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Part V

Khinchin's Law of the Iterated Logarithm:

Quantitative versions

by Bernard Beauzamy

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In this Fifth Part, we first give a quantitative version of Khinchin's law of the iterated logarithm (1924); we then explain the connections and differences with our present work. The comparison between Khinchin's methods and ours lead to the following conclusions:

- Khinchin's methods may handle the case of a single curve, whereas ours may handle only the case of two barriers;

- Quantitative estimates, about the probability to reach a certain curve at a certain time, obtained by means of Khinchin's methods are of probabilistic type, and are much weaker than similar results derived by means of operator theory;

- To say that Khinchin's curve $\varphi(x)=\sqrt{2xLog\left(Log(x)\right)}$ is a "security curve" is incorrect. Take any curve, such as $b(x)=\sqrt{xLog(x)},$ which is above Khinchin's curve, and take two instants $N_1 < N_2$. Then the probability to hit $b(x)$ between these two instants, and thus to go above $k(x)$, is always strictly positive. The statement about "security curve" is only asymptotic.

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I. Khinchin's Law of the Iterated Logarithm: quantitative version

Let us come back to the original setting: the X_n's are independent variables with same law, $P(X_n = \pm 1) = \frac{1}{2}$. We set $S_N = \sum_{n=1}^{N} X_n$ (in other words, we do not consider only even values of the time and, originally, both fortunes are equal).

We introduce Khinchin's curve, or barrier, defined by the equation:

$$\varphi(x) = \sqrt{2x Log(Log(x))},$$

which is a real function, defined for all real $x > e$.

The classical statement of the law of the iterated logarithm is:

$$\textit{almost surely, } \limsup_{n \to +\infty} \frac{S_n}{\varphi(n)} = 1$$

(see https://en.wikipedia.org/wiki/Law_of_the_iterated_logarithm)

The explanation given by Wikipedia, rather obscure, is as follows:
"Thus, although the quantity $\left| \frac{S_n}{\varphi(n)} \right|$ is less than any predefined $\varepsilon > 0$ with probability approaching one, the quantity will nevertheless be greater than ε infinitely often; in fact, the quantity will be visiting the neighborhoods of any point in the interval (-1,1) almost surely."

The difficulty is in the understanding of the words "almost surely", both in theory and in practice.

There is a natural probability on the infinite product $\prod_{n=1}^{\infty} \{-1,1\}^n$, which is simply the product of the elementary probabilities on each layer. With respect to this "global" probability, the words "almost surely" are well-defined. But, for this probability, every elementary path has probability 0, and so does a finite number of paths. Moreover, it is quite hard to obtain quantitative results for this probability, which is well-suited only for probabilistic arguments.

On the contrary, if we stop at time N, we have a precise and intuitive definition of the probability of an event: up to time N, we have 2^N paths, and the probability of any event is : number of paths satisfying the event, divided by 2^N. For instance, the probability of the event $\{\exists n \le N \, ; S_n > \varphi(n)\}$ is perfectly clear: we count the number of paths for which, at some point, the random walk is above the curve, and we divide this number by 2^N.

Since the paths divide into two at each step, an estimate obtained at a given step will remain valid at later stages. For instance, the statement $X_1 = 1$ has the same probability (1/2), no matter whether we consider it at stage 1 or at any later stage.

Let us observe that the statement from Wikipedia may be quite misleading. Indeed, if one reads: "The quantity $\left|\frac{S_n}{\varphi(n)}\right|$ will nevertheless be greater than ε infinitely often", this is true for most curves, and not only for Khinchin's curve. Indeed, we remember (Part I) that the random walk S_n comes back infinitely many times to the x axis, so for instance the value $S_n = 0$ may be expected at time $n = 5000$. But then, consider the situation where S_n increases linearly from this point; it will eventually cross the curve $\varphi(n)$; in fact, this is true for any curve such that $\frac{\varphi(n)}{n} \to 0$ when $n \to +\infty$. We construct this way an infinite number of situations in which S_n exceeds $4\varphi(n)$ or any multiple, as one wishes.

Therefore, we think that, in such statements, a precise definition of the probabilities must be given. This is what we do now. We give Khinchin's results in quantitative settings, which are new, as far as we know.

Theorem 1. Quantitative statement of LIL, 1- *Let* $\varepsilon > 0$ *and* $m \geq 2$. *We set:*

$$\eta(m,\varepsilon) = \frac{2}{\left(Log(1+\varepsilon)\right)^{1+\varepsilon}} \frac{1}{\varepsilon(m-1)^{\varepsilon}}$$

and:

$$B(m,\varepsilon) = \left\{\exists n \geq (1+\varepsilon)^m, S_n > (1+\varepsilon)\varphi(n)\right\}$$

Then, for all m *:*

$$P\left(B(m,\varepsilon)\right) \leq \eta(m,\varepsilon).$$

The set $B(m,\varepsilon)$ is made of the paths which are above the strip $(1+\varepsilon)\varphi$ at least once after the time $(1+\varepsilon)^m$. The Theorem says that the probability of such an event tends to 0 when m increases. In other words, if we fix a width for the strip, that is $\varepsilon > 0$ fixed, it becomes less and less probable to pass above the curve $(1+\varepsilon)\varphi$ when n increases.
Let us take for example $\varepsilon = 1$. Theorem 1 gives the estimate:

$$P\left\{\exists n \geq 2^m, S_n > 2\varphi(n)\right\} \leq \frac{2}{\left(Log(2)\right)^2} \frac{1}{m-1}. \tag{1}$$

For instance, if we want the right-hand side to be ≤ 0.05, we find $m = 85$. Therefore, the probability to have $n \geq 2^{85}$ for which $S_n > 2\varphi(n)$ is < 0.05.

For a better understanding, this statement may be converted into a proportion of paths, as follows:

Fix any $N > 2^{85}$. The proportion of paths, which satisfy $S_n > 2\varphi(n)$ at least in one place, between 2^{85} and N, is smaller than 5%.

Still, it is quite possible that a significant number of paths reach a curve above Khinchin's. Let $b(x)$ be such a curve; an estimate such as:

$$P\left\{\exists n \geq 2^m, S_n > b(n)\right\} \leq \frac{c}{2^m} \tag{2}$$

is compatible with (1).

Proof of Theorem 1

Our proof is a quantitative version of the original "Law of the Iterated Logarithm", by A Khinchin. We adapt the presentation given by [Velenik].

In what follows, $\varepsilon > 0$ (width of the strip) is fixed, so we omit it from most notation. For easy reference, the reader may take $\varepsilon = 1$.

We need several steps. We recall from Part I, Lemma 2, that, for all $n \geq 1$:

$$P\left(S_n \geq 0\right) > \frac{1}{2} \tag{1}$$

We also recall from Part I, Corollary 5, that for any real x and any $n \geq 1$, we have:

$$P\left(\exists k \leq n, S_k > x\right) \leq 2P\left(S_n > x\right) \tag{2}$$

and, from Part I, Lemma IV.1, that for any n and any x, $0 \leq x \leq n$, we have:

$$P\left(S_n \geq x\right) \leq e^{\frac{-x^2}{2n}} \tag{3}$$

We define $\gamma = 1 + \varepsilon$, and, for all $k \geq 1$, integer, we set $n_k = \gamma^k$. The next Lemma gives an estimate on the number of paths which are above the strip at least once, in the interval of time $\left[n_k, n_{k+1}\right]$:

Lemma 2. – *Let* $C_k = \{\exists n, n_k \le n < n_{k+1}, S_n > \gamma\varphi(n_k)\}$. *Then:*

$$P(C_k) \le 2(kLog(\gamma))^{-\gamma}$$

Proof of Lemma 2

Using (2), we have:

$$P(C_k) \le 2P\{S_{n_{k+1}} > \gamma\varphi(n_k)\}$$

and, using (3):

$$P\{S_{n_{k+1}} > \gamma\varphi(n_k)\} \le \exp\left(-\gamma^2 \frac{n_k}{n_{k+1}} Log(Log(n_k))\right)$$

Since $n_k = \gamma^k$, $\frac{n_{k+1}}{n_k} = \gamma$, and we get:

$$P(C_k) \le 2\exp(-\gamma Log(Log(n_k))) = 2(Log(n_k))^{-\gamma} = 2(kLog(\gamma))^{-\gamma}$$

This proves Lemma 2.

We set $D_k = \{\exists n, n_k \le n < n_{k+1}, S_n > \gamma\varphi(n)\}$. We have:

Lemma 3. – *For any* k, $D_k \subset C_k$.

Proof of Lemma 3

Indeed, if there exists an n such that the inequality $S_n > \gamma\varphi(n)$ holds, we have a fortiori $S_n > \gamma\varphi(n_k)$, since the function φ is increasing. This proves Lemma 3. We deduce:

$$P(D_k) \le 2(kLog(\gamma))^{-\gamma} \tag{4}$$

Let $B_m = \bigcup_{k \ge m} D_k$; then the sets B_m are decreasing when m increases. The set B_m is by definition:

$$B_m = \bigcup_{k \ge m} \{\exists n, n_k \le n < n_{k+1}, S_n > \gamma\varphi(n)\} = \{\exists n \ge \gamma^m, S_n > \gamma\varphi(n)\}.$$

Therefore, B_m is the set of all paths which are above the strip $\gamma\varphi(n)$ at least once after time γ^m. We now estimate its probability.

Lemma 4. – *For all* m, *we have:*

$$P(B_m) \leq \frac{2}{(Log\,\gamma)^{\gamma}} \frac{1}{\varepsilon(m-1)^{\varepsilon}}$$

Proof of Lemma 4

By definition of the sets:

$$P(B_m) \leq \sum_{k=m}^{+\infty} P(D_k)$$

and Lemma 4 follows from (4) and the inequality:

$$\sum_{k=m}^{+\infty} \frac{1}{k^{1+\varepsilon}} \leq \int_{m-1}^{+\infty} \frac{dx}{x^{1+\varepsilon}} = \frac{1}{\varepsilon(m-1)^{\varepsilon}}.$$

The proof of Theorem 1 is complete.

We come back here on what we said about the definition of probabilities. In all the statements above, up to Lemma 5, all probabilities refer to a bounded interval for n (for instance $n \leq n_{k+1}$). This is not the case for Lemma 6 ($n \geq \gamma^m$), but we immediately have an upper bound from bounded intervals, deduced from (3).

We now turn to the opposite theorem: many paths enter the given strip. More precisely, let $\alpha > 0$ (small). We will show that there is $N_0(\varepsilon,\alpha)$ such that if $N > N_0(\varepsilon,\alpha)$, then:

$$P\left(\forall n = 1,\dots N \; ; \frac{S_n}{\varphi(n)} < 1-\varepsilon \right) < \alpha \,.$$

The statement is as follows:

Theorem 5. – *Let* $\varepsilon > 0, \alpha > 0$. *Set* $\gamma = \frac{16}{\varepsilon^2}$, $\delta = \frac{2}{(Log(2))^2} \approx 4.2$, $k_0 = 1 + \left(2 + \frac{\delta}{\alpha}\right)\frac{Log(2)}{Log(\gamma)}$ *and* $N_0 = \left(k_0^{\frac{\varepsilon}{2}} + \frac{\varepsilon}{2} Log\left(\frac{2}{\alpha}\right)\right)^{\frac{2}{\varepsilon}}$. *For any* $N \geq N_0$, *we have:*

$$P\left(\exists k, k_0 \le k \le N, \textit{ such that } \frac{S_k}{\varphi(k)} > 1-\varepsilon\right) \ge 1-\alpha.$$

What this statement says, in simple words, is that it is more and more unlikely to stay constantly below $(1-\varepsilon)\varphi(x)$. For a fixed width ε, the probability that S_n enters, at some time, the strip $\left[(1-\varepsilon)\varphi(x), \varphi(x)\right]$ tends to 1 when $n \to +\infty$.

Proof of Theorem 5

We set as before $\varphi(n) = \sqrt{2nLog(Log(n))}$. Let $\varepsilon > 0$: it refers to the width of the strip, is fixed and is most of the time omitted from the notation. We introduce $\gamma = \frac{16}{\varepsilon^2}$ and, for any integer k, $n_k = \gamma^k$. We write the quotient $\frac{S_{n_k}}{\varphi(n_k)}$ under the form of the sum of two terms, which will be treated separately:

$$\frac{S_{n_k}}{\varphi(n_k)} = \frac{S_{n_k} - S_{n_{k-1}}}{\varphi(n_k)} + \frac{S_{n_{k-1}}}{\varphi(n_{k-1})} \frac{\varphi(n_{k-1})}{\varphi(n_k)} \qquad (1)$$

Set $Y_k = \frac{S_{n_k} - S_{n_{k-1}}}{\varphi(n_k)}$ and $Z_k = \frac{S_{n_{k-1}}}{\varphi(n_{k-1})} \frac{\varphi(n_{k-1})}{\varphi(n_k)}$. The general idea of the proof is to show that Y_k is large with large probability, whereas Z_k is small. We start with the study of Z_k.

Lemma 6. – *For all $k \ge 2$, we have:*

$$\frac{\varphi(n_{k-1})}{\varphi(n_k)} \le \frac{1}{\sqrt{\gamma}} = \frac{\varepsilon}{4}$$

Proof of Lemma 6

This is clear, since $\frac{\varphi(n_{k-1})}{\varphi(n_k)} = \frac{\sqrt{\gamma^{k-1} Log((k-1)Log(\gamma))}}{\sqrt{\gamma^k Log(k\, Log(\gamma))}}$ and by the choice of γ.

Lemma 7. – *Let* $k_0 = 1 + \left(\frac{4}{(Log(2))^2}\frac{1}{\alpha} + 1\right)\frac{Log(2)}{Log(\gamma)}$. *Then:*

$$P\left(\forall k \ge k_0, \frac{\left|S_{n_{k-1}}\right|}{\varphi(n_{k-1})} < 2\right) \ge 1 - \frac{\alpha}{2}.$$

Proof of Lemma 7

Let us choose $\varepsilon = 1$ in Theorem 1. We have, for any m :

$$P\left(\left\{\exists n \geq 2^m, S_n > 2\varphi(n)\right\}\right) \leq \frac{2}{\left(Log(2)\right)^2}\frac{1}{m-1}$$

We choose m so that $\frac{2}{\left(Log(2)\right)^2}\frac{1}{m-1} < \frac{\alpha}{2}$, that is $m > \frac{4}{Log^2 2}\frac{1}{\alpha}+1$. So we get:

$$P\left(\exists n > 2^m, \left|S_n\right| > 2\varphi(n)\right) < \frac{\alpha}{2}$$

that is:

$$P\left(\forall n > 2^m, \frac{\left|S_n\right|}{\varphi(n)} \leq 2\right) \geq 1-\frac{\alpha}{2}$$

Here, $n = \gamma^{k-1}$ and the condition $n \geq 2^m$ is satisfied as soon as:

$$k > 1 + m\frac{Log(2)}{Log(\gamma)} > 1 + \left(\frac{4}{\left(Log\,(2)\right)^2}\frac{1}{\alpha}+1\right)\frac{Log(2)}{Log(\gamma)}$$

This proves Lemma 7.

Lemma 8. - Let k_0 *be as before. If* $k \geq k_0$, *we have:*

$$P\left(\forall k \geq k_0, \left|Z_k\right| < \frac{\varepsilon}{2}\right) \geq 1-\frac{\alpha}{2}.$$

Proof of Lemma 8

Indeed, for any k, we have both $\frac{\left|S_{n_{k-1}}\right|}{\varphi(n_{k-1})} < 2$ and $\frac{\varphi(n_{k-1})}{\varphi(n_k)} < \frac{\varepsilon}{4}$ on a set of probability $\geq 1-\frac{\alpha}{2}$. This proves Lemma 8.

We now turn to the term $Y_k = \frac{S_{n_k} - S_{n_{k-1}}}{\varphi(n_k)}$. We recall from Part I, Proposition 9 that, if $k < \sqrt{n}$, we have, with $c = \frac{1}{4\sqrt{2\pi}}$:

$$P\left(S_n \geq k\right) \geq c\exp\left(-\frac{k^2}{2n}\right) \tag{2}$$

We set:

$$D_k = \left\{\frac{S_{n_k} - S_{n_{k-1}}}{\varphi\left(n_k\right)} > 1 - \frac{\varepsilon}{2}\right\}$$

Proposition 9. – *For all* $k \geq 1$, *we have:*

$$P\left(D_k\right) \geq \frac{c}{\left(kLog\left(\gamma\right)\right)^{1-\varepsilon/2}},$$

where $c = \frac{1}{4\sqrt{2\pi}}$ *as before.*

Proof of Proposition 9

Since the definition of D_k relies upon consecutive differences $S_{n_k} - S_{n_{k-1}}$, the events D_k are independent. We have:

$$P\left(D_k\right) = P\left\{S_{n_k} - S_{n_{k-1}} > \left(1 - \frac{\varepsilon}{2}\right)\varphi\left(n_k\right)\right\} = P\left\{S_{n_k - n_{k-1}} > \left(1 - \frac{\varepsilon}{2}\right)\varphi\left(n_k\right)\right\}$$

since $S_{n_k} - S_{n_{k-1}}$ and $S_{n_k - n_{k-1}}$ have the same law.

In the estimate (2) above, we replace n by $n_k - n_{k-1}$ and k by $\left(1 - \frac{\varepsilon}{2}\right)\varphi\left(n_k\right)$; we obtain:

$$P\left(S_{n_k - n_{k-1}} \geq \left(1 - \frac{\varepsilon}{2}\right)\varphi\left(n_k\right)\right) \geq c\exp\left(-\frac{1}{2}\left(1 - \frac{\varepsilon}{2}\right)^2 \frac{\left(\varphi\left(n_k\right)\right)^2}{n_k - n_{k-1}}\right) = c\exp\left(-\left(1 - \frac{\varepsilon}{2}\right)^2 \frac{n_k Log\left(Log\left(n_k\right)\right)}{n_k - n_{k-1}}\right)$$

But:

$$\frac{n_k - n_{k-1}}{n_k} = \frac{\gamma^k - \gamma^{k-1}}{\gamma^k} = \frac{\gamma - 1}{\gamma} > 1 - \frac{\varepsilon}{2},$$ from the choice of γ. Therefore:

$$P\left\{S_{n_k - n_{k-1}} > \left(1 - \frac{\varepsilon}{2}\right)\varphi\left(n_k\right)\right\} \geq c\exp\left(-\left(1 - \frac{\varepsilon}{2}\right)Log\left(Log\left(n_k\right)\right)\right) = c\left(Log\left(n_k\right)\right)^{-\left(1-\frac{\varepsilon}{2}\right)} = \frac{c}{\left(kLog\left(\gamma\right)\right)^{1-\frac{\varepsilon}{2}}}$$

This proves Proposition 9.

We need a quantitative version of the second Borel-Cantelli Lemma:

Lemma 10. – *Let* D_k *be a sequence of independent events; let* D_k^c *be their complements. Then :*

$$P\left(\bigcap_{k=1}^{N} D_k^c\right) \le \exp\left(-\sum_{k=1}^{N} P(D_k)\right)$$

Proof of Lemma 10

We have, for all N :

$$P\left(\bigcap_{k=1}^{N} D_k^c\right) = \prod_{k=1}^{N}\left(1-P(D_k)\right) \le \exp\left(-\sum_{k=1}^{N} P(D_k)\right)$$

using the inequality $1-x \le e^{-x}$, $0 \le x \le 1$. This proves Lemma 10.

Set, for all k, $u_k = \dfrac{c}{\left(Log(\gamma)\right)^{1-\varepsilon/2}} \dfrac{1}{k^{1-\varepsilon/2}}$. We deduce from Proposition 9 and Lemma 10:

$$P\left(\bigcap_{k=1}^{N} D_k^c\right) \le \exp\left(-\sum_{k=1}^{N} u_k\right).$$

Since the series of general term u_k is divergent, the sum $\sum_{k=1}^{N} u_k$ can be made arbitrarily large, choosing N large enough. Then $\exp\left(-\sum_{k=1}^{N} u_k\right)$ is close to 0. So the intersection $\bigcap_{k=1}^{N} D_k^c$ has a very small probability. But this intersection is the set of all paths for which $\left\{\dfrac{S_{n_k}-S_{n_{k-1}}}{\varphi(n_k)} \le 1-\dfrac{\varepsilon}{2}\right\}$ for any $k=1,\dots,N$. More precisely, we have:

Lemma 11. – *Let* k_0 *as in Lemma 7. We have, if* $N > N_0 = \left(k_0^{\frac{\varepsilon}{2}} + \dfrac{\varepsilon}{2} Log\left(\dfrac{2}{\alpha}\right)\right)^{\frac{2}{\varepsilon}}$ *:*

$$P\left(\forall k, k_0 \le k \le N, \frac{S_{n_k}-S_{n_{k-1}}}{\varphi(n_k)} \le 1-\frac{\varepsilon}{2}\right) < \frac{\alpha}{2}.$$

Proof of Lemma 11

We choose N_0 large enough so that, if $N > N_0$, $\exp\left(-\sum_{k=k_0}^{N} u_k\right) < \frac{\alpha}{2}$, that is $\sum_{k_0}^{N} u_k > Log\left(\frac{2}{\alpha}\right)$.

But, for $k > k_0$, we have $u_k > \frac{1}{k^{1-\frac{\varepsilon}{2}}}$, and therefore:

$$\sum_{k_0}^{N} u_k > \sum_{k_0}^{N} k^{-1+\frac{\varepsilon}{2}} \geq \frac{2}{\varepsilon}\left(N^{\frac{\varepsilon}{2}} - k_0^{\frac{\varepsilon}{2}}\right).$$

So we choose N large enough, in order to have:

$$\frac{2}{\varepsilon}\left(N^{\frac{\varepsilon}{2}} - k_0^{\frac{\varepsilon}{2}}\right) > Log\left(\frac{2}{\alpha}\right)$$

that is:

$$N \geq \left(k_0^{\frac{\varepsilon}{2}} + \frac{\varepsilon}{2} Log\left(\frac{2}{\alpha}\right)\right)^{\frac{2}{\varepsilon}}$$

This proves Lemma 11.

Let us finish the proof of Theorem 5. We have:

$$\frac{S_{n_k}}{\varphi(n_k)} = \frac{S_{n_k} - S_{n_{k-1}}}{\varphi(n_k)} + \frac{S_{n_{k-1}}}{\varphi(n_{k-1})} \frac{\varphi(n_{k-1})}{\varphi(n_k)} \tag{3}$$

Let E_1 be the set:

$$E_1 = \left\{ Y_k < 1 - \frac{\varepsilon}{2}, \forall k = k_0, ..., N \right\}$$

Then, Lemma 11 says that $P(E_1) < \frac{\alpha}{2}$.

Let E_2 be the set:

$$E_2 = \left\{ \forall k \geq k_0, |Z_k| < \frac{\varepsilon}{2} \right\}$$

Then $P(E_2) > 1 - \frac{\alpha}{2}$. We have:

$$P\left(\left(E_1^c \cap E_2\right)^c\right) = P\left(E_1 \cup E_2^c\right) \le P\left(E_1\right) + P\left(E_2^c\right) = P\left(E_1\right) + 1 - P\left(E_2\right) < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$$

and therefore:

$$P\left(E_1^c \cap E_2\right) \ge 1 - \alpha$$

But $E_1^c \cap E_2$ is the set for which there is a k, $k_0 \le k \le N$, with $Y_k \ge 1 - \frac{\varepsilon}{2}$ and $\left|Z_k\right| < \frac{\varepsilon}{2}$. We deduce:

$$Y_k + Z_k > Y_k - \left|Z_k\right| > 1 - \varepsilon$$

This finishes the proof of Theorem 5.

II. Comparison with our results

If we consider $\varphi(x) = \sqrt{2xLog\left(Log(x)\right)}$ then the integral $\int \frac{1}{\varphi^2}$ diverges at $x = +\infty$. The same holds for the barrier $b(x) = \sqrt{xLog(x)}$, which is above the previous one. We know that:

$$\int \frac{1}{b^2(x)} dx = \int \frac{1}{xLn(x)} dx = Ln\left(Ln(x)\right)$$

and therefore the integral diverges at $+\infty$. So, the total energy left at time N (probability that the game continues up to time N) tends to zero when $N \to +\infty$, for both barriers.

The comparison between Khinchin's methods and ours lead to the following conclusions:

- Khinchin's methods may handle the case of a single curve, whereas ours may handle only the case of two symmetric barriers;

- Quantitative estimates obtained by means of Khinchin's methods are of probabilistic type, and much weaker than the results derived by means of operator theory;

- To say that Khinchin's curve $\varphi(x) = \sqrt{2xLog\left(Log(x)\right)}$ is a "security curve" is incorrect ; the statement *almost surely,* $\limsup_{n \to +\infty} \frac{S_n}{\varphi(n)} = 1$ is only asymptotic.

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