Société de Calcul Mathématique SA
*Outils d'aide à la décision*
*depuis 1995*

# Simple Random Walks

## Part V

## Khinchin's Curves

by Bernard Beauzamy

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In this Fifth Part, we give the connections between our work and the theories introduced by Alexander Khinchin, in 1924.

## I. Khinchin's Law of the Iterated Logarithm

Let us come back to the original setting: the $X_n$'s are independent variables with same law, $P(X_n = \pm 1) = \frac{1}{2}$. We set $S_N = \sum_{n=1}^{N} X_n$ (in other words, we do not consider only even values of the time and, originally, both fortunes are equal).

We introduce Khinchin's curve, or barrier, defined by the equation:

$$\varphi(x) = \sqrt{2xLog(Log(x))},$$

which is a real function, defined for all real $x > e$.

The classical statement of the law of the iterated logarithm is:

$$\textit{almost surely, } \limsup_{n\to+\infty} \frac{S_n}{\varphi(n)} = 1$$

(see https://en.wikipedia.org/wiki/Law_of_the_iterated_logarithm)

The explanation given by Wikipedia is as follows:

"Thus, although the quantity $\left|\frac{S_n}{\varphi(n)}\right|$ is less than any predefined $\varepsilon > 0$ with probability approaching one, the quantity will nevertheless be greater than $\varepsilon$ infinitely often; in fact, the quantity will be visiting the neighborhoods of any point in the interval (-1,1) almost surely."

The difficulty is in the understanding of the words "almost surely", both in theory and in practice.

There is a natural probability on the infinite product $\prod_{n=1}^{\infty}\{-1,1\}^n$, which is simply the product of the elementary probabilities on each layer. With respect to this "global" probability, the words "almost surely" are well-defined. But, for this probability, every elementary path has probability 0, and so do have a finite number of paths. Moreover, it is quite hard to obtain quantitative results for this probability, which is well-suited only for probabilistic arguments.

On the contrary, if we stop at time $N,$ we have a precise and intuitive definition of the probability of an event: up to time $N,$ we have $2^N$ paths, and the probability of any event is : number of paths satisfying the event, divided by $2^N.$ For instance, the probability of the event $\{\exists n \le N \,;\, S_n > \varphi(n)\}$ is perfectly clear: we count the number of paths for which, at some point, the random walk is above the curve, and we divide this number by $2^N.$ Since the paths divide into two at each step, an estimate obtained at a given step will remain valid at later stages. For instance, the statement $X_1 = 1$ has the same probability (1/2), no matter whether we consider it at stage 1 or at any later stage.

Let us observe that the statement from Wikipedia may be quite misleading. Indeed, if one reads: "The quantity $\left|\frac{S_n}{\varphi(n)}\right|$ will nevertheless be greater than $\varepsilon$ infinitely often", this is true for most curves, and not only for Khinchin's curve. Indeed, we remember (Part I) that the random walk $S_n$ comes back to the $x$ axis (only at even times), so for instance the value $S_n = 0$ may be expected at time $n = 5000.$ But then, consider the situation where $S_n$ increases linearly from this point; it will eventually cross the curve $\varphi(n)$ ; in fact, this is true for any curve such that $\frac{\varphi(n)}{n} \to 0$ when $n \to +\infty.$ We construct this way an infinite number of situations in which $S_n$ exceeds $4\varphi(n)$ or any multiple, as one wishes.

Therefore, we think that in such statements, a precise definition of the probabilities must be given. This is what we do now. We give Khinchin's results in quantitative settings (which are new).

**Theorem 1. Quantitative statement of LIL, 1**- *Let* $\varepsilon > 0$ *and* $m \geq 2$. *We set:*

$$\eta(m,\varepsilon) = \frac{2}{\left(Log(1+\varepsilon)\right)^{1+\varepsilon}} \frac{1}{\varepsilon(m-1)^{\varepsilon}}$$

*and:*

$$B(m,\varepsilon) = \left\{\exists n \geq (1+\varepsilon)^m, S_n > (1+\varepsilon)\varphi(n)\right\}$$

*Then, for all* $m$ *:*

$$P\left(B(m,\varepsilon)\right) \leq \eta(m,\varepsilon)$$

The set $B(m,\varepsilon)$ is made of the paths which are above the "safety strip" $(1+\varepsilon)\varphi$ at least once after the time $(1+\varepsilon)^m$. The Theorem says that the probability of such an event tends to 0 when $m$ increases. In other words, if we fix a width for the safety strip, that is $\varepsilon > 0$ fixed, it becomes less and less probable to pass above the curve $(1+\varepsilon)\varphi$ when $n$ increases.

Let us take for example $\varepsilon = 1$. Theorem 1 gives the estimate:

$$P\left\{\exists n \geq 2^m, S_n > 2\varphi(n)\right\} \leq \frac{2}{\left(Log(2)\right)^2} \frac{1}{m-1}$$

For instance, if we want the right-hand side to be $\leq 0.05$, we find $m = 85$. Therefore, the probability to have $n \geq 2^{85}$ for which $S_n > 2\varphi(n)$ is < 0.05.

For a better understanding, this statement may be converted into a proportion of paths, as follows:

*Fix any* $N > 2^{85}$. *The proportion of paths, which satisfy* $S_n > 2\varphi(n)$ *at least in one place, between* $2^{85}$ *and* $N$, *is smaller than 5%.*

**Proof of Theorem 1**

Our proof is a quantitative version of the original "Law of the Iterated Logarithm", by A Khinchin. We adapt the presentation given by [Velenik].

In all that follows, $\varepsilon > 0$ (width of the strip) is fixed, so we omit it from most notation. For easy reference, the reader may take $\varepsilon = 1$.

We need several steps. We recall from Part I, Lemma 2, that, for all $n \geq 1$:

$$P\left(S_n \geq 0\right) > \frac{1}{2} \tag{1}$$

We also recall from Part I, Corollary 5, that for any real $x$ and any $n \geq 1$, we have:

$$P\left(\exists k \leq n, S_k > x\right) \leq 2P\left(S_n > x\right) \tag{2}$$

and, from Part I, Lemma IV.1, that for any $n$ and any $x$, $0 \leq x \leq n$, we have:

$$P\left(S_n \geq x\right) \leq e^{-\frac{x^2}{2n}} \tag{3}$$

We define $\gamma = 1 + \varepsilon$, and, for all $k \geq 1$, integer, we set $n_k = \gamma^k$. The next Lemma gives an estimate on the number of paths which are above the strip at least once, in the interval of time $\left[n_k, n_{k+1}\right]$:

**Lemma 2.** – *Let* $C_k = \left\{\exists n, n_k \leq n < n_{k+1}, S_n > \gamma\varphi\left(n_k\right)\right\}$. *Then:*

$$P\left(C_k\right) \leq 2\left(kLog\left(\gamma\right)\right)^{-\gamma}$$

**Proof of Lemma 2**

Using (2), we have:

$$P\left(C_k\right) \leq 2P\left\{S_{n_{k+1}} > \gamma\varphi\left(n_k\right)\right\}$$

and, using (3):

$$P\left\{S_{n_{k+1}} > \gamma\varphi\left(n_k\right)\right\} \leq \exp\left(-\gamma^2 \frac{n_k}{n_{k+1}} Log\left(Log\left(n_k\right)\right)\right)$$

Since $n_k = \gamma^k$, $\dfrac{n_{k+1}}{n_k} = \gamma$, and we get:

$$P\left(C_k\right) \leq 2\exp\left(-\gamma Log\left(Log\left(n_k\right)\right)\right) = 2\left(Log\left(n_k\right)\right)^{-\gamma} = 2\left(kLog\left(\gamma\right)\right)^{-\gamma}$$

This proves Lemma 2.

We set $D_k = \left\{\exists n, n_k \leq n < n_{k+1}, S_n > \gamma\varphi\left(n\right)\right\}$. We have:

**Lemma 3.** – *For any* $k$, $D_k \subset C_k$.

**Proof of Lemma 3**

Indeed, if there exists an $n$ such that the inequality $S_n > \gamma\varphi(n)$ holds, we have a fortiori $S_n > \gamma\varphi(n_k)$, since the function $\varphi$ is increasing. This proves Lemma 3. We deduce:

$$P(D_k) \leq 2\left(kLog(\gamma)\right)^{-\gamma} \tag{4}$$

Let $B_m = \bigcup_{k \geq m} D_k$ ; then the sets $B_m$ are decreasing when $m$ increases. The set $B_m$ is by definition:

$$B_m = \bigcup_{k \geq m} \left\{\exists n, n_k \leq n < n_{k+1}, S_n > \gamma\varphi(n)\right\} = \left\{\exists n \geq \gamma^m, S_n > \gamma\varphi(n)\right\}$$

Therefore, $B_m$ is the set of all paths which are above the security strip $\gamma\varphi(n)$ at least once after time $\gamma^m$. We now estimate its probability.

**Lemma 4.** – *For all* $m$, *we have:*

$$P(B_m) \leq \frac{2}{(Log\,\gamma)^{\gamma}} \frac{1}{\varepsilon(m-1)^{\varepsilon}}$$

**Proof of Lemma 4**

By definition of the sets:

$$P(B_m) \leq \sum_{k=m}^{+\infty} P(D_k)$$

and Lemma 4 follows from (4) and the inequality:

$$\sum_{k=m}^{+\infty} \frac{1}{k^{1+\varepsilon}} \leq \int_{m-1}^{+\infty} \frac{dx}{x^{1+\varepsilon}} = \frac{1}{\varepsilon(m-1)^{\varepsilon}}$$

The proof of Theorem 1 is complete.

We come back here on what we said about the definition of probabilities. In all the statements above, up to Lemma 5, all probabilities refer to a bounded interval for $n$ (for instance $n \leq n_{k+1}$). This is not the case for Lemma 6 ($n \geq \gamma^m$), but we immediately have an upper bound from bounded intervals, deduced from (3).

We now turn to the opposite theorem: many paths enter the safety strip. More precisely, let $\alpha > 0$ (small). We will show that there is $N_0(\varepsilon,\alpha)$ such that if $N > N_0(\varepsilon,\alpha)$, then:

$$P\left(\forall n = 1,\dots N\ ; \frac{S_n}{\varphi(n)} < 1-\varepsilon\right) < \alpha .$$

The statement is as follows:

**Theorem 2.** – Let $\varepsilon > 0, \alpha > 0$. Set $\gamma = \frac{16}{\varepsilon^2}$ $\delta = \frac{2}{\left(Log(2)\right)^2} \approx 4.2$, $k_0 = 1 + \left(2 + \frac{\delta}{\alpha}\right)\frac{Log(2)}{Log(\gamma)}$ *and* $N_0 = \left(k_0^{\frac{\varepsilon}{2}} + \frac{\varepsilon}{2} Log\left(\frac{2}{\alpha}\right)\right)^{\frac{2}{\varepsilon}}$. *For any* $N \geq N_0$, *we have:*

$$P\left(\exists k, k_0 \leq k \leq N, \textit{such that } \frac{S_k}{\varphi(k)} > 1-\varepsilon\right) \geq 1-\alpha$$

**Proof of Theorem 2**

We set as before $\varphi(n) = \sqrt{2nLog\left(Log(n)\right)}$. Let $\varepsilon > 0$ : it refers to the width of the safety strip, is fixed and is most of the time omitted from the notation. We introduce $\gamma = \frac{16}{\varepsilon^2}$ and, for any integer $k$, $n_k = \gamma^k$. We write the quotient $\frac{S_{n_k}}{\varphi(n_k)}$ under the form of two terms, which will be treated separately.

$$\frac{S_{n_k}}{\varphi(n_k)} = \frac{S_{n_k} - S_{n_{k-1}}}{\varphi(n_k)} + \frac{S_{n_{k-1}}}{\varphi(n_{k-1})}\frac{\varphi(n_{k-1})}{\varphi(n_k)} \tag{1}$$

Set $Y_k = \frac{S_{n_k} - S_{n_{k-1}}}{\varphi(n_k)}$ and $Z_k = \frac{S_{n_{k-1}}}{\varphi(n_{k-1})}\frac{\varphi(n_{k-1})}{\varphi(n_k)}$. The general idea of the proof is to show that $Y_k$ is large with large probability, whereas $Z_k$ is small.

We start with the study of $Z_k$.

**Lemma 1.** – *For all* $k \geq 2$, *we have:*

$$\frac{\varphi(n_{k-1})}{\varphi(n_k)} \leq \frac{1}{\sqrt{\gamma}} = \frac{\varepsilon}{4}$$

**Proof of Lemma 1**

This is clear, since $\frac{\varphi(n_{k-1})}{\varphi(n_k)} = \frac{\sqrt{\gamma^{k-1} Log((k-1) Log(\gamma))}}{\sqrt{\gamma^k Log(k\ Log(\gamma))}}$ and by the choice of $\gamma$.

**Lemma 2.** – *Let* $k_0 = 1 + \left( \frac{4}{(Log(2))^2} \frac{1}{\alpha} + 1 \right) \frac{Log(2)}{Log(\gamma)}$. *Then:*

$$P\left( \forall k \geq k_0, \frac{|S_{n_{k-1}}|}{\varphi(n_{k-1})} < 2 \right) \geq 1 - \frac{\alpha}{2}$$

**Proof of Lemma 2**

Let us choose $\varepsilon = 1$ in Theorem 1. We have, for any $m$ :

$$P\left( \{ \exists n \geq 2^m, S_n > 2\varphi(n) \} \right) \leq \frac{2}{(Log(2))^2} \frac{1}{m-1}$$

We choose $m$ so that $\frac{2}{(Log(2))^2} \frac{1}{m-1} < \frac{\alpha}{2}$, that is $m > \frac{4}{Log^2 2} \frac{1}{\alpha} + 1$. So we get:

$$P\left( \exists n > 2^m, |S_n| > 2\varphi(n) \right) < \frac{\alpha}{2}$$

that is:

$$P\left( \forall n > 2^m, \frac{|S_n|}{\varphi(n)} \leq 2 \right) \geq 1 - \frac{\alpha}{2}$$

Here, $n = \gamma^{k-1}$ and the condition $n \geq 2^m$ is satisfied as soon as:

$$k > 1 + m \frac{Log(2)}{Log(\gamma)} > 1 + \left( \frac{4}{(Log(2))^2} \frac{1}{\alpha} + 1 \right) \frac{Log(2)}{Log(\gamma)}$$

This proves Lemma 2.

**Lemma 3.** - Let $k_0$ *as before. If* $k \geq k_0$, *we have:*

$$P\left( \forall k \geq k_0, |Z_k| < \frac{\varepsilon}{2} \right) \geq 1 - \frac{\alpha}{2}$$

**Proof of Lemma 3**

Indeed, for any $k,$ we have both $\frac{\left|S_{n_{k-1}}\right|}{\varphi\left(n_{k-1}\right)}<2$ and $\frac{\varphi\left(n_{k-1}\right)}{\varphi\left(n_k\right)}<\frac{\varepsilon}{4}$ on a set of probability $\geq 1-\frac{\alpha}{2}$. This proves Lemma 3.

We now turn to the term $Y_k=\frac{S_{n_k}-S_{n_{k-1}}}{\varphi\left(n_k\right)}$. We recall from Part I, Proposition 9 that, if $k<\sqrt{n}$, we have, with $c=\frac{1}{4\sqrt{2\pi}}$:

$$P\left(S_n\geq k\right)\geq c\exp\left(-\frac{k^2}{2n}\right)$$

We set:

$$D_k=\left\{\frac{S_{n_k}-S_{n_{k-1}}}{\varphi\left(n_k\right)}>1-\frac{\varepsilon}{2}\right\}$$

**Proposition 4.** – *For all* $k\geq 1,$ *we have:*

$$P\left(D_k\right)\geq\frac{c}{\left(kLog\left(\gamma\right)\right)^{1-\varepsilon/2}}$$

where $c$ is as in Lemma 6.

**Proof of Proposition 4**

Since the definition of $D_k$ relies upon consecutive differences $S_{n_k}-S_{n_{k-1}}$, the events $D_k$ are independent. We have:

$$P\left(D_k\right)=P\left\{S_{n_k}-S_{n_{k-1}}>\left(1-\frac{\varepsilon}{2}\right)\varphi\left(n_k\right)\right\}=P\left\{S_{n_k-n_{k-1}}>\left(1-\frac{\varepsilon}{2}\right)\varphi\left(n_k\right)\right\}$$

since $S_{n_k}-S_{n_{k-1}}$ and $S_{n_k-n_{k-1}}$ have the same law.

In the estimate given by Corollary 6 above, we replace $n$ by $n_k-n_{k-1}$ and $k$ by $\left(1-\frac{\varepsilon}{2}\right)\varphi\left(n_k\right)$ ; we obtain:

$$P\left(S_{n_k-n_{k-1}} \geq \left(1-\frac{\varepsilon}{2}\right)\varphi(n_k)\right) \geq c\exp\left(-\frac{1}{2}\left(1-\frac{\varepsilon}{2}\right)^2 \frac{(\varphi(n_k))^2}{n_k-n_{k-1}}\right) = c\exp\left(-\left(1-\frac{\varepsilon}{2}\right)^2 \frac{n_k Log(Log(n_k))}{n_k-n_{k-1}}\right)$$

But:

$\frac{n_k-n_{k-1}}{n_k} = \frac{\gamma^k-\gamma^{k-1}}{\gamma^k} = \frac{\gamma-1}{\gamma} > 1-\frac{\varepsilon}{2}$, from the choice of $\gamma$. Therefore:

$$P\left\{S_{n_k-n_{k-1}} > \left(1-\frac{\varepsilon}{2}\right)\varphi(n_k)\right\} \geq c\exp\left(-\left(1-\frac{\varepsilon}{2}\right)Log(Log(n_k))\right) = c(Log(n_k))^{-\left(1-\frac{\varepsilon}{2}\right)} = \frac{c}{(kLog(\gamma))^{1-\frac{\varepsilon}{2}}}$$

This proves Proposition 7.

We need a quantitative version of the second Borel-Cantelli Lemma:

**Lemma 9.** – *Let* $D_k$ *be a sequence of independent events; let* $D_k^c$ *be their complements. Then :*

$$P\left(\bigcap_{k=1}^N D_k^c\right) \leq \exp\left(-\sum_{k=1}^N P(D_k)\right)$$

**Proof of Lemma 9**

We have, for all $N$ :

$$P\left(\bigcap_{k=1}^N D_k^c\right) = \prod_{k=1}^N \left(1-P(D_k)\right) \leq \exp\left(-\sum_{k=1}^N P(D_k)\right)$$

using the inequality $1-x \leq e^{-x}$, $0 \leq x \leq 1$. This proves Lemma 9.

Set, for all $k$, $u_k = \frac{c}{(Log(\gamma))^{1-\varepsilon/2}} \frac{1}{k^{1-\varepsilon/2}}$. We deduce from Proposition 7 and Lemma 9:

$$P\left(\bigcap_{k=1}^N D_k^c\right) \leq \exp\left(-\sum_{k=1}^N u_k\right)$$

Since the series of general term $u_k$ is divergent, the sum $\sum_{k=1}^N u_k$ can be made arbitrarily large, choosing $N$ large enough. Then $\exp\left(-\sum_{k=1}^N u_k\right)$ is close to 0. So the intersection $\bigcap_{k=1}^N D_k^c$ has a very small probability. But this intersection is the set of all paths for which $\left\{\frac{S_{n_k}-S_{n_{k-1}}}{\varphi(n_k)} \leq 1-\frac{\varepsilon}{2}\right\}$ for any $k = 1,\ldots,N$. More precisely, we have:

**Lemma 11.** – *Let* $k_0$ *as in Lemma 3. We have, if* $N > N_0 = \left(k_0^{\frac{\varepsilon}{2}} + \frac{\varepsilon}{2} Log\left(\frac{2}{\alpha}\right)\right)^{\frac{2}{\varepsilon}}$ *:*

$$P\left(\forall k, k_0 \le k \le N, \frac{S_{n_k} - S_{n_{k-1}}}{\varphi(n_k)} \le 1 - \frac{\varepsilon}{2}\right) < \frac{\alpha}{2}$$

**Proof of Lemma 11**

Using Lemma 6, we choose $N_0$ large enough so that $\exp\left(-\sum_{k=k_0}^{N} u_k\right) < \frac{\alpha}{2}$, which means $\sum_{k_0}^{N} u_k > Log\left(\frac{2}{\alpha}\right)$. But, for $k > k_0$, we have $u_k > \frac{1}{k^{1-\frac{\varepsilon}{2}}}$, and therefore:

$$\sum_{k_0}^{N} u_k > \sum_{k_0}^{N} k^{-1+\frac{\varepsilon}{2}} \ge \frac{2}{\varepsilon}\left(N^{\frac{\varepsilon}{2}} - k_0^{\frac{\varepsilon}{2}}\right)$$

So we choose $N$ large enough for :

$$\frac{2}{\varepsilon}\left(N^{\frac{\varepsilon}{2}} - k_0^{\frac{\varepsilon}{2}}\right) > Log\left(\frac{2}{\alpha}\right)$$

that is:

$$N \ge \left(k_0^{\frac{\varepsilon}{2}} + \frac{\varepsilon}{2} Log\left(\frac{2}{\alpha}\right)\right)^{\frac{2}{\varepsilon}}$$

This proves Lemma 11.

Let us finish the proof of Theorem 2. We have:

$$\frac{S_{n_k}}{\varphi(n_k)} = \frac{S_{n_k} - S_{n_{k-1}}}{\varphi(n_k)} + \frac{S_{n_{k-1}}}{\varphi(n_{k-1})} \frac{\varphi(n_{k-1})}{\varphi(n_k)} \qquad (5)$$

Let $E_1$ be the set:

$$E_1 = \left\{Y_k < 1 - \frac{\varepsilon}{2}, \forall k = k_0, \ldots, N\right\}$$

Then, Lemma 7 says that $P(E_1) < \frac{\alpha}{2}$.

Let $E_2$ be the set:

$$E_2 = \left\{ \forall k \geq k_0, \left| Z_k \right| < \frac{\varepsilon}{2} \right\}$$

Then $P(E_2) > 1 - \frac{\alpha}{2}$ . We have:

$$P\left( \left( E_1^c \cap E_2 \right)^c \right) = P\left( E_1 \cup E_2^c \right) \leq P\left( E_1 \right) + P\left( E_2^c \right) = P\left( E_1 \right) + 1 - P\left( E_2 \right) < \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha$$

and therefore:

$$P\left( E_1^c \cap E_2 \right) \geq 1 - \alpha$$

But $E_1^c \cap E_2$ is the set for which there is a $k$, $k_0 \leq k \leq N$, with $Y_k \geq 1 - \frac{\varepsilon}{2}$ and $\left| Z_k \right| < \frac{\varepsilon}{2}$. We deduce:

$$Y_k + Z_k > Y_k - \left| Z_k \right| > 1 - \varepsilon$$

This finishes the proof of Theorem 2.

## References

[Velenik] Y. Velenik, Université de Genève "Chapitres choisis de Théorie des Probabilités" : http://www.unige.ch/math/folks/velenik/papers/LN-CC1.pdf