4}\right)^{n-1}$ . The quadratic energy is computed the same way, and Proposition IV.1 is proved.

The same definitions of  $f(n, i)$  and  $x(n, i)$  will be used later.

## V. General case: $\xi > 1$

We need several steps in order to investigate the behavior of the energy.

### 1. A first change of variables

We set, as before  $f(n, i) = e(2n, 2i)$ , with the initial values  $f(0, 0) = 1$ ,  $f(0, i) = 0$  for  $i = 1, \dots, \xi$ . Recall that the barrier is set at  $\pm(2\xi + 1)$ , so the last non-zero value for  $f$  on each vertical is  $f(n, \xi)$ . The recurrence equations are:

$$\begin{cases} f(n+1, 0) = \frac{1}{2}(f(n, 0) + f(n, 1)) \\ f(n+1, i) = \frac{1}{4}(f(n, i-1) + 2f(n, i) + f(n, i+1)), i = 1, \dots, \xi - 1 \\ f(n+1, \xi) = \frac{1}{4}(f(n, \xi-1) + f(n, \xi)) \end{cases} \quad (\text{V.1.1})$$

We now study the variation of energy, at a given time, on each vertical.

### 2. Decrease of the energy on each vertical

**Lemma V.2.1.** - *For a given time  $n$ , the energy is decreasing as a function of  $i$ :*

$$f(n, i) \geq f(n, i+1), \quad i \geq 0.$$

#### Proof of Lemma V.2.1

This is true for  $n = 0$ ; let us admit the result for  $n$  and prove it for  $n + 1$ .

We have:

$$f(n+1, 1) = \frac{1}{4}f(n, 0) + \frac{1}{2}f(n, 1) + \frac{1}{4}f(n, 2) \leq \frac{1}{2}f(n, 0) + \frac{1}{2}f(n, 1) = f(n+1, 0)$$

since  $f(n, 2) \leq f(n, 0)$  by the recurrence assumption. For  $1 \leq i \leq \xi - 2$ :

$$\begin{aligned}
f(n+1, i+1) &= \frac{1}{4}f(n, i) + \frac{1}{2}f(n, i+1) + \frac{1}{4}f(n, i+2) \\
&\leq \frac{1}{4}f(n, i-1) + \frac{1}{2}f(n, i) + \frac{1}{4}f(n, i+1) = f(n+1, i)
\end{aligned}$$

Finally, the property  $f(n+1, \xi) \leq f(n+1, \xi-1)$  comes from:

$$\frac{1}{4}f(n, \xi-1) + \frac{1}{4}f(n, \xi) \leq \frac{1}{4}f(n, \xi-2) + \frac{1}{2}f(n, \xi-1) + \frac{1}{4}f(n, \xi)$$

which is clear. So Lemma V.2.1 is proved.

**Corollary V.2.2.** - Let  $m < n$  be two instants; let  $A(m, i)$   $i = 0, \dots, \xi$  be points on the  $m^{\text{th}}$  vertical and let  $B = B(n, 0)$  be the point on the  $x$  axis at time  $n$ . Assume we put the energy 1 at one of the points  $A(m, i)$   $i = 0, \dots, \xi$ . The energy received by  $B$  will be maximal if this energy is put at  $A(m, 0)$ . In fact, the energy received by  $B$  is a decreasing function of  $i$ .

### Proof of Corollary V.2.2

This is a simple consequence of Lemma V.2.1, because if we put energy 1 at  $A(m, i)$ , the energy received by  $B$  is the same as the energy received by  $A(m, i)$  if we put energy 1 at  $B$ .

**Corollary V.2.3.** - Assume we have any distribution of energy  $E_m$  on the vertical  $W_m$ . Then the energy received by  $B$  will be larger if all this energy is concentrated at the single point  $A_0$ .

This is a clear consequence of the previous Corollary. There is a more general statement, which will be useful in Part III:

**Corollary V.2.4.** - Let  $m < n$  be two instants, and let  $f_1(m, i), f_2(m, i)$  be two distributions of energy on the vertical  $W_m$ . Assume that the first one is more concentrated near the origin, which means that, for all  $k = 0, \dots, \xi$ :

$$\sum_{i=0}^k f_1(m, i) \geq \sum_{i=0}^k f_2(m, i)$$

Then, for any  $n > m$ , on the vertical  $W_n$ , the energy coming from the first distribution is larger than the second, which means that the loss of energy is larger in the second case.

This corollary is quite intuitive. The second distribution is globally closer to the barrier, so the loss of energy is larger. Another way to say this is as follows: take any distribution of energy, and move any quantity closer to the  $x$  axis: this is a "protective" move, in the sense that there will be less loss of energy.

We now turn to the behaviour on the horizontal direction.

### 3. Decrease of the energy with time

**Lemma V.3.1** - *On the  $x$  axis, the energy is decreasing: for all  $n$ ,*

$$f(n,0) \geq f(n+1,0).$$

#### Proof of Lemma V.3.1

We have  $f(0,0)=1$  and  $f(1,0)<1$ ; by Lemma V.2.1, we have the induction formula

$$f(n,0) - f(n+1,0) = \frac{1}{2}(f(n,0) - f(n,1)) > 0. \text{ This proves Lemma V.3.1.}$$

However, it is not true that the energy is decreasing on all horizontal lines  $y = j$ ; indeed, if  $j > 1$ , it first increases and then decreases.

### 4. A second change in coordinates

We set, for any  $n \geq 1$  and  $i \geq 2$  :

$$x(n,i) = \frac{1}{2}(f(n,i-1) + f(n,i))$$

This definition makes sense even if there is no barrier. We have  $x(n,i) = 0$  if  $i > n+1$ .

**Lemma V.4.1.** - *If there is no barrier, we have, for any  $n$ ,  $i \leq n+1$  :*

$$x(n,i) = \frac{1}{2^{2n+1}} \binom{2n+1}{n+i}$$

#### Proof of Lemma V.4.1

Indeed, we have:

$$\begin{aligned} x(n,i) &= \frac{1}{2}(f(n,i-1) + f(n,i)) = \frac{1}{2}(e(2n,2i-2) + e(2n,2i)) \\ &= \frac{1}{2^{2n+1}} \left( \binom{2n}{n+i-1} + \binom{2n}{n+i} \right) = \frac{1}{2^{2n+1}} \binom{2n+1}{n+i} \end{aligned}$$

using Pascal's formula. This proves Lemma V.4.1.

**Lemma V.4.2.** - Let  $X_n = (x(n,1), \dots, x(n,n))$ . We have, for all  $n \geq 1$ :

$$|X_n|_1 = \frac{1}{2}, \quad |X_n|_2^2 = \frac{1}{2^{4n+3}} \binom{4n+2}{2n+1};$$

asymptotically, when  $n \rightarrow +\infty$ :

$$|X_n|_2^2 \sim \frac{1}{2\sqrt{2\pi}} \frac{1}{\sqrt{n}}$$

### Proof of Lemma V.4.2

The first statement is just the sum of binomial coefficients. The second is an identity about the sum of squares of binomial coefficients, and the third is an application of Stirling's formula.

From now on, we will work mostly with the new coordinates.

### 5. The propagation problem in the new coordinates

Equations (V.1.1) become:

$$\begin{cases} f(n+1,0) = x(n,1) \\ f(n+1,i) = \frac{1}{2}(x(n,i) + x(n,i+1)), i = 1, \dots, \xi - 1 \\ f(n+1,\xi) = \frac{1}{2}x(n,\xi) \end{cases} \quad (\text{V.5.1})$$

We observe that:

$$E_1(W_{n+1}) = \sum_{i=0}^{\xi} e(2n, 2i) = \sum_{i=0}^{\xi} f(n, i) \leq \frac{3}{2} \sum_{i=1}^{\xi} x(n, i) = \frac{3}{2} |X_n|_1 \quad (\text{V.5.2})$$

and:

$$E_2^2(W_{n+1}) = \sum_{i=0}^{\xi} f^2(n, i) \leq \frac{3}{2} \sum_{i=1}^{\xi} (x(n, i))^2 = \frac{3}{2} |X_n|_2^2 \quad (\text{V.5.3})$$

From (V.5.1), we deduce:

$$\begin{aligned} x(n+1,1) &= \frac{1}{2}(f(n+1,0) + f(n+1,1)) = \frac{1}{2} \left( x(n,1) + \frac{1}{2}(x(n,1) + x(n,2)) \right) \\ &= \frac{3}{4}x(n,1) + \frac{1}{4}x(n,2) \end{aligned}$$

$$\begin{aligned}
x(n+1,i) &= \frac{1}{2} \left( \frac{1}{2} (x(n,i-1) + x(n,i)) + \frac{1}{2} (x(n,i) + x(n,i+1)) \right) \\
&= \frac{1}{4} x(n,i-1) + \frac{1}{2} x(n,i) + \frac{1}{4} x(n,i+1)
\end{aligned}$$

for  $i = 2, \dots, \xi - 1$ , and:

$$x(n+1,\xi) = \frac{1}{4} x(n,\xi-1) + \frac{1}{2} x(n,\xi)$$

So we have the system:

$$\begin{cases}
x(n+1,1) = \frac{3}{4} x(n,1) + \frac{1}{4} x(n,2) \\
x(n+1,i) = \frac{1}{4} x(n,i-1) + \frac{1}{2} x(n,i) + \frac{1}{4} x(n,i+1), \text{ for } i = 2, \dots, \xi - 1 \\
x(n+1,\xi) = \frac{1}{4} x(n,\xi-1) + \frac{1}{2} x(n,\xi)
\end{cases} \quad (\text{V.5.4})$$

with the initial values:

$$x(0,1) = \frac{1}{2} (f(0,0) + f(0,1)) = \frac{1}{2}, \quad x(0,i) = \frac{1}{2} (f(0,i-1) + f(0,i)) = 0 \text{ for } i \geq 2.$$

These initial values may be written as a vector:

$$X_0 = \left( \frac{1}{2}, 0, \dots, 0 \right).$$

## 6. A matrix approach

The system of equations (V.5.4) may be written as a matrix, under the form:

$$\begin{pmatrix}
x(n+1,1) \\
x(n+1,2) \\
\vdots \\
x(n+1,\xi-1) \\
x(n+1,\xi)
\end{pmatrix}
=
\begin{pmatrix}
\frac{3}{4} & \frac{1}{4} & 0 & \dots & \dots & \dots & 0 \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \dots & \dots & 0 \\
0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \dots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \dots & \dots & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
0 & \dots & \dots & \dots & 0 & \frac{1}{4} & \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
x(n,1) \\
x(n,2) \\
\vdots \\
x(n,\xi-1) \\
x(n,\xi)
\end{pmatrix} \quad (\text{V.6.1})$$

We have a real symmetric matrix, of size  $\xi$ , which is denoted by  $M$ .

We observe that, in this matrix representation, things are opposite to the physical representation: the first element of the vector  $X$  and the first row of the matrix correspond to what happens on the  $Ox$  axis; the last element of  $X$  and the last row of the matrix correspond to what happens close to the barrier.

We may consider that this is also a propagation problem, with the following properties:

A point may move upwards, horizontally or downwards ; all horizontal arrows have probability  $\frac{1}{2}$  except the first one (the one on the  $x$  axis) which has probability  $\frac{3}{4}$  ; all oblique arrows (up or down) have probability  $\frac{1}{4}$ . In this representation, two paths with same origin and same destination do not need to have the same probability. Therefore, on the  $x(n,i)$  coordinates, a matrix-oriented approach is appropriate, but an approach counting the number of paths is not.

## 7. Properties of the matrix $M$

**Lemma V.7.1.** - *The matrix  $M$  is positive defined.*

### Proof of Lemma V.7.1

We have to show that, for all non-zero column-vector  $X$  of size  $\xi$ , we have:

$$X^t M X > 0$$

Let  $X = \begin{pmatrix} x_1 \\ \vdots \\ x_\xi \end{pmatrix}$ ; we have:

$$MX = \begin{pmatrix} \frac{3}{4}x_1 + \frac{1}{4}x_2 \\ \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3 \\ \vdots \\ \frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1} \\ \vdots \\ \frac{1}{4}x_{\xi-2} + \frac{1}{2}x_{\xi-1} + \frac{1}{4}x_\xi \\ \frac{1}{4}x_{\xi-1} + \frac{1}{2}x_\xi \end{pmatrix}$$



and therefore:

$$\begin{aligned} X^T M X &= \left( \frac{3}{4} x_1 + \frac{1}{4} x_2 \right) x_1 + \left( \frac{1}{4} x_1 + \frac{1}{2} x_2 + \frac{1}{4} x_3 \right) x_2 + \dots + \left( \frac{1}{4} x_{i-1} + \frac{1}{2} x_i + \frac{1}{4} x_{i+1} \right) x_i + \dots + \\ &\quad + \left( \frac{1}{4} x_{\xi-2} + \frac{1}{2} x_{\xi-1} + \frac{1}{4} x_{\xi} \right) x_{\xi-1} + \left( \frac{1}{4} x_{\xi-1} + \frac{1}{2} x_{\xi} \right) x_{\xi} \\ &= \frac{1}{4} x_1^2 + b \end{aligned}$$

with  $b = \frac{1}{2} \left( x_1^2 + x_1 x_2 + x_2^2 + x_2 x_3 + x_3^2 + \dots + x_{i-1} x_i + x_i^2 + \dots + x_{\xi-2} x_{\xi-1} + x_{\xi-1}^2 + x_{\xi-1} x_{\xi} + x_{\xi}^2 \right)$

and:

$$\begin{aligned} 4b &= x_1^2 + x_1^2 + 2x_1 x_2 + x_2^2 + x_2^2 + 2x_2 x_3 + x_3^2 + \dots + 2x_{i-1} x_i + x_i^2 + x_i^2 + \dots + \\ &\quad + 2x_{\xi-2} x_{\xi-1} + x_{\xi-1}^2 + x_{\xi-1}^2 + 2x_{\xi-1} x_{\xi} + x_{\xi}^2 + x_{\xi}^2 \\ &= x_1^2 + (x_1 + x_2)^2 + (x_2 + x_3)^2 + \dots + (x_{i-1} + x_i)^2 + \dots + (x_{\xi-1} + x_{\xi})^2 + x_{\xi}^2 \end{aligned}$$

So clearly  $X^T M X > 0$  if the  $x_i$  are not all equal to 0. This proves Lemma V.7.1.

From Lemma V.7.1 follows that all eigenvalues of  $M$  are real and  $> 0$  and that  $M$  can be diagonalized in an orthogonal basis made of eigenvectors.

We now study the operator norm of  $M$  :

**Proposition V.7.2.** - From  $l_1$  into  $l_1$ , the operator norm of  $M$  is 1. From  $l_2$  into  $l_2$ , this norm is  $< 1$ .

### Proof of Proposition V.7.2

In order to prove the first part, let us take a vector  $X$  with  $l_1$  norm equal to 1. We want to find the maximum value of  $|Y|_1$  with  $Y = MX$ . This maximum value is obtained when there is no cancellation, that is assuming that all coefficients of  $X$  are positive and satisfy  $\sum_{i=1}^{\xi} x_i = 1$ . But then:

$$|Y|_1 = \sum_{i=1}^{\xi} y_i = \frac{3}{4} x_1 + \frac{1}{4} x_2 + \sum_{i=2}^{\xi-1} \frac{1}{4} x_{i-1} + \frac{1}{2} x_i + \frac{1}{4} x_{i+1} + \frac{1}{4} x_{\xi-1} + \frac{1}{2} x_{\xi} = x_1 + \dots + x_{\xi-1} + \frac{3}{4} x_{\xi} \leq 1,$$

which proves our claim ; we have  $|Y|_1 = |X|_1$  for any vector  $X$  for which  $x_{\xi} = 0$ .

Let us now turn to the  $l_2$  norm. We assume  $\sum_{i=1}^{\xi} x_i^2 = 1$  and we want to prove that  $\sum_{i=1}^{\xi} y_i^2 < 1$ . But,

since  $f(t) = t^2$  is a convex function, we have, for all  $\alpha_i \geq 0$  with  $\sum_i \alpha_i = 1$ ,  $\left(\sum_i \alpha_i x_i\right)^2 \leq \sum_i \alpha_i x_i^2$

and in particular :

$$\left(\frac{3}{4}x_1 + \frac{1}{4}x_2\right)^2 \leq \frac{3}{4}x_1^2 + \frac{1}{4}x_2^2, \quad \left(\frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1}\right)^2 \leq \frac{1}{4}x_{i-1}^2 + \frac{1}{2}x_i^2 + \frac{1}{4}x_{i+1}^2.$$

For the last term,  $y_{\xi} = \frac{1}{4}x_{\xi-1} + \frac{1}{2}x_{\xi}$ , the sum of the coefficients is  $< 1$ , so we write:

$$\left(\frac{\frac{1}{4}x_{\xi-1} + \frac{1}{2}x_{\xi}}{3/4}\right)^2 \leq \frac{1}{3}x_{\xi-1}^2 + \frac{2}{3}x_{\xi}^2$$

and therefore:

$$\left(\frac{1}{4}x_{\xi-1} + \frac{1}{2}x_{\xi}\right)^2 \leq \left(\frac{3}{4}\right)^2 \left(\frac{1}{3}x_{\xi-1}^2 + \frac{2}{3}x_{\xi}^2\right) = \frac{3}{16}x_{\xi-1}^2 + \frac{6}{16}x_{\xi}^2 < \frac{1}{4}x_{\xi-1}^2 + \frac{1}{2}x_{\xi}^2$$

and we add up all terms as we did previously:

$$\sum_{i=1}^{\xi} y_i^2 \leq \sum_{i=1}^{\xi-1} x_i^2 + \frac{3}{4}x_{\xi}^2$$

which proves that the operator norm of  $M$ , from  $l_2$  to  $l_2$ , is  $\leq 1$ . If the operator norm was equal to 1, there would be a case where all inequalities above would be equalities, which implies  $x_1 = \dots = x_{\xi-1}$  and  $x_{\xi} = 0$ . But, for the vector  $X = (1, 1, \dots, 1, 0)$  ; we have  $|X|_2^2 = \xi - 1$  and

$$|Y|_2^2 = \xi - 2 + \left(\frac{3}{4}\right)^2 + \left(\frac{1}{4}\right)^2 < \xi - 1.$$

**Lemma V.7.3.** - All eigenvalues of  $M$  are  $< 1$ .

### Proof of Lemma V.7.3

Let us write the system of equations defining the eigenvalues and eigenvectors:

$$MX = \lambda X$$

$$\left\{ \begin{array}{l} \frac{3}{4}x_1 + \frac{1}{4}x_2 = \lambda x_1 \\ \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3 = \lambda x_2 \\ \vdots \\ \frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1} = \lambda x_i \\ \vdots \\ \frac{1}{4}x_{\xi-2} + \frac{1}{2}x_{\xi-1} + \frac{1}{4}x_{\xi} = \lambda x_{\xi-1} \\ \frac{1}{4}x_{\xi-1} + \frac{1}{2}x_{\xi} = \lambda x_{\xi} \end{array} \right. \quad (\text{V.7.1})$$

It may be written:

$$\left\{ \begin{array}{l} x_2 = (4\lambda - 3)x_1 \\ x_3 = (4\lambda - 2)x_2 - x_1 \\ \vdots \\ x_{i+1} = (4\lambda - 2)x_i - x_{i-1} \\ \vdots \\ x_{\xi} = (4\lambda - 2)x_{\xi-1} - x_{\xi-2} \\ x_{\xi-1} = (4\lambda - 2)x_{\xi} \end{array} \right. \quad (\text{V.7.2})$$

We know that  $x_1 \neq 0$  (if  $x_1 = 0$ , all  $x_i = 0$ ), so we may assume  $x_1 = 1$ . Assume  $\lambda \geq 1$ . From the first equation  $x_2 = (4\lambda - 3)x_1$  we deduce  $x_2 > x_1 > 0$ .

More generally, the equation  $x_{i+1} = (4\lambda - 2)x_i - x_{i-1}$  gives:

$$x_{i-1} - x_i = (4\lambda - 3)x_i - x_{i+1} \geq x_i - x_{i+1}$$

that is  $x_i - x_{i-1} \leq x_{i+1} - x_i$ . So the sequence of consecutive differences is increasing. Since  $x_2 > x_1$ , all differences are positive, the  $x_i$  are increasing and are  $> 0$ . Set  $S = \sum_{i=1}^{\xi} x_i$ ; summing all equations, we get  $S - \frac{1}{4}x_{\xi} = \lambda S$ , that is  $-\frac{1}{4}x_{\xi} = (\lambda - 1)S$ . But this is a contradiction:  $\lambda > 1$ ,  $S > 0$  and  $x_{\xi} > 0$ . Lemma V.7.3 is proved.

## 8. Precise results on the eigenvalues and eigenvectors

**Proposition V.8.1.** - For  $j = 1, \dots, \xi$ , let  $\vartheta_j = \frac{2j-1}{2\xi+1}\pi$ . The  $j^{\text{th}}$  eigenvalue  $\lambda_j$  is:

$$\lambda_j = \frac{1 + \cos(\vartheta_j)}{2} = \cos^2 \frac{\vartheta_j}{2}$$

and the  $j^{\text{th}}$  eigenvector has components :

$$V_j = (\sin(\xi \vartheta_j), \sin((\xi - 1) \vartheta_j), \dots, \sin(\vartheta_j))$$

All vectors have the same  $l_1$  - norm and the same quadratic norm, which are:

$$|V_j|_1 = \frac{1}{2} \tan(\xi \vartheta_1) = \frac{1}{2 \tan \frac{\vartheta_1}{2}}$$

$$|V_j|_2^2 = \frac{\xi}{2} + \frac{1}{4}$$

### Proof of Proposition V.8.1

We have  $x_\xi \neq 0$  (otherwise all  $x_j$ 's are 0), so we may assume  $x_\xi = 1$ .

We set  $\mu = 2\lambda - 1$ , then  $\mu < 1$ . The system (V.7.2) becomes :

$$\left\{ \begin{array}{l} x_2 = (2\mu - 1)x_1 \\ x_3 = 2\mu x_2 - x_1 \\ \vdots \\ x_{i+1} = 2\mu x_i - x_{i-1} \\ \vdots \\ x_\xi = 2\mu x_{\xi-1} - x_{\xi-2} \\ x_{\xi-1} = 2\mu x_\xi \end{array} \right. \quad (\text{V.8.1})$$

We set  $y_j = x_{\xi-j}$  for  $j = 0, \dots, \xi - 1$ . The system (V.8.1) becomes:

$$\left\{ \begin{array}{l} y_0 = 1 \\ y_1 = 2\mu \\ y_2 = 2\mu y_1 - y_0 \\ \vdots \\ y_j = 2\mu y_{j-1} - y_{j-2} \\ \vdots \\ y_{\xi-1} = 2\mu y_{\xi-2} - y_{\xi-3} \\ y_{\xi-2} = (2\mu - 1)y_{\xi-1} \end{array} \right. \quad (\text{V.8.2})$$

Therefore,  $y_j = U_j(\mu)$  where  $U_j$  is the  $j^{\text{th}}$  Chebychev's polynomial of second kind, for  $j = 0, \dots, \xi - 1$ . The final equation in (V.8.2) may be written:

$$U_{\xi-2}(\mu) = (2\mu - 1)U_{\xi-1}(\mu) \quad (\text{V.8.3})$$

that is, with  $\mu = \cos(\vartheta)$ :

$$\frac{\sin((\xi - 1)\vartheta)}{\sin(\vartheta)} = (2\cos(\vartheta) - 1) \frac{\sin(\xi\vartheta)}{\sin(\vartheta)}.$$

By Lemma V.7.3,  $\sin(\vartheta) \neq 0$ , so the above equation is equivalent to:

$$\sin((\xi - 1)\vartheta) = (2\cos(\vartheta) - 1)\sin(\xi\vartheta) \quad (\text{V.8.4})$$

We have :

$$\sin((\xi - 1)\vartheta) - (2\cos(\vartheta) - 1)\sin(\xi\vartheta) = -\sin(\xi\vartheta)\cos(\vartheta) - \cos(\xi\vartheta)\sin(\vartheta) + \sin(\xi\vartheta)$$

Therefore, equation (V.8.4) is equivalent to:

$$\sin(\xi\vartheta)(1 - \cos(\vartheta)) = \cos(\xi\vartheta)\sin(\vartheta)$$

or :

$$\tan(\xi\vartheta) = \frac{\sin(\vartheta)}{1 - \cos(\vartheta)} \quad (\text{V.8.5})$$

which may be written:

$$\tan(\xi\vartheta) = \frac{1}{\tan \frac{\vartheta}{2}} \quad (\text{V.8.6})$$

Therefore:

$$\cos(\xi\vartheta) \cos \frac{\vartheta}{2} - \sin(\xi\vartheta) \sin \frac{\vartheta}{2} = 0$$

which gives:

$$\cos\left(\xi\vartheta + \frac{\vartheta}{2}\right) = 0$$

and this equation has the solutions  $\frac{2\xi+1}{2}\vartheta = \frac{\pi}{2} + (j-1)\pi$ ,  $j = 1, \dots, \xi$ ,

that is:

$$\vartheta = \frac{(2j-1)\pi}{2\xi+1} \quad (\text{V.8.7})$$

as we announced.

Since  $U_j(\cos \vartheta) = \frac{\sin((j+1)\vartheta)}{\sin \vartheta}$ , after multiplication, we may take, for  $j = 1, \dots, \xi$ :

$$V_j = \left(\sin(\xi\vartheta_j), \sin((\xi-1)\vartheta_j), \dots, \sin(\vartheta_j)\right) \quad (\text{V.8.8})$$

and  $\lambda_j = \frac{1 + \cos(\vartheta_j)}{2} = \cos^2 \frac{\vartheta_j}{2}$ .

We observe that:

$$\xi\vartheta_j = (2j-1)\frac{\pi}{2} - \frac{2j-1}{2\xi+1}\frac{\pi}{2} = j\pi - \frac{\pi}{2} - \frac{2j-1}{2\xi+1}\frac{\pi}{2} \quad (\text{V.8.9})$$

and therefore:

$$\sin(\xi\vartheta_j) = \sin\left(j\pi - \frac{\pi}{2} - \frac{2j-1}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1} \cos\left(\frac{2j-1}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1} \cos \frac{\vartheta_j}{2} \quad (\text{V.8.10})$$

$$\cos(\xi\vartheta_j) = \cos\left(j\pi - \frac{\pi}{2} - \frac{2j-1}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1} \sin\left(\frac{2j-1}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1} \sin \frac{\vartheta_j}{2} \quad (\text{V.8.11})$$

More generally:

$$\begin{aligned}\sin((\xi - i + 1)\mathcal{G}_j) &= \sin\left(j\pi - \frac{\pi}{2} - \frac{(2i-1)(2j-1)\pi}{2\xi+1}\right) \\ &= (-1)^{j-1} \cos\left(\frac{(2i-1)(2j-1)\pi}{2\xi+1}\right) = (-1)^{j-1} \cos\frac{(2i-1)\mathcal{G}_j}{2}\end{aligned}\tag{V.8.12}$$

$$\begin{aligned}\cos((\xi - i + 1)\mathcal{G}_j) &= \cos\left(j\pi - \frac{\pi}{2} - \frac{(2i-1)(2j-1)\pi}{2\xi+1}\right) \\ &= (-1)^{j-1} \sin\left(\frac{(2i-1)(2j-1)\pi}{2\xi+1}\right) = (-1)^{j-1} \sin\frac{(2i-1)\mathcal{G}_j}{2}\end{aligned}\tag{V.8.13}$$

In order to compute the  $l_1$  and  $l_2$  norms of the eigenvectors, let us first observe that all of them are, up to changes of signs, reorderings of the terms of  $V_1$ . Indeed, when  $j$  changes, the  $\xi$  numbers  $\sin(k\mathcal{G}_j)$  ( $k=1, \dots, \xi$ ) are reorderings of the  $\xi$  numbers  $\sin(k\mathcal{G}_1)$ , except for the sign, which may become minus (this does not affect the norms).

So, let us compute  $|V_1|_1$ . Apply the matrix  $M$  to the eigenvector  $V_1$ : by definition, we get  $MV_1 = \lambda_1 V_1$  and the loss of energy is  $(1 - \lambda_1)|V_1|_1$ . But this loss of energy is also  $\frac{1}{4}\sin(\mathcal{G}_1)$ , since the last coordinate of the eigenvector is  $\sin(\mathcal{G}_1)$ . So we get:

$$|V_1|_1 = \frac{1}{2} \frac{\sin(\mathcal{G}_1)}{1 - \cos(\mathcal{G}_1)} = \frac{1}{2} \tan(\xi\mathcal{G}_1)$$

$$(1 - \lambda_1)|V_1|_1 = \frac{1}{4} \sin(\mathcal{G}_1)$$

But  $\lambda_1 = \frac{1 + \cos(\mathcal{G}_1)}{2}$ , which gives:

$$|V_1|_1 = \frac{1}{2} \tan(\xi\mathcal{G}_1) = \frac{1}{2} \frac{\sin(\mathcal{G}_1)}{1 - \cos(\mathcal{G}_1)} = \frac{1}{2 \tan \frac{\mathcal{G}_1}{2}}$$

and this result is valid for all eigenvectors.

Let us now compute  $|V_1|_2^2$ . We have:

$$\begin{aligned}
|V_1|_2^2 &= \sum_{k=1}^{\xi} \sin^2\left(\frac{k\pi}{2\xi+1}\right) = \frac{\xi \sin(\vartheta_1) + \sin(\vartheta_1) \cos^2((\xi+1)\vartheta_1) - \cos(\vartheta_1) \sin((\xi+1)\vartheta_1) \cos((\xi+1)\vartheta_1)}{2 \sin(\vartheta_1)} \\
&= \frac{\xi}{2} + \frac{\cos^2((\xi+1)\vartheta_1)}{2} - \frac{\cos(\vartheta_1) \sin((\xi+1)\vartheta_1) \cos((\xi+1)\vartheta_1)}{2 \sin(\vartheta_1)} \\
&= \frac{\xi}{2} - \frac{\cos((\xi+1)\vartheta_1) \sin(\xi\vartheta_1)}{2 \sin(\vartheta_1)}
\end{aligned}$$

We need to prove that  $\frac{\cos((\xi+1)\vartheta_1) \sin(\xi\vartheta_1)}{\sin(\vartheta_1)} = \frac{-1}{2}$ . This is equivalent to:

$$2 \cos\left(\frac{\xi+1}{2\xi+1}\pi\right) \sin\left(\frac{\xi}{2\xi+1}\pi\right) + \sin\left(\frac{1}{2\xi+1}\pi\right) = 0$$

But:

$$2 \cos\left(\frac{\xi+1}{2\xi+1}\pi\right) \sin\left(\frac{\xi}{2\xi+1}\pi\right) = \sin\left(\frac{(\xi+1)\pi}{2\xi+1} + \frac{\xi\pi}{2\xi+1}\right) - \sin\left(\frac{(\xi+1)\pi}{2\xi+1} - \frac{\xi\pi}{2\xi+1}\right) = -\sin\left(\frac{\pi}{2\xi+1}\right)$$

This proves Proposition V.8.1.

So the eigenvector satisfies the equations:

$$\frac{3}{4} \sin(\xi\vartheta) + \frac{1}{4} \sin((\xi-1)\vartheta) = \frac{1+\cos(\vartheta)}{2} \sin(\xi\vartheta)$$

$$\frac{1}{4} \sin((\xi-j)\vartheta) + \frac{1}{2} \sin((\xi-j+1)\vartheta) + \frac{1}{4} \sin((\xi-j+2)\vartheta) = \frac{1+\cos(\vartheta)}{2} \sin((\xi-j+1)\vartheta)$$

for  $j=1, \dots, \xi-1$

and finally:

$$\frac{\sin(2\vartheta)}{4} + \frac{\sin(\vartheta)}{2} = \frac{1+\cos(\vartheta)}{2} \sin(\vartheta)$$

**Remark 1.** - For any  $n$  and  $t$ , we have the identity :

$$\frac{\sin((n-1)t)}{4} + \frac{\sin(nt)}{2} + \frac{\sin((n+1)t)}{4} = \frac{1+\cos(t)}{2} \sin(nt)$$

including the case  $n=1$  :



$$\frac{\sin(t)}{2} + \frac{\sin(2t)}{4} = \frac{1 + \cos(t)}{2} \sin(t)$$

Therefore, the value of  $\mathcal{G}$  is determined by the first equation only, that is:

$$\frac{3}{4} \sin(\xi \mathcal{G}) + \frac{1}{4} \sin((\xi - 1)\mathcal{G}) = \frac{1 + \cos(\mathcal{G})}{2} \sin(\xi \mathcal{G})$$

which is equivalent to the equation:

$$\tan(\xi \mathcal{G}) = \frac{\sin(\mathcal{G})}{1 - \cos(\mathcal{G})}.$$

The first eigenvector,  $V_1$ , has all its components real and  $> 0$ , but all other eigenvectors have some negative component.

**Remark 2.** - It follows from the general theory of symmetric matrices, positive defined, that any two eigenvectors  $V_{j_1}, V_{j_2}$  are mutually orthogonal, that is:

$$\sum_{l=1}^{\xi} \sin(l \mathcal{G}_{j_1}) \sin(l \mathcal{G}_{j_2}) = 0 \quad (\text{V.8.14})$$

where  $\mathcal{G}_{j_1} = \frac{(2j_1 - 1)\pi}{2\xi + 1}$ ,  $\mathcal{G}_{j_2} = \frac{(2j_2 - 1)\pi}{2\xi + 1}$ . This can be checked directly.

## 9. Energy profile at each step

The main result of this section is :

**Theorem V.9.1.** - *At each step, we have :*

$$x(n, i) = \frac{2}{2\xi + 1} \sum_{j=1}^{\xi} \cos\left(\frac{(2i-1)\mathcal{G}_j}{2}\right) \cos^{2n+1} \frac{\mathcal{G}_j}{2}$$

with  $\mathcal{G}_j = \frac{2j-1}{2\xi+1} \pi$ ,  $j = 1, \dots, \xi$ .

### Proof of Theorem V.9.1

If we want to compute the energy at the  $n^{\text{th}}$  step, we start with the initial value  $f(0, 0) = 1$ ,  $f(0, i) = 0$ ,  $i = 1, \dots, \xi$ . This gives for the initial vector :

$$X_0 = \left( \frac{1}{2}, 0, \dots, 0 \right)$$

We decompose this vector on the basis of eigenvectors. We write :

$$X_0 = \sum_{j=1}^{\xi} \alpha_j V_j$$

Since the eigenvectors are orthogonal, the coefficients  $\alpha_j$  may be computed simply:

$$\alpha_j = \frac{\langle X_0, V_j \rangle}{|V_j|_2^2} = \frac{\sin(\xi \vartheta_j)}{\xi + \frac{1}{2}}$$

Then, at the  $n^{\text{th}}$  step (time  $2n$ ), the vector  $X_n$  is :

$$X_n = M^n X_0 = M^n \sum_{j=1}^{\xi} \alpha_j V_j = \sum_{j=1}^{\xi} \alpha_j M^n V_j = \sum_{j=1}^{\xi} \alpha_j \lambda_j^n V_j$$

Using the identity  $\frac{1 + \cos(\vartheta_j)}{2} = \cos^2\left(\frac{\vartheta_j}{2}\right)$ , we obtain the formula:

$$x(n, i) = \frac{2}{2^{\xi+1}} \sum_{j=1}^{\xi} \sin(\xi \vartheta_j) \sin((\xi - i + 1) \vartheta_j) \cos^{2n} \frac{\vartheta_j}{2}$$

using formulas (V.8.12) and (V.8.13), this proves Theorem V.9.1. In this formula, all terms are known, so the energy at each step is explicit.

We can study the decrease of the first coordinate :

**Proposition V.9.2.** - *For every  $n$ , we have :*

$$X_n(1) = \frac{1}{\xi + \frac{1}{2}} \sum_{j=1}^{\xi} \cos^{2n+2} \left( \frac{2j-1}{2} \frac{\pi}{2} \right)$$

### Proof of Proposition V.9.2

We start with  $X_0 = \left( \frac{1}{2}, 0, \dots, 0 \right)$  and we want to investigate the behaviour of the first coordinate of  $X_n = M^n X_0$ , denoted by  $X_n(1)$ . We have:

$$X_0 = \sum_{j=1}^{\xi} \alpha_j V_j \text{ with } \alpha_j = \frac{\sin(\xi \vartheta_j)}{\xi + \frac{1}{2}} = \frac{2}{2\xi + 1} (-1)^{j-1} \cos \frac{\vartheta_j}{2}, \quad j = 1, \dots, \xi.$$

We have  $X_n = M^n X_0 = \sum_{j=1}^{\xi} \alpha_j \lambda_j^n V_j$  and therefore, since  $V_j(1) = \sin(\xi \vartheta_j)$ :

$$X_n(1) = \frac{1}{\xi + \frac{1}{2}} \sum_{j=1}^{\xi} \sin^2(\xi \vartheta_j) \lambda_j^n,$$

and we obtain the explicit formula:

$$X_n(1) = \frac{2}{2\xi + 1} \sum_{j=1}^{\xi} \cos^{2n+2} \left( \frac{2j-1}{2\xi+1} \frac{\pi}{2} \right)$$

and Proposition V.9.2 is proved.

### 10. The energy at each step

Using the previous representation, we now investigate the behaviour of the energy as a function of  $n$ .

**Lemma V.10.1.** - *The largest eigenvalue of the matrix  $M$  satisfies:*

$$\lambda_1 = \frac{1}{2} \left( 1 + \cos \left( \frac{\pi}{2\xi + 1} \right) \right) \geq 1 - \frac{1}{4} \frac{\pi^2}{(2\xi + 1)^2} \sim 1 - \frac{\pi^2}{16 \xi^2} \text{ when } \xi \rightarrow +\infty.$$

#### Proof of Lemma V.10.1

This follows from the estimate  $\cos(x) \geq 1 - \frac{x^2}{2}$ . This proves Lemma V.10.1.

Let, for any  $\xi$ ,  $V_{\xi,j}$ ,  $j = 1, \dots, \xi$  be the eigenvectors of the matrix  $M$  of size  $\xi$ .

**Lemma V.10.2.** - *We have, when  $\xi \rightarrow +\infty$  :*

$$\|V_{\xi,j}\|_1 \sim \frac{2\xi}{\pi}$$

#### Proof of Lemma V.10.2

It is enough to prove the Lemma for the first eigenvector, since all eigenvectors have the same  $l_1$  norm. We have  $\vartheta_1 = \frac{\pi}{2\xi+1}$ , and:

$$|V_{\xi,1}|_1 = \frac{1}{2} \tan(\xi\vartheta_1) = \frac{\sin(\vartheta_1)}{2(1-\cos(\vartheta_1))} = \frac{\sin\left(\frac{\pi}{2\xi+1}\right)}{2\left(1-\cos\left(\frac{\pi}{2\xi+1}\right)\right)} \sim \frac{2\xi}{\pi}$$

as we announced.

From Lemma V.7.3 follows that the operator  $M$  is a strict contraction in the  $l_2$  norm ; indeed we know (see for instance [BB\_Op]) that:

$$|MX|_2 \leq \max(\lambda_j) |X|_2.$$

We may now state a Theorem describing the decrease of energy when  $n$  increases:

**Theorem V.10.3** - *The quadratic energy at step  $2n$  satisfies the estimate:*

$$E_2(W_{2n}) \leq \frac{1}{2} \sqrt{\frac{3}{2}} \left(1 - \frac{\pi^2}{16\xi^2}\right)^{n-1}$$

and the energy satisfies:

$$E(W_{2n}) \leq \sqrt{\frac{3\xi}{8}} \left(1 - \frac{\pi^2}{16\xi^2}\right)^{n-1}$$

### Proof of Theorem V.10.3

From the expression  $X_0 = \sum_{k=1}^{\xi} \alpha_j V_j$ , we deduce, with  $\alpha_j = \frac{\sin(\xi\vartheta_j)}{\xi + \frac{1}{2}}$  and  $|V_j|_2^2 = \frac{\xi}{2} + \frac{1}{4}$  :

$$|M^n X_0|_2^2 = \sum_{j=1}^{\xi} \alpha_j^2 \lambda_j^{2n} |V_j|_2^2.$$

We have  $|X_0|_2^2 = \frac{1}{4}$  and  $\sum_{j=1}^{\xi} \alpha_j^2 = \frac{\xi}{2} + \frac{1}{4}$ . Therefore:

$$|M^{n-1} X_0|_2^2 = \frac{1}{2\xi+1} \sum_{j=1}^{\xi} \sin^2(\xi\vartheta_j) \lambda_j^{2n-2} \leq \lambda_1^{2n-2} \frac{1}{2\xi+1} \sum_{j=1}^{\xi} \sin^2(\xi\vartheta_j) = \frac{\lambda_1^{2n-2}}{4}$$

that is:

$$|X_{n-1}|_2 \leq \frac{1}{2} \left( 1 - \frac{\pi^2}{16\xi^2} \right)^{n-1} \quad (\text{V.10.1})$$

Now, we observe that:

$$E_2(W_{2n}) \leq \sqrt{\frac{3}{2}} |X_{n-1}|_2 \quad (\text{V.10.2})$$

Indeed,

$$\begin{aligned} f(n,0)^2 + \sum_{i=1}^{\xi-1} f(n,i)^2 + f(n,\xi)^2 &= x(n-1,1)^2 + \frac{1}{4} \sum_{i=1}^{\xi-1} (x(n-1,i) + x(n-1,i+1))^2 + \frac{1}{4} x(n-1,\xi)^2 \\ &\leq x(n-1,1)^2 + \frac{1}{2} \sum_{i=1}^{\xi-1} x(n-1,i)^2 + \frac{1}{2} \sum_{i=1}^{\xi-1} x(n-1,i+1)^2 + \frac{1}{4} x(n-1,\xi)^2 \\ &\leq \frac{3}{2} |X_{n-1}|_2^2 \end{aligned}$$

which proves (V.10.2).

We deduce from (V.10.1) and (V.10.2):

$$E_2(W_{2n}) \leq \frac{1}{2} \sqrt{\frac{3}{2}} \left( 1 - \frac{\pi^2}{16\xi^2} \right)^{n-1}$$

which proves the first part of Theorem V.10.3.

Now, in order to prove the second part, we use Cauchy-Schwarz inequality:

$$\sum_{i=0}^{\xi-1} x(n-1,i) = |X_{n-1}|_1 \leq \sqrt{\xi} |X_{n-1}|_2$$

using the computation above. This proves Theorem V.10.3.

We now turn to the study of asymptotic estimates of the profile on any vertical, when  $n \rightarrow +\infty$ , fixed  $\xi$ .

## 11. Asymptotic estimates on the energy profile

**Proposition V.11.1.** - *Asymptotically when  $n \rightarrow +\infty$ , we have the estimates, with  $\mathfrak{G}_1 = \frac{\pi}{2\xi+1}$ :*

$$e(2n+2,0) \sim \frac{2}{2\xi+1} \cos^{2n+2} \frac{\vartheta_1}{2}$$

For  $j=1, \dots, \xi-1$  :

$$e(2n+2,2j) = \frac{2}{2\xi+1} \sin((\xi-j)\vartheta_1) \cos^{2n+2} \frac{\vartheta_1}{2}$$

and for  $j=\xi$  :

$$e(2n+2,2\xi) \sim \frac{2}{2\xi+1} \sin(\vartheta_1) \cos^{2n+1} \frac{\vartheta_1}{2}$$

### Proof of Proposition V.11.1

We work on the vector  $X_n = M^n X_0$  ; writing the decomposition  $X_0 = \sum_{j=1}^{\xi} \alpha_j V_j$  on the basis of eigenvectors, we obtain:

$$X_n = \sum_{j=1}^{\xi} \alpha_j \lambda_j^n V_j$$

We have  $\lambda_1 > \dots > \lambda_{\xi}$ , so, when  $n \rightarrow +\infty$ ,  $X_n \sim \alpha_1 \lambda_1^n V_1$ .

Asymptotically when  $n \rightarrow +\infty$ , the energy distribution, on the variables  $x(n, j)$ , is therefore proportional to the first column of the change of basis matrix.

Returning to the energy  $f(n, j) = e(2n, 2j)$  using formulas (V.1.1), we find:

$$e(2n+2,0) = x(n,1) \sim \lambda_1^n \frac{\sin^2(\xi\vartheta_1)}{\xi + \frac{1}{2}} = \frac{\sin^2(\xi\vartheta_1)}{\xi + \frac{1}{2}} \cos^{2n} \frac{\vartheta_1}{2}$$

But  $\sin(\xi\vartheta_1) = \cos \frac{\vartheta_1}{2}$ , which gives the announced formula.

For  $j=1, \dots, \xi-1$  :

$$\begin{aligned} e(2n+2,2j) &= \frac{1}{2} (x(n, j) + x(n, j+1)) \sim \frac{\lambda_1^n \sin(\xi\vartheta_1)}{\xi + \frac{1}{2}} \frac{\sin((\xi-j+1)\vartheta_1) + \sin((\xi-j)\vartheta_1)}{2} \\ &= \frac{\lambda_1^n \sin(\xi\vartheta_1)}{\xi + \frac{1}{2}} \sin((\xi-j)\vartheta_1) \cos \frac{\vartheta_1}{2} = \frac{2}{2\xi+1} \cos^{2n+2} \frac{\vartheta_1}{2} \sin((\xi-j)\vartheta_1) \end{aligned}$$

and finally:

$$e(2n+2, 2\xi) = \frac{1}{2}x(n, \xi) \sim \lambda_1^n \frac{\sin(\mathcal{G}_1) \sin(\xi \mathcal{G}_1)}{\xi + \frac{1}{2}} = \frac{2}{2\xi+1} \cos^{2n+1} \frac{\mathcal{G}_1}{2} \sin(\mathcal{G}_1)$$

which proves Proposition V.11.1.

**Corollary V.11.2.** - *For any starting point  $X_0$ , the energy profile at time  $n$ , for the vector  $X_n$ , is asymptotically equivalent to the first eigenvector  $V_1$  : this profile is proportional to the vector :*

$$V_1 = (\sin(\xi \mathcal{G}_1), \dots, \sin((\xi - i + 1) \mathcal{G}_1), \dots, \sin(\mathcal{G}_1))$$

*This profile is concave, which means that:*

$$g_i \geq \frac{1}{2}g_{i-1} + \frac{1}{2}g_{i+1}$$

### Proof of Corollary V.11.2

The first part follows immediately from Proposition V.11.1. In order to prove the concavity, let us compute:

$$\begin{aligned} P_j &= g_i - \frac{1}{2}g_{i-1} - \frac{1}{2}g_{i+1} = \sin((\xi - i + 1) \mathcal{G}_1) - \frac{1}{2}\sin((\xi - i + 2) \mathcal{G}_1) - \frac{1}{2}\sin((\xi - i) \mathcal{G}_1) \\ &= \sin((\xi - i + 1) \mathcal{G}_1) - \sin((\xi - i + 1) \mathcal{G}_1) \cos \mathcal{G}_1 > 0. \end{aligned}$$

This proves Corollary V.11.2.

## 12. The energy on the boundary

Since we have seen that the energy on each vertical tends to zero, one might wonder where this energy has gone. More precisely, let us assume now that the barrier is not a "black hole", but it keeps the energy it receives, without propagating it further. In other words, it stores the energy it receives. Then we may wonder about the distribution of this energy. Recall that the barrier is assumed to be on the  $y = \xi + 1$  axis, for the  $X$  coordinates.

**Proposition V.12.1.** - *Assume we start with the eigenvector:*

$$V_j = (\sin(\xi \mathcal{G}_j), \dots, \sin(\mathcal{G}_j))$$

*Then the point of the barrier with coordinates  $(n, \xi + 1)$  receives the energy:*

$$e_{n,\xi+1} = \frac{1}{4} \lambda_j^{n-1} \sin(\mathcal{G}_j)$$

### Proof of Proposition V.12.1

By definition, at the first step, the distribution of energy is

$$E_1 = \left( \lambda_j \sin(\xi \mathcal{G}_j), \dots, \lambda_j \sin(\mathcal{G}_j), \frac{1}{4} \sin(\mathcal{G}_j) \right)$$

and the loss, kept by the barrier, is  $\frac{1}{4} \sin(\mathcal{G}_j)$ . At the second step, the distribution of energy is :

$$E_2 = \left( \lambda_j^2 \sin(\xi \mathcal{G}_j), \dots, \lambda_j^2 \sin(\mathcal{G}_j), \frac{1}{4} \lambda_j \sin(\mathcal{G}_j) \right)$$

and the loss, kept by the barrier, is  $\frac{1}{4} \lambda_j \sin(\mathcal{G}_j)$  ; and so on at further steps and this proves Proposition V.12.1.

For the first eigenvector, we check immediately that the total loss of energy is equal to the total initial energy. Indeed, the total initial energy was, by Proposition V.8.1:

$$TIE = \sum_{k=1}^{\xi} \sin(k \mathcal{G}_1) = \frac{1}{2} \tan(\xi \mathcal{G}_1)$$

and the total loss is, by Proposition V.12.1:

$$TLE = \frac{\sin(\mathcal{G}_1)}{4} \sum_{n=1}^{+\infty} \lambda_1^{n-1} = \frac{\sin(\mathcal{G}_1)}{4} \frac{1}{1-\lambda_1} = \frac{\sin(\mathcal{G}_1)}{2(1-\cos(\mathcal{G}_1))}$$

and both quantities coincide.

If we start with another vector, we have to decompose it on the basis of eigenvectors:

$$V = \sum_{j=1}^{\xi} \alpha_j V_j$$

and the loss at step  $n$  will be  $e_{n,\xi+1} = \frac{1}{4} \sum_{j=1}^{\xi} \alpha_j \sin(\mathcal{G}_j) \lambda_j^{n-1}$ .

For instance, if we start with:

$$X_1 = (1, 0, \dots, 0)$$



we have (see above):

$$\alpha_{1,j} = \frac{\sin(\xi \mathcal{G}_j)}{\frac{\xi}{2} + \frac{1}{4}}$$

and the loss at step  $n$  will be:

$$e_{n,\xi+1} = \frac{1}{2\xi+1} \sum_{j=1}^{\xi} \sin(\xi \mathcal{G}_j) \sin(\mathcal{G}_j) \lambda_j^{n-1} = \frac{1}{2\xi+1} \sum_{j=1}^{\xi} (-1)^{j-1} \sin(\mathcal{G}_j) \cos^{2n-1} \frac{\mathcal{G}_j}{2}.$$

The sum  $\sum_{n=1}^{+\infty} e_{n,\xi+1}$  is equal to 1. We also observe that  $e_{n,\xi+1} = 0$  if  $n < \xi$ .

If we start with:

$X_i = (0, \dots, 0, 1, 0, \dots, 0)$  (1 at the  $i^{\text{th}}$  place), we have:

$$\alpha_{i,j} = \frac{\sin((\xi - i + 1) \mathcal{G}_j)}{\frac{\xi}{2} + \frac{1}{4}}$$

and the loss at step  $n$  will be:

$$e_{n,\xi+1} = \frac{1}{2\xi+1} \sum_{j=1}^{\xi} \sin((\xi - i + 1) \mathcal{G}_j) \sin(\mathcal{G}_j) \lambda_j^{n-1} = \frac{1}{2\xi+1} \sum_{j=1}^{\xi} (-1)^{j-1} \cos \frac{(2i-1) \mathcal{G}_j}{2} \sin(\mathcal{G}_j) \cos^{2(n-1)} \frac{\mathcal{G}_j}{2}.$$

### Part III : Variable barriers

In this Third Part, we investigate the case of variable barriers : the barrier is represented by a function of time; say for instance  $b(n) = \pm\sqrt{n}$ , to start with. The main theorem is as follows:

**Theorem.** - *The probability  $P_N$  that the game continues after  $N$  steps tends to 0 when  $N \rightarrow +\infty$*

*if and only if the integral  $\int \frac{dx}{b^2(x)}$  diverges at  $+\infty$ . More precisely, this probability satisfies the*

*estimate:*

$$E_N \leq \exp\left(-\frac{\pi^2}{16} \int_1^{t_{N+1}} \frac{dx}{b^2(x)}\right)$$

where  $t_n$  is the unique number such that  $b(t_n) = n$ .

During the  $n^{\text{th}}$  period (see definition below), the profile of fortune is proportional to the vector:

$$(\sin(\vartheta), \sin(2\vartheta), \dots, \sin(n\vartheta), \sin((n-1)\vartheta), \dots, \sin(\vartheta))$$

where  $\vartheta = \frac{\pi}{2n+1}$ .

The case of the barrier  $f(n) = \pm\sqrt{n \text{Log}(n)}$  is of special interest, because it lies above Khinchine's safety curve  $\varphi(n) = \sqrt{2n \text{Log}(\text{Log}(n))}$ . Still, the main Theorem shows that the probability that the game continues after  $N$  steps tends to 0 when  $N \rightarrow +\infty$ , and gives quantitative estimates for this probability.

## I. Transition between two periods

### 1. Energy propagation

We have a continuous barrier, but our game uses only integer values. So we have to convert our barrier into a succession of constant segments, with integer values.

Let us describe this representation in detail in the case of the barrier  $\pm\sqrt{n}$ .

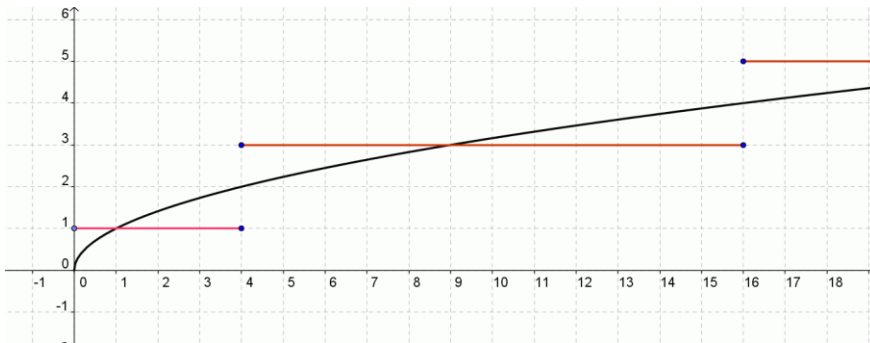


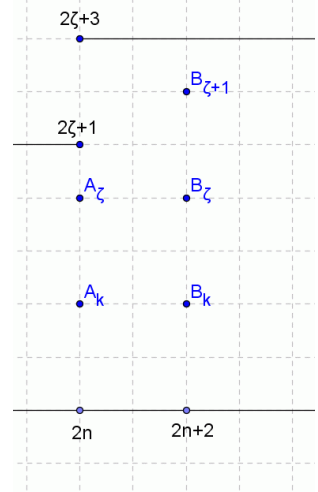
Figure 1: discretisation of the barrier

The changes will occur at times  $4n^2$ ,  $n = 0, 1, \dots$ . On the interval  $4n^2 \leq x < 4(n+1)^2$ , the barrier is at  $2n+1$  (recall from Part II that we want even values of time and odd values for the barrier).

So, in the notation introduced in Part II,  $\xi = n$  and this value is used on an interval of length  $l_n = 4(n+1)^2 - 4n^2$ , that is  $l_n = 8n + 4$ .

The interval of time during which  $\xi = n$  is called the  $n^{\text{th}}$  period. From now on, we forget about the continuous curve and remember only the segments.

We have now to investigate the transition between two periods. The barrier was at  $2\xi + 1$  and moves to  $2\xi + 3$ . Let us first consider the transition on the energy, that is the variables  $e(2n, 2k)$  (see Part II). It will be convenient to have a simple notation just for the transition. On the vertical corresponding to time  $2n$ , we have  $\xi + 1$  points  $A_0, \dots, A_\xi$ ; at time  $2n + 2$ , we have  $\xi + 2$  points  $B_0, \dots, B_{\xi+1}$ . We denote by  $a_k$  the energy at the point  $A_k$  and similarly  $b_k$  for the  $B_k$ .



For the first  $\xi$  points, we have the usual transition equations:

$$b_0 = \frac{1}{2}(a_0 + a_1) \quad (1.1)$$

$$b_k = \frac{1}{4}a_{k-1} + \frac{1}{2}a_k + \frac{1}{4}a_{k+1} \text{ for } k = 1, \dots, \xi - 1 \quad (1.2)$$

The last two equations are different from the constant case; they are:

$$b_\xi = \frac{1}{4}a_{\xi-1} + \frac{1}{2}a_\xi \quad (1.3)$$

$$b_{\xi+1} = \frac{1}{4}a_\xi \quad (1.4)$$

If the barrier was constantly at  $2\xi + 1$ , instead of (1.3), we would have:

$$b_\xi = \frac{1}{4}a_{\xi-1} + \frac{1}{4}a_\xi \quad (1.5)$$

and instead of (1.4) :

$$b_{\xi+1} = 0 \quad (1.6)$$

So, the fact that the barrier moves one step higher means more energy: for  $b_\xi$ , increase of  $\frac{1}{4}a_\xi$ , for  $b_{\xi+1}$ , increase of  $\frac{1}{4}a_\xi$ , which represents a total increase of energy equal to  $\frac{1}{2}a_\xi$ .

Let us now turn to the variables  $x(n, k)$  and describe the transition on these variables. Recall that, for  $k = 0, \dots, \xi - 1$  and  $n \geq 2$ :

$$x_k = \frac{1}{2}(a_k + a_{k+1})$$

We have (see Part II):

$$b_0 = x_0 \quad (1.7)$$

$$b_k = \frac{1}{2}(x_{k-1} + x_k) \text{ for } k = 1, \dots, \xi - 1 \quad (1.8)$$

$$b_\xi = \frac{1}{4}(a_{\xi-1} + a_\xi) + \frac{1}{4}(a_\xi + a_{\xi+1}) \text{ with } a_{\xi+1} = 0$$

which gives:

$$b_\xi = \frac{1}{2}x_{\xi-1} + \frac{1}{2}x_\xi \quad (1.9)$$

$$b_{\xi+1} = \frac{1}{4}(a_\xi + a_{\xi+1}) = \frac{1}{2}x_\xi \quad (1.10)$$

Let us define  $y_k = \frac{1}{2}(b_k + b_{k+1})$ ,  $k = 0, \dots, \xi$ . We get:

$$y_0 = \frac{1}{2}\left(x_0 + \frac{1}{2}(x_0 + x_1)\right) = \frac{3}{4}x_0 + \frac{1}{4}x_1$$

$$y_k = \frac{1}{4}(x_{k-1} + 2x_k + x_{k+1}), \quad k = 1, \dots, \xi - 1$$

$$y_\xi = \frac{1}{2}\left(\frac{1}{2}x_{\xi-1} + \frac{1}{2}x_\xi + \frac{1}{2}x_\xi\right) = \frac{1}{4}x_{\xi-1} + \frac{1}{2}x_\xi$$

The equations are the same as in the constant case; we simply have one more intermediate equation. With the original notation, we have:

$$\begin{cases} x(n+1, 0) = \frac{3}{4}x(n, 0) + \frac{1}{4}x(n, 1) \\ x(n+1, k) = \frac{1}{4}x(n, k-1) + \frac{1}{2}x(n, k) + \frac{1}{4}x(n, k+1), \text{ for } k = 1, \dots, \xi - 1 \\ x(n+1, \xi) = \frac{1}{4}x(n, \xi-1) + \frac{1}{2}x(n, \xi) \end{cases} \quad (1.11)$$

We have proved:

**Proposition I.1** - *On the variables  $x(n,k)$ , the fact that the barrier is shifted one step higher leads simply to a new intermediate equation in the transition equations.*

This is quite important in practice, because it means that the theory developed in Part II will apply, despite the changes of position for the barrier. We have simply to take into account the fact that the matrix  $M$  will increase by one dimension at the transition between two periods and the corresponding eigenvalues will change accordingly.

We note here that this result applies to any transition, where the barrier is shifted one step up, and does not depend on the particular function (here  $\pm\sqrt{n}$ ).

## 2. Changes in the eigenvalues and in the eigenvectors

We now work constantly on the variables  $X_n = x(n,i)$ . In dimension  $\xi$ , we know (see Part II) that the eigenvalues are of the form:

$$\lambda_j = \frac{1 + \cos(\vartheta_j)}{2}, \quad \vartheta_j = \frac{2j-1}{2\xi+1}\pi, \quad j = 1, \dots, \xi,$$

and the same expressions will remain in dimension  $\xi+1$ , with  $\xi$  replaced by  $\xi+1$ . In dimension  $\xi$ , the eigenvectors were:

$$V_{\xi,j} = \left( \sin(\xi\vartheta_j), \dots, \sin(\vartheta_j) \right), \quad j = 1, \dots, \xi$$

and in dimension  $\xi+1$ , they will be:

$$V_{\xi+1,j} = \left( \sin((\xi+1)\vartheta_j), \sin(\xi\vartheta_j), \dots, \sin(\vartheta_j) \right), \quad j = 1, \dots, \xi+1.$$

The natural embedding from dimension  $\xi$  to dimension  $\xi+1$  (simply adding a zero as the last coordinate) preserves the eigenvectors. Indeed, we have:

**Proposition I.2.1.** - *The image of the vector  $V_{\xi,j} = \left( \sin(\xi\vartheta_j), \dots, \sin(\vartheta_j), 0 \right)$ , with  $\vartheta_j = \frac{2j-1}{2\xi+1}\pi$ ,*

*by the matrix  $M_{\xi+1}$  is the vector:*

$$M_{\xi+1}V_{\xi,j} = \left( \lambda_{\xi,j} \sin(\xi\vartheta_j), \dots, \lambda_{\xi,j} \sin(\vartheta_j), \frac{1}{4} \sin(\vartheta_j) \right)$$

and we have  $|M_{\xi+1}V_{\xi,j}|_1 = |V_{\xi,j}|_1$ .

### Proof of Proposition I.2.1

This is obvious for the first coordinates, so we have to look only at the last 3. The image of 0 (last coordinate) is  $\frac{1}{4}\sin(\vartheta_j)$ . The image of  $\sin(2\vartheta_j)$  is  $\lambda_{\xi,j}\sin(2\vartheta_j)$ . The image of  $\sin(\vartheta_j)$  is by definition  $\frac{1}{4}\sin(2\vartheta_j) + \frac{1}{2}\sin(\vartheta_j) = \sin(\vartheta_j)\left(\frac{1+\cos(\vartheta_j)}{2}\right) = \lambda_{\xi,j}\sin(\vartheta_j)$ . The assertion on the  $l_1$  norm is obvious, since there is no loss of energy in the transition.

This proves Proposition I.2.1.

The matrix  $M$ , at any stage, operates only on three coordinates (and only on 2, at the first and last coordinates). So, if these three coordinates are those of an eigenvector of a previous situation, the result is a multiplication by the corresponding eigenvalue.

### 3. Energy transition

Assume we have  $l_1$  times the value  $\xi_1$  for the barrier. After  $l_1$  steps, the vector of energies will be  $X_{l_1}$ , with  $|X_{l_1}|_2 \leq a(l_1, \xi_1)|X_0|_2$ , where  $a(l_1, \xi_1)$  is the attenuation of quadratic energy during the episode of length  $l_1$  where the barrier is at  $\xi_1$ . We have  $a(l_1, \xi_1) \leq \lambda_1$ , where  $\lambda_1$  is the largest eigenvalue of the matrix  $M$  with size  $\xi_1$ .

Assume that, after the first  $l_1$  steps, we have  $l_2$  times the value  $\xi_2$  for the barrier. After  $l_1 + l_2$  steps, the vector of energies will be  $X_{l_1+l_2}$  with  $|X_{l_1+l_2}|_2 \leq a(l_2, \xi_2)|X_{l_1}|_2$ , where  $a(l_2, \xi_2)$  is the attenuation of quadratic energy during the episode of length  $l_2$  where the barrier is at  $\xi_2$ , and  $a(l_2, \xi_2) \leq \lambda_2$ , where  $\lambda_2$  is the largest eigenvalue of the matrix  $M$  with size  $\xi_2$ . Therefore:

$$|X_{l_1+l_2}|_2 \leq a(l_1, \xi_1)a(l_2, \xi_2)|X_0|_2$$

Let us now consider a sequence of periods of respective lengths  $l_1, \dots, l_N$ . We get, at the end:

$$|X_{l_1+\dots+l_N}|_2 \leq a(l_1, \xi_1) \cdots a(l_N, \xi_N)|X_0|_2 \tag{I.3.1}$$

The quadratic energy will tend to 0, when  $N \rightarrow +\infty$ , as soon as  $a(l_1, \xi_1) \cdots a(l_N, \xi_N) \rightarrow 0$ . This will be satisfied as soon as:

$$S_N = \sum_{n=1}^N l_n \text{Log}(\lambda_n) \rightarrow -\infty \text{ when } N \rightarrow +\infty.$$

### 4. Estimates for a square root barrier

**Theorem I.4.1** - Consider the symmetric barrier  $b(x) = \pm\sqrt{x}$ . During each period, we have the estimates:

$$\text{Quadratic energy : } E_2\left(W_{2(4N+4)^2+2}\right) \leq \frac{1}{2}\sqrt{\frac{3}{2}}N^{-\frac{\pi^2}{2}}$$

$$\text{Total energy : } E_1\left(W_{2(4N+4)^2+2}\right) \leq \frac{1}{2}\sqrt{\frac{3}{2}}N^{-\frac{\pi^2}{2}+\frac{1}{2}}$$

**Proof of Theorem I.4.1**

The duration of the  $n^{\text{th}}$  period is  $l_n = 8n + 4$  with  $\xi_n = n$ ; during this period, we have seen the estimate (Lemma V.10.1, Part II), for the largest eigenvalue:

$$\lambda_n \approx 1 - \frac{\pi^2}{16n^2}$$

which gives, using the inequality  $\text{Log}(1-x) \leq -x$ ,  $0 \leq x \leq 1$ :

$$S_N = \sum_{n=1}^N l_n \text{Log}(\lambda_n) \approx \sum_{n=1}^N (8n+4) \text{Log}\left(1 - \frac{\pi^2}{16n^2}\right) \leq -\sum_{n=1}^N (8n+4) \frac{\pi^2}{16n^2} \approx -\frac{\pi^2}{2} \sum_{n=1}^N \frac{1}{n} \approx -\frac{\pi^2}{2} \text{Log}(N)$$

So this series diverges. At the end of the  $N^{\text{th}}$  period, we have:

$$\left|X_{(4N+4)^2}\right|_2 \leq \left|X_0\right|_2 \exp(S_N) = \frac{1}{2}N^{-\frac{\pi^2}{2}}.$$

At the end of the  $N^{\text{th}}$  period, the barrier is at  $2N + 1$  and we have  $N$  points on the corresponding vertical. Therefore:

$$\left|X_{(4N+4)^2}\right|_1 \leq \frac{1}{2}N^{-\frac{\pi^2}{2}+\frac{1}{2}}$$

This quantity tends to 0 when  $N$  increases. In order to convert these estimates in terms of energy, we use the estimates (V.5.2) and (V.5.3) in Part II, that is:

$$E_1(W_{2n+2}) \leq \frac{3}{2}|X_n|_1 ; E_2(W_{2n+2}) \leq \sqrt{\frac{3}{2}}|X_n|_2$$

This proves Theorem I.4.1. The proof is rather simple here, because we may work directly with the  $l_2$  norm, and convert the result at the end to the  $l_1$  norm. But such a rough argument does not work for other barriers, such as for instance  $b(n) = \pm 8\sqrt{n}$ ; we now treat this case.

## II. The case of the barrier $b(n) = \pm c\sqrt{n}$

In order to fix ideas, we will take the case of  $b(n) = \pm 8\sqrt{n}$ . As we did earlier, we work on the variable  $X_n$  rather than on the energy directly.

### 1. Transition times

Let  $t_n = \left\lceil \left(\frac{n}{8}\right)^2 \right\rceil$ . On the interval  $[t_n, t_{n+1}]$ , the barrier takes the value  $b_n = n$ ; the duration of this period is  $l_n \approx \left(\frac{n+1}{8}\right)^2 - \left(\frac{n}{8}\right)^2 = \frac{n}{32}$ . On the original setting, this corresponds to a barrier at  $2n+1$  on the interval  $[2t_n, 2t_{n+1}]$ . From  $n=0$  to  $n=64$ , the value of the barrier is 64 and this barrier is useless. On the interval from 64 to  $\left\lceil \left(\frac{65}{8}\right)^2 \right\rceil = 66$  the value of the barrier is 64, on  $\left[\left(\frac{65}{8}\right)^2\right] \rightarrow \left[\left(\frac{66}{8}\right)^2\right]$ , that is  $66 \rightarrow 68$ , the barrier is 65, and so on.

### 2. The previous estimates are insufficient

Let us first observe that the simple proof we saw for the barrier  $\sqrt{x}$  does not work here. Indeed, with the previous notation  $\lambda_n \approx 1 - \frac{\pi^2}{16n^2}$   $l_n = \frac{n}{32}$ ,

$$S_N = \sum_{n=1}^N l_n \text{Log}(\lambda_n) \approx \sum_{n=1}^N \frac{n}{32} \text{Log}\left(1 - \frac{\pi^2}{16n^2}\right) \leq -\frac{\pi^2}{512} \sum_{n=1}^N \frac{1}{n} \approx -\frac{\pi^2}{512} \text{Log}(N).$$

At the end of the  $N^{\text{th}}$  step, we have:

$$\left| X_{\frac{N^2}{64}} \right|_2 \leq |X_0|_2 \exp(S_N) = \frac{1}{2} N^{-\frac{\pi^2}{512}}$$

This tends to 0, but very slowly, and for the energy we get the estimate:

$$\left| X_{(4N+4)^2} \right|_1 \leq \frac{1}{2} N^{-\frac{\pi^2+1}{512}}$$

which is useless. In the previous case, we had a sharp estimate for the  $l_2$ -norm, which proved to be sufficient for the  $l_1$ -norm. But this is not the case anymore for other barriers.



### 3. Initial step

We have the energy 1 at the origin (time 0). We first introduce the barrier at altitude  $\xi_1 = 64$ ,  $n_1 = 64$ . At distance  $n_1$ , the energy 1 at O becomes  $X_1$  with components:

$$x(n_1, i) = \frac{1}{2^{2n_1}} \binom{2n_1 + 1}{n_1 + i} \quad (\text{II.3.1})$$

The total amount of energy is still 1: there is no loss during the first  $n_1$  steps. Let  $M$  be the operator reflecting the propagation of energy. It operates first on a space of dimension  $\xi_1$ , then  $\xi_1 + 1$ , and so on.

The vector  $X_1$  does not have a satisfactory shape, in the sense that we have no information at all on the iterates  $M^n X_1$ ; we have such an information only for the eigenvectors of the matrix  $M$  (and these eigenvectors depend on the dimension, of course). Therefore, we want to replace  $X_1$  by the vector  $V_{1,1}$ , first eigenvector of the matrix  $M$  in dimension  $n_1$ . We know that:

$$V_{1,1} = \frac{2(\sin(\xi_1 \vartheta_{1,1}), \dots, \sin(\vartheta_{1,1}))}{\tan(\xi_1 \vartheta_{1,1})} \quad (\text{II.3.2})$$

with  $\xi_1 = n_1 = 64$ ,  $\vartheta_{1,1} = \frac{\pi}{2\xi_1 + 1}$ . This vector is normalized in  $l_1$  norm: it has positive coefficients,

with sum equal to 1. The replacement of the vector  $X_1$  by  $V_{1,1}$  is done, using the following Lemma:

**Lemma II.3.1.** - For every  $n$ ,  $|M^n X_1|_1 \leq c |M^n V_{n_1,1}|_1$  with  $c = \frac{\binom{2n_1 + 1}{n_1 + i}}{2^{2n_1} \cos(\vartheta_{1,1})}$ .

#### Proof of Lemma II.3.1

The first coefficient of  $X_1$  is  $x(n_1, 1) = \frac{1}{2^{2n_1+1}} \binom{2n_1 + 1}{n_1 + 1}$ . We have  $X_1(i) \leq c V_{n_1,1}(i)$  for every  $i = 1, \dots, n_1$ , with this choice of  $c$ . The operator  $M$  has positive coefficients, so it respects the order: during each period, we will have  $|M^n X_1|_1 \leq c |M^n V_{n_1,1}|_1$ , because the  $l_1$  is simply the sum of all coefficients (the coefficients are positive). This proves Lemma II.3.1.

### 4. Study of the first transition

We are now with an eigenvector,  $V_{1,1}$ , of the matrix  $M$  in dimension  $n_1$ . Unfortunately, this vector, when we pass to dimension  $n_1 + 1$ , does not become an eigenvector of the next matrix. More precisely (without normalisation):

$$V_{1,1} = (\sin(n_1 \mathcal{G}_{1,1}), \dots, \sin(\mathcal{G}_{1,1})) \quad (n_1 \text{ coordinates}),$$

becomes, after embedding :

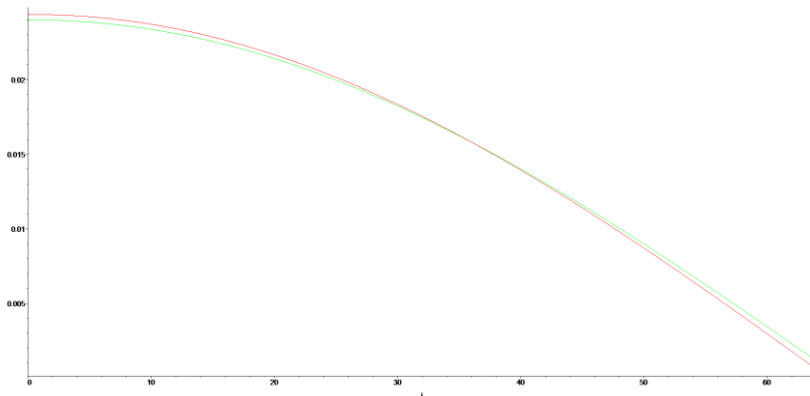
$$V'_{1,1} = (\sin(n_1 \mathcal{G}_{1,1}), \dots, \sin(\mathcal{G}_{1,1}), 0) \quad (n_1 + 1 \text{ coordinates})$$

where as the first eigenvector of the new matrix is :

$$V_{2,1} = (\sin((n_1 + 1) \mathcal{G}_{2,1}), \dots, \sin(\mathcal{G}_{2,1})) \quad (n_1 + 1 \text{ coordinates})$$

with  $\mathcal{G}_{2,1} = \frac{\pi}{2\xi_1 + 3}$ .

There is no simple connection between  $V'_{1,1}$  and  $V_{2,1}$  : both are very close, as the following picture shows ( $V'_{1,1}$  is in red and  $V_{2,1}$  is in green), for  $\xi = 64$  :



We cannot simply say that we replace  $V'_{1,1}$  by  $V_{2,1}$ , because  $V_{2,1}$  is slightly closer to the barrier. We may compute the loss in this replacement, but the sum of such losses, over all transitions, is infinite, so such an approach, keeping only one eigenvector at each step, must be abandoned: we have to keep all eigenvectors.

First, what we do is to expand  $V'_{1,1}$  on the basis of eigenvectors of the matrix  $M_2$  in dimension  $\xi_2$ . We describe the general stage of the process.

### 5. General transition

We go from dimension  $\xi - 1$  to dimension  $\xi$ . In order to simplify the notation, we denote by  $V_i$  the eigenvectors in dimension  $\xi - 1$  ( $i = 1, \dots, \xi - 1$ ) and by  $W_j$  the vectors in dimension  $\xi$ ,

$j = 1, \dots, \xi$ . Also in order to simplify the notation, we identify  $V_i$  and  $V_i'$  (one more coordinate, equal to 0).

We have the decomposition on the basis of eigenvectors in dimension  $\xi$  :

$$V_i = \sum_{j=1}^{\xi} \alpha_{i,j} W_j = \sum_{j=1}^{\xi} \frac{\langle V_i, W_j \rangle}{\|W_j\|_2} W_j = \frac{1}{\frac{\xi}{2} + \frac{1}{4}} \sum_{j=1}^{\xi} \langle V_i, W_j \rangle W_j \quad (\text{II.5.1})$$

In this decomposition, some coefficients are negative, so they do not represent an "energy" in the previous sense of the word. Still, the above decomposition is valid, and represents an algebraic (Hilbert space) decomposition, in which the total energy is the sum of all components of all vectors. There is a conceptual difficulty here, because we have to leave the framework of "ordinary energy" (every component is positive), and to adopt the framework of "algebraic energy" (some components may be negative).

We introduce a notation for the sum of components of a vector :

$$s(V_i) = \sum_{l=1}^{\xi-1} V_i(l)$$

and the same for  $s(W_j)$ . With this notation, the total energy at the beginning of the  $\xi^{\text{th}}$  period (just after the embedding  $\xi - 1 \rightarrow \xi$ ) is :

$$E_{\xi} = \sum_{i=1}^{\xi-1} s(V_i) \quad (\text{II.5.2})$$

which may be written:

$$\begin{aligned} E_{\xi} &= \sum_{i=1}^{\xi-1} s(V_i) = \frac{1}{\frac{\xi}{2} + \frac{1}{4}} \sum_{i=1}^{\xi-1} s\left(\sum_{j=1}^{\xi} \langle V_i, W_j \rangle W_j\right) \\ &= \frac{1}{\frac{\xi}{2} + \frac{1}{4}} \sum_{i=1}^{\xi-1} \sum_{j=1}^{\xi} \langle V_i, W_j \rangle s(W_j) = \frac{1}{\frac{\xi}{2} + \frac{1}{4}} \left\langle \sum_{i=1}^{\xi-1} V_i, \sum_{j=1}^{\xi} s(W_j) W_j \right\rangle \end{aligned} \quad (\text{II.5.3})$$

Let  $l = l_{\xi}$  be the duration of the  $\xi^{\text{th}}$  period, and let  $\lambda_j$ , instead of  $\lambda_{\xi,j}$  ( $j = 1, \dots, \xi$ ) be the eigenvalues during this period. The total energy at the end of the  $\xi^{\text{th}}$  period is:

$$F_{\xi} = \frac{1}{\frac{\xi}{2} + \frac{1}{4}} \sum_{i=1}^{\xi-1} s\left(\sum_{j=1}^{\xi} \langle V_i, W_j \rangle \lambda_j W_j\right) \quad (\text{II.5.4})$$

Indeed, during this period, each  $W_j$  is transformed into  $\lambda_j^l W_j$ . So we have:

$$F_\xi = \frac{1}{\frac{\xi}{2} + \frac{1}{4}} \sum_{i=1}^{\xi-1} s \left( \sum_{j=1}^{\xi} \langle V_i, W_j \rangle \lambda_j^l W_j \right) = \frac{1}{\frac{\xi}{2} + \frac{1}{4}} \sum_{i=1}^{\xi-1} \sum_{j=1}^{\xi} \langle V_i, W_j \rangle \lambda_j^l s(W_j) = \frac{1}{\frac{\xi}{2} + \frac{1}{4}} \left\langle \sum_{i=1}^{\xi-1} V_i, \sum_{j=1}^{\xi} s(W_j) \lambda_j^l W_j \right\rangle$$

We want to compare  $F_\xi$  and  $E_\xi$ . The following Theorem answers our question and is the key point in our approach.

## 6. Sharp transition estimates

**Theorem II.6.1.** - The total energy at the end of the  $\xi^{th}$  period, denoted by  $F_\xi$ , and the total energy at the beginning of the  $\xi^{th}$  period, denoted by  $E_\xi$ , are linked by the inequality:

$$F_\xi \leq \lambda_1^l E_\xi$$

where  $l = l_\xi$  is the duration of the  $\xi^{th}$  period, and  $\lambda_1 = \lambda_{\xi,1}$  is the largest eigenvalue during this period.

### Proof of Theorem II.6.1

The statement is equivalent to:

$$\left\langle \sum_{i=1}^{\xi-1} V_i, \sum_{j=1}^{\xi} s(W_j) \lambda_j^l W_j \right\rangle \leq \lambda_1^l \left\langle \sum_{i=1}^{\xi-1} V_i, \sum_{j=1}^{\xi} s(W_j) W_j \right\rangle \quad (\text{II.6.1})$$

We will study all terms separately.

**Lemma II.6.2.** - All components of the vector  $\sum_{i=1}^{\xi-1} V_i$  are positive.

### Proof of Lemma II.6.2

In dimension  $\xi - 1$ , the components of the  $i^{th}$  vecteur,  $V_i$ , are  $\sin((\xi - 1)\mathcal{G}_i), \dots, \sin(\mathcal{G}_i)$ , with

$\mathcal{G}_i = \mathcal{G}_{\xi-1,i} = \frac{2i-1}{2\xi-1} \pi$ . The  $k^{th}$  component (starting from the right) of  $\sum_{i=1}^{\xi-1} V_i$  is therefore:

$$C_k = \sum_{i=1}^{\xi-1} \sin \left( \frac{k(2i-1)}{2\xi-1} \pi \right).$$

We use the identity:

$$\sum_{i=1}^{\xi-1} \sin((2i-1)\alpha) = \frac{2 \sin(\alpha)(1 - \cos^2((\xi-1)\alpha))}{1 - \cos(2\alpha)}$$

in which we take  $\alpha = \frac{k\pi}{2\xi-1}$ ; since  $1 \leq k \leq \pi$ ,  $0 < \alpha < \frac{\pi}{2}$ , we have  $\sin(\alpha) > 0$ , and this proves

Lemma II.6.2.

We now compute the sum of the components of each eigenvector. By definition, it is:

$$s(W_j) = \sum_{k=1}^{\xi} \sin\left(\frac{k(2j-1)}{2\xi+1} \pi\right)$$

**Lemma II.6.3.** - For each  $j = 1, \dots, \xi$ , we have:

$$s(W_j) = \frac{\sin \vartheta_j}{2(1 - \cos \vartheta_j)} = \frac{\tan(\xi \vartheta_j)}{2}, \text{ with } \vartheta_j = \vartheta_{\xi, j} = \frac{2j-1}{2\xi+1} \pi.$$

### Proof of Lemma II.6.3

We have the identity, for any  $\vartheta$  and any  $\xi$ :

$$\sum_{k=1}^{\xi} \sin(k\vartheta) = \frac{\sin(\xi\vartheta) - \sin((\xi+1)\vartheta) + \sin(\vartheta)}{2(1 - \cos(\vartheta))} \quad (\text{II.6.2})$$

The numerator is:

$$\text{Num} = \sin\left(\xi \frac{(2j-1)\pi}{2\xi+1}\right) - \sin\left((\xi+1) \frac{(2j-1)\pi}{2\xi+1}\right) + \sin\left(\frac{(2j-1)\pi}{2\xi+1}\right)$$

But in fact:

$$\sin\left(\xi \frac{(2j-1)\pi}{2\xi+1}\right) = \sin\left((\xi+1) \frac{(2j-1)\pi}{2\xi+1}\right) \quad (\text{II.6.3})$$

Indeed, this follows from the equality:

$$\xi \frac{(2j-1)\pi}{2\xi+1} = \pi - (\xi+1) \frac{(2j-1)\pi}{2\xi+1} + 2k\pi \quad (\text{II.6.4})$$

with  $k = 2j - 2$ .

So we get simply:

$$Num = \sin\left(\frac{(2j-1)\pi}{2\xi+1}\right)$$

We have finally:

$$s(W_j) = \frac{\sin\left(\frac{(2j-1)\pi}{2\xi+1}\right)}{2\left(1 - \cos\left(\frac{(2j-1)\pi}{2\xi+1}\right)\right)} = \frac{1}{2} \frac{1}{\tan\frac{2j-1}{2\xi+1} \frac{\pi}{2}} = \frac{1}{2} \frac{1}{\tan\frac{\mathcal{G}_j}{2}}$$

which proves Lemma II.6.3.

It follows from Lemma II.6.3 that, since  $j \leq \xi$ ,  $2j-1 \leq 2\xi+1$ , the term  $s(W_j)$  is positive. We

now study the vector  $T = \sum_{j=1}^{\xi} s(W_j)W_j$ . The  $k^{th}$  component (starting from the right) is, using

Lemma II.6.3:

$$T_k = \sum_{j=1}^{\xi} s(W_j) \sin(k\mathcal{G}_j) = \sum_{j=1}^{\xi} \frac{\sin(k\mathcal{G}_j)}{2 \tan\frac{\mathcal{G}_j}{2}} = \sum_{j=1}^{\xi} \frac{\sin\left(k \frac{2j-1}{2\xi+1} \pi\right)}{2 \tan\frac{2j-1}{2\xi+1} \frac{\pi}{2}}$$

We observe that the coefficient  $\tan\left(\frac{\mathcal{G}_j}{2}\right)$  is positive and increasing with  $j$ .

**Proposition II.6.4.** - For each  $k$ , we have the identity :

$$T_k = \frac{2^\xi - 1}{4} > 0$$

**Proof of Proposition II.6.4**

Set  $\varphi_j = \frac{2j-1}{2\xi+1} \frac{\pi}{2} = \frac{\mathcal{G}_j}{2}$ . Then :

$$T_k = \frac{1}{2} \sum_{j=1}^{\xi} \frac{\sin(2k\varphi_j)}{\tan(\varphi_j)}$$

We use the identities:

$$\frac{\sin(2kx)}{\tan x} = \frac{1}{2} \left( \frac{\sin((2k+1)x)}{\sin x} + \frac{\sin((2k-1)x)}{\sin x} \right)$$

$$\frac{\sin((2k+1)x)}{\sin x} - \frac{\sin((2k-1)x)}{\sin x} = 4 \cos^2(kx) - 2 = 2(\cos^2(kx) - 1) = 2 \cos(2kx)$$

They give :

$$T_k = \frac{1}{4} \left( \sum_{j=1}^{\xi} \frac{\sin((2k+1)\varphi_j)}{\sin \varphi_j} + \sum_{j=1}^{\xi} \frac{\sin((2k-1)\varphi_j)}{\sin \varphi_j} \right)$$

Set  $W_k = \sum_{j=1}^{\xi} \frac{\sin((2k+1)\varphi_j)}{\sin \varphi_j}$ . Then  $T_k = \frac{1}{4}(W_k + W_{k-1})$  and:

$$\frac{\sin((2k+1)\varphi_j)}{\sin \varphi_j} - \frac{\sin((2k-1)\varphi_j)}{\sin \varphi_j} = 2 \cos(2k\varphi_j) = 2 \cos\left(k \frac{2j-1}{2\xi+1} \pi\right).$$

We use the identity  $\sum_{j=1}^{\xi} \cos((2j-1)\vartheta) = \frac{\sin(2\xi\vartheta)}{2 \sin(\vartheta)}$ , which gives  $W_k - W_{k-1} = \frac{\sin\left(2\xi \frac{k\pi}{2\xi+1}\right)}{\sin\left(\frac{k\pi}{2\xi+1}\right)}$ .

But  $\sin\left(2\xi \frac{k\pi}{2\xi+1}\right) = (-1)^{k-1} \sin\left(\frac{k\pi}{2\xi+1}\right)$ , and therefore  $W_k - W_{k-1} = (-1)^{k-1}$ . Since  $W_0 = \xi$ , this gives  $W_{2k} = \xi$ ,  $W_{2k-1} = \xi - 1$ ,  $T_k = \frac{1}{4}(W_k + W_{k-1}) = \frac{2\xi - 1}{4}$ , which proves Proposition II.6.4.

Let us now finish the proof of Theorem II.6.1. We want to show that:

$$\left\langle \sum_{i=1}^{\xi-1} V_i, \sum_{j=1}^{\xi} s(W_j) \lambda_j' W_j \right\rangle \leq \lambda_1' \left\langle \sum_{i=1}^{\xi-1} V_i, \sum_{j=1}^{\xi} s(W_j) W_j \right\rangle \quad (\text{II.6.5})$$

The scalar product on the left hand-side is, by definition:

$$\begin{aligned} \left\langle \sum_{i=1}^{\xi-1} V_i, \sum_{j=1}^{\xi} s(W_j) \lambda_j' W_j \right\rangle &= \sum_{l=1}^{\xi} \sum_{i=1}^{\xi-1} V_i(l) \sum_{j=1}^{\xi} s(W_j) \lambda_j' W_j(l) \\ &\leq \lambda_1' \sum_{l=1}^{\xi} \sum_{i=1}^{\xi-1} V_i(l) \sum_{j=1}^{\xi} s(W_j) W_j(l) \\ &= \lambda_1' \left\langle \sum_{i=1}^{\xi-1} V_i, \sum_{j=1}^{\xi} s(W_j) W_j \right\rangle \end{aligned}$$

since all terms are positive. This finishes the proof of Theorem II.6.1.

## 7. Combining several periods

From Theorem II.6.1 follows that the loss of total energy, during a period of altitude  $\xi$  for the barrier, and duration  $l$ , is  $\leq \lambda_1^l$ , where  $\lambda_1$  is the largest eigenvalue of the matrix  $M$  in dimension  $\xi$ .

Now, assume that the barrier is  $f(n) = \pm 8\sqrt{n}$ . As we saw, the duration of the  $n^{\text{th}}$  period is  $l_n = \frac{n}{32}$ . The first eigenvalue,  $\lambda_{n,1}$ , satisfies  $\lambda_n = \cos^2\left(\frac{1}{2n+1} \frac{\pi}{2}\right)$ , and therefore, since  $\text{Log}(x) \leq x-1$  when  $x > 0$  :

$$\text{Log}(\lambda_n^{l_n}) = l_n \text{Log}\left(\cos^2\left(\frac{1}{2n+1} \frac{\pi}{2}\right)\right) \leq -l_n \left(1 - \cos^2\left(\frac{1}{2n+1} \frac{\pi}{2}\right)\right) = -l_n \sin^2\left(\frac{1}{2n+1} \frac{\pi}{2}\right) \approx -l_n \frac{\pi^2}{16n^2}$$

We have, with  $\delta = \frac{\pi^2}{512}$ ,  $\sum_{n=1}^N \text{Log}(\lambda_n^{l_n}) \approx -\delta \sum_{n=1}^N \frac{1}{n} \approx -\delta \text{Log}(N)$  and therefore  $\prod_{n=1}^N \lambda_n^{l_n} \approx \frac{1}{N^\delta} \rightarrow 0$  when  $N \rightarrow +\infty$ , which proves that the energy left at the  $N^{\text{th}}$  period tends to zero.

## 8. Quantitative statements

Let us summarize the results we obtained.

**Theorem II.8.1.** - Let  $b(x) = \pm 8\sqrt{x}$  be the barrier. At the end of the period  $t_n = \left[\left(\frac{n}{8}\right)^2\right]$ , the total energy satisfies:

$$E_{t_n} \leq \frac{3}{2} n^{-\frac{\pi^2}{512}}$$

### Proof of Theorem II.8.1

The factor  $3/2$  comes (see the estimates (V.5.2) and (V.5.3) in Part II) from the conversion of the variable  $X$  to the energy.

Similar results hold for any barrier of the form  $f(x) = \pm c\sqrt{x}$ . The statement is:

**Theorem II.8.2.** - Let  $b(x) = \pm c\sqrt{x}$  be the barrier, with  $c \geq 1$ . At the end of the period  $t_n = \left[\left(\frac{n}{c}\right)^2\right]$ , the total energy satisfies:



$$E_{t_n} \leq \frac{3}{2} n^{-\frac{\pi^2}{8c^2}}$$

In other words, at any instant  $N$ , the energy satisfies:

$$E_N \leq \frac{3}{2} c^{-\frac{\pi^2}{8c^2}} N^{-\frac{\pi^2}{16c^2}}$$

### III. Study of the barrier $b(x) = \sqrt{x \text{Log}(x)}$

#### 1. Transition times

In this case,  $t_n$  is defined by  $b(t_n) = n$  ;  $t_n \text{Log}(t_n) = n^2$  ;  $t_n = b^{-1}(n)$ . We denote by  $\beta$  the inverse function of  $b$  (which is well-defined, since  $b$  is increasing). The duration of the  $n^{\text{th}}$  period is, with the notation of the previous paragraphs:

$$l_n = t_{n+1} - t_n = \beta(n+1) - \beta(n) \approx \beta'(n)$$

#### 2. Decrease of energy

For this barrier, we prove:

**Theorem III.2.1.** - *The energy left after the  $n^{\text{th}}$  period tends to 0 when  $n \rightarrow +\infty$ . More precisely, at the end of the  $n^{\text{th}}$  period, the total energy satisfies:*

$$E_{t_n} \leq \frac{3}{2} (\text{Log}(N))^{-\frac{\pi^2}{16}}$$

#### Proof of Theorem III.2.1

Let us first see the qualitative version. We know (see II.7 above) that  $\text{Log}(\lambda_n^{l_n}) \leq -\frac{\pi^2 l_n}{16n^2}$ , so we

have to show that  $\sum \frac{l_n}{n^2} = +\infty$ .

We have  $b'(x) = \frac{1}{2} \frac{1 + \text{Log}(x)}{\sqrt{x \text{Log}(x)}}$ ,  $\beta'(n) = \frac{1}{b'(t_n)} = \frac{2n}{1 + \text{Log}(t_n)}$ . Therefore  $\frac{l_n}{n^2} \approx \frac{2}{n(1 + \text{Log}(t_n))}$ .

We have  $t_n < n^2$ , which implies  $\text{Log}(t_n) < 2\text{Log}(n)$ , and finally:

$\frac{2}{n(1 + \text{Log}(t_n))} > \frac{2}{n(1 + 2\text{Log}(n))} \approx \frac{1}{n \text{Log}(n)}$ , general term of a divergent series.

More quantitatively, the energy left at the end of the  $N^{\text{th}}$  period (in the variable  $X$ ) satisfies:

$$E_N = \exp\left(\sum_{n=2}^N \text{Log}(\lambda_n^{l_n})\right) \leq \exp\left(-\sum_{n=2}^N \frac{\pi^2 l_n}{16n^2}\right) \leq \exp\left(-\frac{\pi^2}{16} \sum_{n=2}^N \frac{1}{n \text{Log}(n)}\right) \\ \approx \exp\left(-\frac{\pi^2}{16} \text{Log}(\text{Log}(N))\right) = (\text{Log}(N))^{\frac{-\pi^2}{16}}$$

which proves Theorem III.2.1.

## IV. Energy profile

In all these cases, the energy profile during the  $n^{\text{th}}$  period is approximately proportional to the first eigenvector, and this approximation is more and more accurate when  $n \rightarrow +\infty$ . This means that, during the  $n^{\text{th}}$  period, the energy profile (in the  $X$  variable) is proportional to  $V_n = (\sin(n\vartheta), \dots, \sin(\vartheta))$ , with  $\vartheta = \frac{\pi}{2n+1}$ .

## V. Comparison with Khinchine's curves

Let  $b(x)$  be the barrier ; here  $b(x) = \sqrt{x \text{Log}(x)}$ . Let  $N$  be any instant. The fact that the energy left after the instant  $N$  tends to 0 when  $N \rightarrow +\infty$  may at first sight look contradictory with Khinchine's result, according to which  $k(x) = \sqrt{2x \text{Log}(\text{Log}(x))}$  is a security curve, since our barrier  $b$  is above Khinchine's barrier  $k$ . But in fact, there is no contradiction. Let us explain the situation more in detail.

We distinguish between 4 types of paths:

- $A$  : all paths which never touch  $b$  nor  $-b$  before the instant  $N$  ;
- $B^+$  : all paths which touch  $b$  but do not touch  $-b$  before the instant  $N$  ;
- $B^-$  : all paths which touch  $-b$  but do not touch  $b$  before the instant  $N$  ;
- $C$  : all paths which touch both  $b$  and  $-b$  before the instant  $N$ .

Of course, these four sets are disjoint, and their union represents all possible paths. What we saw, for  $b(x) = \sqrt{x \text{Log}(x)}$ , is that  $P(B^+ \cup B^- \cup C) \rightarrow 1$  when  $N \rightarrow +\infty$ . In other words, it becomes more and more unlikely that a path never touches the barrier or its opposite.

This barrier is above Khinchine's security curve, which is  $k(x) = \sqrt{2x \text{Log}(\text{Log}(x))}$ , which means that the probability to touch  $b$  after the time  $N$  tends to 0 when  $N \rightarrow +\infty$ . In other words, almost every path returns near Khinchine's curve  $k$  infinitely many times, but this is not so for the barrier  $b$ . This is not contradictory with our result. It means simply that, for instance  $P(B^+) \rightarrow 0.1$ ,  $P(B^-) \rightarrow 0.1$ ,  $P(C) \rightarrow 0.8$  when  $N \rightarrow +\infty$ . So the total probability of

hitting  $\pm b$  tends to 1 when  $N \rightarrow +\infty$  (our result), but the probability to hit either  $b$  or  $-b$  after time  $N$  tends to 0 when  $N \rightarrow +\infty$ .

## VI. Proof of the main Theorem, first statement

We have given all ingredients for the proof, except that we have to convert a statement given in terms of integral into a statement given in terms of a series. This is done by means of the following Proposition.

**Proposition VI.1.** - *Let  $b(x)$  be a barrier, that is a positive, differentiable, strictly increasing function, tending to  $+\infty$  when  $x \rightarrow +\infty$ . Let, for each  $n$ ,  $[t_n, t_{n+1}[$  be the interval on which the discretization of  $b$  takes the value  $n$  (this is the  $n^{\text{th}}$  period), and let  $l_n = t_{n+1} - t_n$  be the duration of this period. Then the integral  $\int_A^{+\infty} \frac{dx}{b^2(x)}$  diverges at infinity if and only if the series  $\sum_{n=1}^{+\infty} \frac{l_n}{n^2}$*

*diverges.*

### Proof of Proposition VI.1

Let  $\beta = b^{-1}$  be the inverse function of the function  $b$  (this inverse exists since  $b$  is strictly increasing). It is also positive, differentiable and strictly increasing. We have, by definition

$t_n = \beta(n)$  and therefore  $l_n = t_{n+1} - t_n = \beta(n+1) - \beta(n) \approx \beta'(n) = \frac{1}{b'(t_n)}$ . So we may write:

$$\sum_{n=1}^{+\infty} \frac{l_n}{n^2} \approx \sum_{n=1}^{+\infty} \frac{1}{n^2 b'(t_n)} \approx \int_1^{+\infty} \frac{1}{y^2 b'(\beta(y))} dy$$

Set  $y = b(x)$ ,  $dy = b'(x)dx$ . The above integral becomes:

$$\int_1^{+\infty} \frac{1}{y^2 b'(\beta(y))} dy = \int_A^{+\infty} \frac{b'(x)}{b^2(x) b'(x)} dx = \int_A^{+\infty} \frac{dx}{b^2(x)},$$

which proves Proposition VI.1. Let us now give the corresponding quantitative statement. We

saw that the energy left at the end of the  $N^{\text{th}}$  period satisfies  $E_N \leq \exp\left(-\frac{\pi^2}{16} \sum_{n=2}^N \frac{l_n}{n^2}\right)$ . But :

$$\sum_{n=1}^N \frac{l_n}{n^2} \approx \int_1^{N+1} \frac{dy}{y^2 b'(\beta(y))} = \int_{\beta(1)}^{\beta(N+1)} \frac{dx}{b^2(x)} = \int_1^{t_{N+1}} \frac{dx}{b^2(x)}$$

and finally:

$$E_N \leq \exp\left(-\frac{\pi^2}{16} \int_1^{t_{N+1}} \frac{dx}{b^2(x)}\right)$$

which proves our claim.

## VII. Proof of the Main Theorem, converse statement

We want to show that if the integral diverges at infinity, the game may continue indefinitely (the remaining energy does not tend to zero). This part is much simpler than the previous one.

First of all, we have seen (§ above) that the convergence of the integral is equivalent to the convergence of the series  $\sum_{n=1}^{+\infty} \frac{l_n}{n^2} < +\infty$ , which corresponds to the energy, assuming it is carried by the first eigenvector only at each period.

The energy left after  $N$  periods will be  $E_{t_N} \geq \exp\left(-\sum_{n=1}^N \frac{l_n}{n^2}\right)$  if we can prove that this energy is larger than the energy carried, during each period, by the first eigenvector of the corresponding matrix.

**Proposition VII.1.** - *We penalize ourselves if, at the end of each period, we replace the first eigenvector of this period by the first eigenvector of the next period, with same normalization.*

### Proof of Proposition VII.1

Let us explain the statement more in detail. Let simply  $V$  be the first eigenvector during the  $n^{\text{th}}$  period. Recall that, when normalized in  $l_1$  norm, we have, with  $\mathcal{G}_1 = \frac{\pi}{2n+1}$ :

$$V = 2 \tan \frac{\mathcal{G}_1}{2} (\sin(n\mathcal{G}_1), \dots, \sin(\mathcal{G}_1)).$$

When we start the  $(n+1)^{\text{st}}$  period, it becomes:

$$V = 2 \tan \frac{\mathcal{G}_1}{2} (\sin(n\mathcal{G}_1), \dots, \sin(\mathcal{G}_1), 0).$$

The first eigenvector of the  $(n+1)^{\text{st}}$  period is, with  $\mathcal{G}_2 = \frac{\pi}{2n+3}$ :

$$W = 2 \tan \frac{\mathcal{G}_2}{2} (\sin((n+1)\mathcal{G}_2), \dots, \sin(\mathcal{G}_2))$$

When we say that we "penalize ourselves", it means that the energy will be likely to disappear more easily with  $W$  than with  $V$ . So, if we perform this replacement at each step and, at the end, get a non-zero energy, it means that the whole game produces a non-zero energy.

In practice, using Part II, Corollary V.2.4, it means that the "tail" of  $W$  contains more energy than the tail of  $V$ ; in simpler terms,  $W$  is globally closer to the barrier. More precisely, for any  $k \leq n+1$ , let us define the tails made of the last  $k$  terms:

$$V_k = 2 \tan \frac{\mathcal{G}_1}{2} (0, \sin(\mathcal{G}_1), \dots, \sin((k-1)\mathcal{G}_1)), \quad W_k = 2 \tan \frac{\mathcal{G}_2}{2} (\sin(\mathcal{G}_2), \dots, \sin(k\mathcal{G}_2))$$

In order to prove the Theorem, all we need to show is :

**Lemma VII.2.** - For any  $k$ ,  $k = 1, \dots, n+1$ ,  $|W_k|_1 \geq |V_k|_1$ .

**Proof of Lemma VII.2**

We use the identity  $\sum_{j=1}^k \sin(j\mathcal{G}) = \frac{\sin(k\mathcal{G}) - \sin((k+1)\mathcal{G}) + \sin(\mathcal{G})}{2(1 - \cos(\mathcal{G}))}$ . We have to show that:

$$\tan\left(\frac{\mathcal{G}_1}{2}\right) \frac{\sin((k-1)\mathcal{G}_1) - \sin(k\mathcal{G}_1) + \sin(\mathcal{G}_1)}{1 - \cos(\mathcal{G}_1)} \leq \tan\left(\frac{\mathcal{G}_2}{2}\right) \frac{\sin(k\mathcal{G}_2) - \sin((k+1)\mathcal{G}_2) + \sin(\mathcal{G}_2)}{1 - \cos(\mathcal{G}_2)} \quad (\text{VII.1})$$

Using the identity  $\frac{\tan \frac{t}{2}}{1 - \cos(t)} = \frac{1}{\sin(t)}$ , (VII.1) is equivalent to:

$$\frac{\sin((k-1)\mathcal{G}_1) - \sin(k\mathcal{G}_1) + \sin(\mathcal{G}_1)}{\sin(\mathcal{G}_1)} \leq \frac{\sin(k\mathcal{G}_2) - \sin((k+1)\mathcal{G}_2) + \sin(\mathcal{G}_2)}{\sin(\mathcal{G}_2)} \quad (\text{VII.2})$$

Or:

$$\frac{\sin((k-1)\mathcal{G}_1) - \sin(k\mathcal{G}_1)}{\sin(\mathcal{G}_1)} \leq \frac{\sin(k\mathcal{G}_2) - \sin((k+1)\mathcal{G}_2)}{\sin(\mathcal{G}_2)} \quad (\text{VII.3})$$

Using the identity  $\sin(p) - \sin(q) = 2 \cos \frac{p+q}{2} \sin\left(\frac{p-q}{2}\right)$ , (VII.3) becomes:

$$\cos((2k-1)\mathcal{G}_1) \geq \cos((2k+1)\mathcal{G}_2) \quad (\text{VII.4})$$

But the angles in (VII.4) are smaller than  $\pi$ , so the cosine is decreasing. Therefore, (VII.4) is equivalent to  $(2k-1)\mathcal{G}_1 \leq (2k+1)\mathcal{G}_2$ , that is  $\frac{(2k-1)\pi}{2n+1} \leq \frac{(2k+1)\pi}{2n+3}$ , which itself is equivalent to

$4k \leq 4n + 2$ , which is satisfied for  $k \leq n$ . For  $k = n + 1$ , the  $l_1$  norms are equal by definition. This proves Lemma VII.2, Proposition VII.1, and finishes the proof of the Main Theorem.

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