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Simple Random Walks in the plane:

An energy based approach

Part IV: Variable Fortunes

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In this Fourth Part, we investigate the case of variable barriers: the barrier is represented by a function of time; say for instance $b(n) = \pm\sqrt{n}$.

We consider symmetric barriers, which means that the rules are the same for both players: if at some time n the fortune of one of the players reaches the barrier, the game stops. The question is: what is the probability that the game continues after N steps ?

We obtain a necessary and sufficient condition:

Theorem. - *The probability E_N that the game continues after N steps tends to zero when $N \to +\infty$ if and only if the integral*

$$\int_1^{+\infty} \frac{dx}{b^2(x)}$$

diverges at infinity.

For instance, for $b(x)=\sqrt{x}$, $\int \frac{1}{b^2}=Log(x)$ diverges at infinity, so the probability that the game continues indefinitely is zero. The same holds for $b(x)=\sqrt{xLog(x)}$, since $\int \frac{1}{b^2}=Log(Log(x))$.

More quantitatively, what we prove is that the remaining energy at stage N, denoted by E_N , satisfies:

$$E_N \approx c\exp\left(-\frac{\pi^2}{8}\int_1^{t_{N+1}} \frac{dx}{b^2(x)}\right)$$

where t_N is the unique number such that $b(t_N)=N$. The constant $0<c<1$ depends on the particular barrier and on its discretization, but is independent from N.

During the n^{th} period (see definition below), the profile of fortune is proportional to the vector:

$$\left(\sin(\vartheta),\sin(2\vartheta),...,\sin(n\vartheta),\sin((n-1)\vartheta),...,\sin(\vartheta)\right)$$

where $\vartheta=\frac{\pi}{2n+1}$.

We use the "energy based" approach, described in Parts 1, 2, 3. In terms of energy, what our results show is that, for instance for $b(x)=\sqrt{xLog(x)}$, the energy is more and more absorbed by the boundary, and the remaining part on each vertical becomes smaller and smaller. In probabilistic terms, it means that almost all games will eventually touch this curve. This seems to contradict Khinchin's results (Law of the Iterated Logarithm), which is often presented under the form "the curve $b(x)=\sqrt{2xLog(Log(x))}$ is a security curve". Such a presentation is incorrect. The curve $\sqrt{xLog(x)}$ is above Khinchin's curve, and still the probability to hit it between two instants $N_1 < N_2$ is always strictly positive.

In order to use the tools defined in the previous Chapters, we must first investigate what happens when the barrier is not constant.

I. Transition between two periods

1. Energy propagation

For us, a "barrier" is a positive function $b(x)$, defined on $x \geq 0$, differentiable, increasing, and satisfying $b(x) \to +\infty$ when $x \to +\infty$. A simple example is $b(x) = \sqrt{x}$.

We define a continuous barrier, but our game uses only integer values. So we have to convert our barrier into a succession of constant segments, with integer values.

Let us describe this representation in detail in the case of the barrier $\pm\sqrt{n}$.

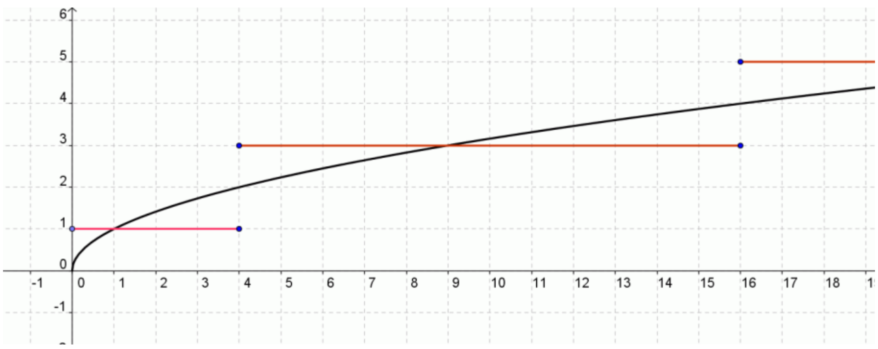


Figure 1: Discretization of the barrier

The changes will occur at times $4n^2$, $n = 0,1,\ldots$. On the interval $4n^2 \leq x < 4(n+1)^2$, the barrier is at $2n+1$ (recall from Part II that we want even values of time and odd values for the barrier). So, in the notation introduced in Part II, $\xi = n$ and this value is used on an interval of length $l_n = 4(n+1)^2 - 4n^2$, that is $l_n = 8n+4$.

Let us give a rough description of the meaning of l_n , forgetting about the requirements "even times and odd values for the barrier". Then, l_1 is the time that the barrier needs to go from 0 to 1, and l_n is the time it needs to go from $n-1$ to n. If x_n is defined by the equation $b(x_n) = n$, then $x_n = l_1 + \cdots + l_n$. If β is the inverse function of b (which exists, since b is strictly increasing), we have:

$$l_n = x_n - x_{n-1} = \beta(n) - \beta(n-1) \approx \beta'(n) = \frac{1}{b'(\beta(n))}.$$

In the case of the function $b(x) = \sqrt{x}$, we have $l_n \sim 2n$; this rough description is correct within a multiplicative constant.

The interval of time during which $\xi = n$ is called the n^{th} period. From now on, we forget about the continuous curve and remember only the segments.

We have now to investigate the transition between two periods. The barrier was at $2\xi+1$ and moves to $2\xi+3$.

Let us first consider the transition on the energy, that is the variables $e(2n,2k)$. As we did in Part II, we restrict ourselves to the upper half-plane, using the symmetry of the problem.

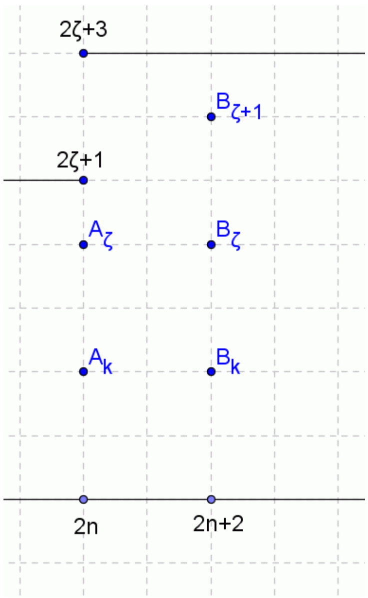


Figure 2: Notation for the transition

It will be convenient to have a simple notation just for the transition. On the vertical corresponding to time $2n$, we have $\xi+1$ points $A_0,\ldots,A_\xi$; at time $2n+2$, we have $\xi+2$ points $B_0,\ldots,B_{\xi+1}$. We denote by a_k the energy at the point A_k and similarly b_k for the B_k.

For the first ξ points, we have the usual transition equations:

$$b_0 = \frac{1}{2}(a_0 + a_1) \qquad (1)$$

$$b_k = \frac{1}{4}a_{k-1} + \frac{1}{2}a_k + \frac{1}{4}a_{k+1} \text{ for } k = 1,\ldots,\xi-1 \qquad (2)$$

The last two equations are different from the constant case; they are:

$$b_\xi = \frac{1}{4}a_{\xi-1} + \frac{1}{2}a_\xi \qquad (3)$$

$$b_{\xi+1} = \frac{1}{4} a_\xi \qquad (4)$$

If the barrier was constantly at $2\xi + 1$, instead of (3), we would have:

$$b_\xi = \frac{1}{4} a_{\xi-1} + \frac{1}{4} a_\xi \qquad (5)$$

and instead of (4) :

$$b_{\xi+1} = 0 \qquad (6)$$

So, the fact that the barrier moves one step higher means that less energy is lost:

- for b_ξ, increase of $\frac{1}{4} a_\xi$,
- for $b_{\xi+1}$, increase of $\frac{1}{4} a_\xi$,

which represents a total increase of energy equal to $\frac{1}{2} a_\xi$.

Let us now turn to the variables $x(n,k)$ and describe the transition on these variables. Recall that, for $k = 0, \dots, \xi - 1$ and $n \geq 2$:

$$x_k = \frac{1}{2}\left(a_k + a_{k+1}\right)$$

We have (see Part II):

$$b_0 = x_0 \qquad (7)$$

$$b_k = \frac{1}{2}\left(x_{k-1} + x_k\right) \text{ for } k = 1, \dots, \xi - 1 \qquad (8)$$

$$b_\xi = \frac{1}{4}\left(a_{\xi-1} + a_\xi\right) + \frac{1}{4}\left(a_\xi + a_{\xi+1}\right) \text{ with } a_{\xi+1} = 0$$

which gives:

$$b_\xi = \frac{1}{2} x_{\xi-1} + \frac{1}{2} x_\xi \qquad (9)$$

$$b_{\xi+1} = \frac{1}{4}\left(a_\xi + a_{\xi+1}\right) = \frac{1}{2} x_\xi \qquad (10)$$

Let us define $y_k = \frac{1}{2}\left(b_k + b_{k+1}\right)$, $k = 0,...,\xi.$

We get:

$$y_0 = \frac{1}{2}\left(x_0 + \frac{1}{2}\left(x_0 + x_1\right)\right) = \frac{3}{4}x_0 + \frac{1}{4}x_1$$

$$y_k = \frac{1}{4}\left(x_{k-1} + 2x_k + x_{k+1}\right),\ k = 1,...,\xi - 1$$

$$y_\xi = \frac{1}{2}\left(\frac{1}{2}x_{\xi-1} + \frac{1}{2}x_\xi + \frac{1}{2}x_\xi\right) = \frac{1}{4}x_{\xi-1} + \frac{1}{2}x_\xi$$

The equations are the same as in the constant case; we simply have one more intermediate equation. With the original notation, we have:

$$\begin{cases} x(n+1,0) = \frac{3}{4}x(n,0) + \frac{1}{4}x(n,1) \\ x(n+1,k) = \frac{1}{4}x(n,k-1) + \frac{1}{2}x(n,k) + \frac{1}{4}x(n,k+1), \text{for } k = 1,...,\xi - 1 \\ x(n+1,\xi) = \frac{1}{4}x(n,\xi-1) + \frac{1}{2}x(n,\xi) \end{cases} \quad (11)$$

We have proved:

Proposition 1- *On the variables* $x(n,k)$, *the fact that the barrier is shifted one step higher leads simply to a new intermediate equation in the transition equations.*

This is quite important in practice, because it means that the theory developed in Part II will apply, despite the changes of position for the barrier. We have simply to take into account the fact that the matrix M_n will increase by one dimension at the transition between two periods and the corresponding eigenvalues will change accordingly.

We note here that this result applies to any transition, where the barrier is shifted one step up, and does not depend on the particular function $b(x)$.

2. Changes in the eigenvalues and in the eigenvectors

We now work constantly with the variables $X_n = x(n,i)$. In dimension ξ, we know (see Part II) that the eigenvalues are of the form:

$$\lambda_j = \frac{1+\cos\left(\vartheta_{\xi,j}\right)}{2} = \cos^2\left(\frac{\vartheta_{\xi,j}}{2}\right), \text{ with } \vartheta_{\xi,j} = \frac{2j-1}{2\xi+1}\pi,\ j = 1,...,\xi,$$

and the same expressions will remain in dimension $\xi+1$, with ξ replaced by $\xi+1$.

In dimension ξ, the eigenvectors were:

$$V_{\xi,j}=\left(\sin\left(\xi\vartheta_{\xi,j}\right),\dots,\sin\left(\vartheta_{\xi,j}\right)\right),\ j=1,\dots,\xi.$$

Recall that:

$$\left|V_{\xi,j}\right|_2^2=\frac{\xi}{2}+\frac{1}{4},\ s\left(V_{\xi,j}\right)=\frac{1}{2}\frac{1}{\tan\left(\frac{\vartheta_{\xi,j}}{2}\right)},$$

where $s(V)$ is the sum of the components of the vector V. In dimension $\xi+1$, they will be:

$$V_{\xi+1,j}=\left(\sin\left(\left(\xi+1\right)\vartheta_{\xi+1,j}\right),\sin\left(\xi\vartheta_{\xi+1,j}\right),\dots,\sin\left(\vartheta_{\xi+1,j}\right)\right),\ j=1,\dots,\xi+1,\ \text{and:}$$

$$\left|V_{\xi+1,j}\right|_2^2=\frac{\xi}{2}+\frac{3}{4},\ s\left(V_{\xi+1,j}\right)=\frac{1}{2}\frac{1}{\tan\left(\frac{\vartheta_{\xi+1,j}}{2}\right)},$$

where $s(V)$ is the sum of the components of the vector.

We first investigate an upper estimate: what properties of the barrier will ensure that the energy left at the n^{th} stage tends to 0 when $n\to+\infty$? Recall that we always work on the variables $x(n,k)$.

II. Upper estimates

We will show:

Theorem. 1 - *Let l_n be the length of the n^{th} period. If the series $\sum_{n=1}^{+\infty}\frac{l_n}{n^2}$ is divergent, then the probability E_N that the game continues after n steps tends to 0 when $n\to+\infty$.*

Proof of Theorem 1

We start with an energy equal to 1 at the origin. Initially, the barrier is set at some arbitrary value $2n_0+1$, and remains so until some time t_1 is reached. Then, the barrier increases to $2n_0+3$, and so on. More generally, the barrier takes the value $2n+1$ at the n^{th} stage, which starts at t_n and finishes at t_{n+1}; its duration is $l_n=t_{n+1}-t_n$.

Let us first see how to deal with the first stage. As we saw in Part II, at time t_1 the energy 1 at O becomes the vector X_1 with components:

$$x(t_1,i)=\frac{1}{2^{2t_1}}\binom{2t_1+1}{t_1+i} \tag{1}$$

The total amount of energy is still 1: there is no loss during the first t_1 steps.

Let M be the operator reflecting the propagation of energy. It operates first on a space of dimension n_0, then n_0+1, and so on (we keep the same notation, independent of the dimension).

The vector X_1 does not have a satisfactory shape, in the sense that we have no information at all on the iterates $M^n X_1$; we have such an information only for the eigenvectors of the matrix M (and these eigenvectors depend on the dimension, of course). Therefore, we want to replace X_1 by the vector $V_{0,1}$, first eigenvector of the matrix M in dimension n_0. We know that:

$$V_{0,1}=\frac{2\left(\sin\left(n_0\vartheta_0\right),\ldots,\sin\left(\vartheta_0\right)\right)}{\tan\left(n_0\vartheta_0\right)} \tag{2}$$

with $\vartheta_0=\dfrac{\pi}{2n_0+1}$. This vector is normalized in l_1 norm: it has positive coefficients, with sum equal to 1.

The replacement of the vector X_1 by $V_{0,1}$ is done using the following Lemma:

Lemma 2 . - *For every* n, $\left|M^n X_1\right|_1 \le c\left|M^n V_{0,1}\right|_1$ with $c=\dfrac{\binom{2n_0+1}{n_0+1}}{2^{2n_0}\cos\left(\vartheta_0\right)}$.

Proof of Lemma 2

The first coefficient of X_1 is $x(1,1)=\dfrac{1}{2^{2n_0+1}}\dbinom{2n_0+1}{n_0+1}$. We have $X_1(i)\le cV_{0,1}(i)$ for every $i=1,\ldots,n_0$, with this choice of c. The operator M has positive coefficients, so it respects the order: during each period, we will have $\left|M^n X_1\right|_1 \le c\left|M^n V_{0,1}\right|_1$, because the l_1 norm is simply the sum of all coefficients (the coefficients are positive). This proves Lemma 2.

The replacement of the vector X_1 by $V_{0,1}$ is not important in the process, because it is done only once. From now on, we constantly work with eigenvectors.

Assume now we have reached the end of the n^{th} period and start we $n+1^{st}$ period. Assume we have obtained an information about the energy carried by the first eigenvector in dimension $n,$ that is $V_{n,1}$; we would like to pass it to the first eigenvector in dimension $n+1,$ that is $V_{n+1,1}$. Unfortunately, this is not possible directly: this transition is not a conservative move.

Indeed, we are now with an eigenvector, V_1, of the matrix M in dimension $n.$ Unfortunately, this vector, when we pass to dimension $n+1,$ does not become an eigenvector of the next matrix. More precisely (without normalization),

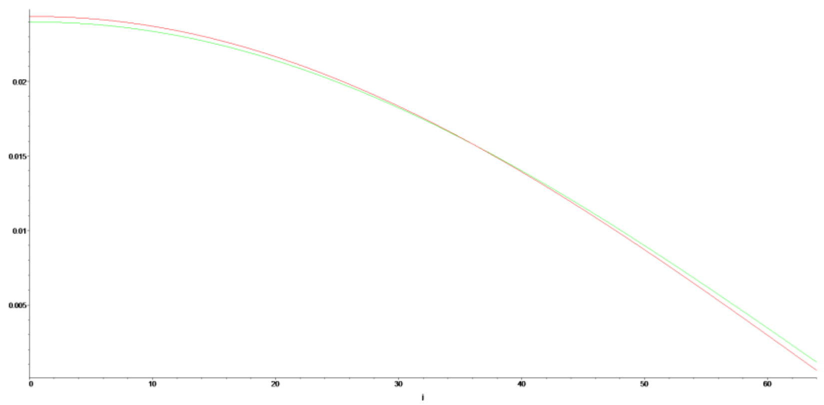
$V_1=\left(\sin\left(n\vartheta_1\right),...,\sin\left(\vartheta_1\right)\right)$ (n coordinates), with $\vartheta_1=\dfrac{\pi}{2n+1}$, becomes, after embedding :

$V_1'=\left(\sin\left(n\vartheta_1\right),...,\sin\left(\vartheta_1\right),0\right)$ ($n+1$ coordinates),

whereas the first eigenvector of the new matrix is:

$W_1=\left(\sin\left(\left(n+1\right)\vartheta_2\right),...,\sin\left(2\vartheta_2\right),\sin\left(\vartheta_2\right)\right)$ ($n+1$ coordinates), with $\vartheta_2=\dfrac{\pi}{2n+3}$.

There is no simple connection between V_1' and W_1 : both are very close, as the following picture shows (V_1' is in red and W_1 is in green) :



Still, we cannot simply say that we replace V_1' by W_1, because W_1 is slightly closer to the barrier. We may compute the loss in this replacement, but the sum of such losses, over all transitions, in infinite, so such an approach, keeping only one eigenvector at each step, must be refined: we have to take into account the whole decomposition of V_1' on the basis of eigenvectors in dimension $n+1$.

So we need to compute the decomposition of the eigenvectors in dimension n on the basis made of the eigenvectors in dimension $n+1$. Quite surprisingly, there is a closed form for the coefficients, and this closed form is rather simple.

In order to simplify our notation, we write $V_i = V_{n,i}$ for the vectors in dimension n, embedded in the space of dimension $n+1$ (a zero is added as the last coordinate, see above), and $W_j = V_{n+1,j}$ for the eigenvectors in dimension $n+1$.

Recall that the i^{th} first vector, normalized so that $s(V_i)=1$, imbedded in dimension $n+1$, is:

$$V_i = 2\tan\frac{\vartheta_i}{2}\left(\sin(n\vartheta_i),...,\sin(\vartheta_i),0\right), \text{ with } \vartheta_i = \frac{(2i-1)\pi}{2n+1}.$$

The eigenvectors of the next step are:

$$W_j = 2\tan\frac{\eta_j}{2}\left(\sin((n+1)\eta_j),...,\sin(\eta_j)\right),\ j=1,...,n+1,\ \eta_j = \frac{2j-1}{2n+3}\pi$$

With this normalization,

$$|W_j|_2^2 = 4\tan^2\frac{\eta_j}{2}\sum_{k=1}^{n+1}\sin^2(k\eta_j) = \left(\frac{n+1}{2}+\frac{1}{4}\right)4\tan^2\frac{\eta_j}{2} = (2n+3)\tan^2\frac{\eta_j}{2}.$$

Proposition 3. - *Let* $\alpha_{i,j}$, $j=1,...,n+1$ *be the coefficients of the decomposition of* V_i *on the basis* W_j. *Then we have the closed form:*

$$\alpha_{i,j} = \frac{2}{2n+3}\frac{\left(1-\cos\left(\frac{2i-1}{2n+1}\pi\right)\right)\left(1+\cos\left(\frac{2j-1}{2n+3}\pi\right)\right)}{\cos\left(\frac{2j-1}{2n+3}\pi\right)-\cos\left(\frac{2i-1}{2n+1}\pi\right)}$$

Proof of Proposition 3

We write the orthogonal decomposition of V_i on the basis of eigenvectors W_j in dimension $n+1$:

$$V_i = \sum_{j=1}^{n+1}\alpha_{i,j}W_j = \sum_{j=1}^{n+1}\frac{\langle V_i,W_j\rangle}{|W_j|_2^2}W_j = \frac{1}{2n+3}\sum_{j=1}^{n+1}\frac{\langle V_i,W_j\rangle}{\tan^2\frac{\eta_j}{2}}W_j \qquad (1)$$

We have:

$$\langle V_i, W_j \rangle = \sum_{l=1}^{n+1} V_i(l) W_j(l) = 4 \tan\frac{\vartheta_i}{2} \tan\frac{\eta_j}{2} \sum_{l=1}^{n} \left(\sin\left((n-l+1)\vartheta_i\right)\right)\left(\sin\left((n-l+2)\eta_j\right)\right)$$

We know (cf. Part II) that:

$$\sin\left((n-l+1)\vartheta_i\right) = (-1)^{i-1} \cos\frac{(2l-1)\vartheta_i}{2}$$

and:

$$\sin\left((n-l+2)\eta_j\right) = (-1)^{j-1} \cos\frac{(2l-1)\eta_j}{2}$$

which gives:

$$\alpha_{i,j} = \frac{\langle V_i, W_j \rangle}{\left|W_j\right|_2^2} = \frac{4 \tan\frac{\vartheta_i}{2} \tan\frac{\eta_j}{2}}{(2n+3)\tan^2\frac{\eta_j}{2}} \sum_{l=1}^{n} \left(\sin\left((n-l+1)\vartheta_i\right)\right)\left(\sin\left((n-l+2)\eta_j\right)\right)$$

$$\alpha_{i,j} = (-1)^{i+j} \frac{4}{2n+3} \frac{\tan\left((2i-1)t_1\right)}{\tan\left((2j-1)t_2\right)} \sum_{l=1}^{n} \cos\left((2l-1)(2i-1)t_1\right) \cos\left((2l-1)(2j-1)t_2\right)$$

Since, with this normalization, $s(W_j) = 1$ for all j, α_j represents the part of energy carried by W_j.

We write, with $t_1 = \frac{1}{2n+1}\frac{\pi}{2}$, $t_2 = \frac{1}{2n+3}\frac{\pi}{2}$:

$$\cos\left((2l-1)(2i-1)t_1\right)\cos\left((2l-1)(2j-1)t_2\right) =$$
$$= \frac{1}{2}\cos\left((2l-1)\left((2i-1)t_1 - (2j-1)t_2\right)\right) + \frac{1}{2}\cos\left((2l-1)\left((2i-1)t_1 + (2j-1)t_2\right)\right)$$

So $\alpha_{i,j} = A_{i,j} + B_{i,j}$ with:

$$A_{i,j} = (-1)^{i+j} \frac{2}{2n+3} \frac{\tan\left((2i-1)t_1\right)}{\tan\left((2j-1)t_2\right)} \sum_{l=1}^{n} \cos\left((2l-1)\left((2i-1)t_1 - (2j-1)t_2\right)\right)$$

$$B_{i,j} = (-1)^{i+j} \frac{2}{2n+3} \frac{\tan\left((2i-1)t_1\right)}{\tan\left((2j-1)t_2\right)} \sum_{l=1}^{n} \cos\left((2l-1)\left((2i-1)t_1 + (2j-1)t_2\right)\right)$$

We use the identity:

$$\sum_{l=1}^{n} \cos\left((2l-1)t\right) = \frac{1}{2}\frac{\sin(2tn)}{\sin(t)}$$

So we get:

$$A_{i,j} = (-1)^{i+j}\frac{1}{2n+3}\frac{\tan\left((2i-1)t_1\right)}{\tan\left((2j-1)t_2\right)}\frac{\sin\left(2n\left((2i-1)t_1-(2j-1)t_2\right)\right)}{\sin\left(\left((2i-1)t_1-(2j-1)t_2\right)\right)}$$

$$B_{i,j} = (-1)^{i+j}\frac{1}{2n+3}\frac{\tan\left((2i-1)t_1\right)}{\tan\left((2j-1)t_2\right)}\frac{\sin\left(2n\left((2i-1)t_1+(2j-1)t_2\right)\right)}{\sin\left(\left((2i-1)t_1+(2j-1)t_2\right)\right)}$$

One checks easily that:

$$2n\left((2i-1)t_1-(2j-1)t_2\right) = (i-j)\pi-(2i-1)t_1+3(2j-1)t_2$$

$$2n\left((2i-1)t_1+(2j-1)t_2\right) = (i+j-1)\pi-(2i-1)t_1-3(2j-1)t_2$$

$$\sin\left(2n\left((2i-1)t_1-(2j-1)t_2\right)\right) = (-1)^{i-j}\sin\left(-(2i-1)t_1+3(2j-1)t_2\right)$$

$$\sin\left(2n\left((2i-1)t_1+(2j-1)t_2\right)\right) = (-1)^{i+j}\sin\left((2i-1)t_1+3(2j-1)t_2\right)$$

so we obtain:

$$A_{i,j} = \frac{1}{2n+3}\frac{\tan\left((2i-1)t_1\right)}{\tan\left((2j-1)t_2\right)}\frac{\sin\left(-(2i-1)t_1-3(2j-1)t_2\right)}{\sin\left(\left((2i-1)t_1-(2j-1)t_2\right)\right)}$$

$$B_{i,j} = \frac{1}{2n+3}\frac{\tan\left((2i-1)t_1\right)}{\tan\left((2j-1)t_2\right)}\frac{\sin\left((2i-1)t_1+3(2j-1)t_2\right)}{\sin\left(\left((2i-1)t_1+(2j-1)t_2\right)\right)}$$

and therefore:

$$\alpha_{i,j} = \frac{1}{2n+3}\frac{\tan\left((2i-1)t_1\right)}{\tan\left((2j-1)t_2\right)}\left(\frac{\sin\left(-(2i-1)t_1+3(2j-1)t_2\right)}{\sin\left((2i-1)t_1-(2j-1)t_2\right)}+\frac{\sin\left((2i-1)t_1+3(2j-1)t_2\right)}{\sin\left(\left((2i-1)t_1+(2j-1)t_2\right)\right)}\right)$$

After simplification, we obtain:

$$\alpha_{i,j}=\frac{2}{2n+3}\frac{\tan\left((2i-1)t_1\right)}{\tan\left((2j-1)t_2\right)}\frac{\sin\left(2(2j-1)t_2\right)\sin\left(2(2i-1)t_1\right)}{\cos\left(2(2j-1)t_2\right)-\cos\left(2(2i-1)t_1\right)}$$
$$=\frac{8}{2n+3}\frac{\cos^2\left((2j-1)t_2\right)\sin^2\left((2i-1)t_1\right)}{\cos\left(2(2j-1)t_2\right)-\cos\left(2(2i-1)t_1\right)}$$

and finally:

$$\alpha_{i,j}=\frac{2}{2n+3}\frac{\left(1+\cos\left(2(2j-1)t_2\right)\right)\left(1-\cos\left(2(2i-1)t_1\right)\right)}{\cos\left(2(2j-1)t_2\right)-\cos\left(2(2i-1)t_1\right)}$$

which proves Proposition 3.

The sign of $\alpha_{i,j}$ is the sign of $\cos\left(2(2j-1)t_2\right)-\cos\left(2(2i-1)t_1\right)$; it is positive if:

$$\cos\left((2j-1)\frac{\pi}{2n+3}\right)\geq\cos\left((2i-1)\frac{\pi}{2n+1}\right),\text{ that is }\frac{2j-1}{2n+3}\leq\frac{2i-1}{2n+1}.$$

For a given vector V_i, this is satisfied for all $j\leq i$.

For $j=i+1$, the sign is negative, since $\dfrac{2(i+1)-1}{2n+3}>\dfrac{2i-1}{2n+1}$.

We deduce:

Corollary. 4. - *For the first eigenvector* V_1, *the first coefficient* $\alpha_{1,1}$ *is strictly positive, all others are strictly negative.*

In order to make a proper investigation, we have to start with the first eigenvector U_1 in dimension $n-1$, then decompose it on the basis of eigenvectors V_i in dimension n, then again decompose the V_i on the basis of eigenvectors W_j in dimension $n+1$.

Notation. - In dimension $n-1$, the eigenvectors will be denoted by U_j, $j=1,\dots,n-1$; the first one, normalized in order to have energy 1, is:

$$U_1=2\tan\left(\frac{\vartheta_1}{2}\right)\left(\sin\left((n-1)\vartheta_1\right),\sin\left((n-2)\vartheta_1\right),\dots,\sin\left(\vartheta_1\right)\right),\ \vartheta_1=\frac{\pi}{2n-1}$$

The associated eigenvalue is: $\lambda_1=\cos^2\dfrac{\vartheta_1}{2}$.

We have : $\left|U_1\right|_2^2=\tan^2\left(\dfrac{\vartheta_1}{2}\right)(2n-1)$.

In dimension n, with $\varphi_i = \frac{2i-1}{2n+1}\pi$, we have the eigenvectors $V_i, i = 1,\dots,n$:

$V_i = 2\tan\left(\frac{\varphi_i}{2}\right)\left(\sin(n\varphi_i), \sin((n-1)\varphi_i), \dots, \sin(\varphi_i)\right)$, with the eigenvalues $\mu_i = \cos^2\frac{\varphi_i}{2}$, and

$|V_i|_2^2 = \tan^2\left(\frac{\varphi_i}{2}\right)(2n+1)$.

From Proposition 3, we deduce the decomposition:

$$U_1 = \sum_{i=1}^{n} v_i V_i$$

where v_i is the energy carried by V_i ; we have, for $i = 1,\dots,n$:

$$v_i = \frac{2}{2n+1}\frac{\left(1-\cos\left(\frac{1}{2n-1}\pi\right)\right)\left(1+\cos\left(\frac{2i-1}{2n+1}\pi\right)\right)}{\cos\left(\frac{2i-1}{2n+1}\pi\right)-\cos\left(\frac{1}{2n-1}\pi\right)}.$$

Using the identity $1-\cos(A) = 2\sin^2\left(\frac{A}{2}\right)$, we may write v_i under the form:

$$v_i = \frac{4}{2n+1}\sin^2\left(\frac{1}{2n-1}\frac{\pi}{2}\right)\frac{1+\cos\left(\frac{2i-1}{2n+1}\pi\right)}{\cos\left(\frac{2i-1}{2n+1}\pi\right)-\cos\left(\frac{\pi}{2n-1}\right)}.$$

We saw above that $v_1 > 0$, $v_i < 0$ for $i = 2,\dots,n$. The function $f(x) = \frac{1+x}{\alpha - x}$, $x < \alpha$, satisfies $f'(x) = \frac{1+\alpha}{(\alpha-x)^2} > 0$ so f is increasing. Therefore $|v_i|$ is a decreasing function of $i = 2,\dots,n$.

We now turn to explicit computations of the energies carried by the V_i's :

– Explicit computation of v_1 :

$$v_1 = \frac{4}{2n+1}\sin^2\left(\frac{1}{2n-1}\frac{\pi}{2}\right)\frac{1+\cos\left(\frac{\pi}{2n+1}\right)}{\cos\left(\frac{\pi}{2n+1}\right)-\cos\left(\frac{\pi}{2n-1}\right)}$$

Indeed, using the identity:

$$\cos\left(\frac{\pi}{2n+1}\right)-\cos\left(\frac{\pi}{2n-1}\right)=2\sin\left(\frac{\pi}{4n^2-1}\right)\sin\left(\frac{2\pi n}{4n^2-1}\right)$$

we obtain:

$$v_1=\frac{2}{2n+1}\sin^2\left(\frac{1}{2n-1}\frac{\pi}{2}\right)\frac{1+\cos\left(\frac{\pi}{2n+1}\right)}{\sin\left(\frac{\pi}{4n^2-1}\right)\sin\left(\frac{2\pi n}{4n^2-1}\right)}$$

which can be written:

$$v_1=\frac{4}{2n+1}\sin^2\left(\frac{1}{2n-1}\frac{\pi}{2}\right)\frac{\cos^2\left(\frac{1}{2n+1}\frac{\pi}{2}\right)}{\sin\left(\frac{\pi}{4n^2-1}\right)\sin\left(\frac{2\pi n}{4n^2-1}\right)}$$

This gives the decomposition:

$$v_1=1+\frac{1}{2n}-\frac{\pi^2}{24n^2}+\frac{\pi^2}{48n^3}+O\left(\frac{1}{n^4}\right)$$

– Explicit computation of v_i for $i\geq 2$:

We know that $v_i<0$ for $i\geq 2$. Therefore:

$$|v_i|=\frac{8}{2n+1}\sin^2\left(\frac{1}{2n-1}\frac{\pi}{2}\right)\frac{\cos^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2n-1}\right)-\cos\left(\frac{2i-1}{2n+1}\pi\right)}$$

Using the identity:

$$\cos\left(\frac{\pi}{2n-1}\right)-\cos\left(\frac{2i-1}{2n+1}\pi\right)=2\left(\cos^2\left(\frac{1}{2n-1}\frac{\pi}{2}\right)-\cos^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)\right)$$

we obtain:

$$|v_i|=\frac{4}{2n+1}\sin^2\left(\frac{1}{2n-1}\frac{\pi}{2}\right)\frac{\cos^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\cos^2\left(\frac{1}{2n-1}\frac{\pi}{2}\right)-\cos^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}$$

which gives the representation:

$$|v_i| = \frac{1}{2i(i-1)n} + \frac{2i^2-2i+1}{4i^2(i-1)^2 n^2} - \frac{c(i)}{n^3}$$

with $0 < c(i) < 1$.

Therefore:

$$|v_i| \le \frac{1}{2i(i-1)n} + \frac{2i^2-2i+1}{4i^2(i-1)^2 n^2}$$

We know that $\sum_{i=1}^{n} v_i = 1$, so $\sum_{i=2}^{n} v_i = 1 - v_1 < 0$,

which gives:

$$\sum_{i=2}^{n} |v_i| = v_1 - 1 = \frac{1}{2n} - \frac{\pi^2}{24n^2} + \frac{\pi^2}{48n^3} + O\left(\frac{1}{n^4}\right)$$

when $n \to +\infty$.

The decomposition of v_1 looks strange at first sight: in order to pass from the first eigenvector in dimension $n-1$ to its homologue in dimension n, one has to multiply by a coefficient of the size $1+\frac{1}{2n}$; our result explains this, since all coefficients in the decomposition are negative. However, this corrective coefficient is the main technical difficulty in our approach.

3. Transition from step n to step $n+1$

We now study the transition from step n to step $n+1$. We write the decomposition of the i^{th} eigenvector V_i in dimension n on the basis of eigenvectors W_j in dimension $n+1$.

After embedding, V_i becomes:

$$V_i = 2\tan\frac{\vartheta_{n,i}}{2}\left(\sin(n\vartheta_{n,i}), \sin((n-1)\vartheta_{n,i}), \dots, \sin(\vartheta_{n,i}), 0\right), \ \vartheta_{n,i} = \frac{2i-1}{2n+1}\pi$$

The decomposition is:

$$V_i = \sum_{j=1}^{n+1} w_{i,j} W_j$$

where, for $i=1,\ldots,n,\quad j=1,\ldots,n+1$, by Proposition 3:

$$w_{i,j}=\frac{8}{2n+3}\frac{\sin^2\left(\frac{\vartheta_{n,i}}{2}\right)\cos^2\left(\frac{\vartheta_{n+1,j}}{2}\right)}{\cos\left(\vartheta_{n+1,j}\right)-\cos\left(\vartheta_{n,i}\right)}$$

We have the equivalent formulas:

$$w_{i,j}=\frac{4}{2n+3}\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)\frac{1+\cos\left(\frac{2j-1}{2n+3}\pi\right)}{\cos\left(\frac{2j-1}{2n+3}\pi\right)-\cos\left(\frac{2i-1}{2n+1}\pi\right)}$$

$$=\frac{8}{2n+3}\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)\frac{\cos^2\left(\frac{2j-1}{2n+3}\frac{\pi}{2}\right)}{\cos\left(\frac{2j-1}{2n+3}\pi\right)-\cos\left(\frac{2i-1}{2n+1}\pi\right)}$$

and:

$$w_{i,j}=\frac{4}{2n+3}\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)\frac{\cos^2\left(\frac{2j-1}{2n+3}\frac{\pi}{2}\right)}{\cos^2\left(\frac{2j-1}{2n+3}\frac{\pi}{2}\right)-\cos^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}$$

The coefficient $w_{i,j}$ may be viewed as the part of the energy on W_j which comes from V_i. This is a transfer coefficient.

– Estimates of $w_{i,j}$:

If $j\le i$, $w_{i,j}>0$; if $j>i$, $w_{i,j}<0$.

For $i=j$:

$$w_{i,i}=1+\frac{1}{2n}-\frac{1}{24}\frac{(2i-1)^2\pi^2}{n^2}+O\left(\frac{1}{n^3}\right)$$

For $i\ne j$, we have the Taylor expansion:

$$w_{i,j}=\frac{1}{2}\frac{(2i-1)^2}{i(i-1)-j(j-1)}\frac{1}{n}-\frac{1}{4}\frac{(2i-1)^2\left(3i^2-3i+j^2-j+1\right)}{i(i-1)-j(j-1)}\frac{1}{n^2}+O\left(\frac{1}{n^3}\right)$$

For $i>j$, $w_{i,j}>0$ and we have the upper estimate:

$$w_{i,j} \le \frac{1}{2}\frac{(2i-1)^2}{i(i-1)-(i-1)(i-2)}\frac{1}{n} \le \frac{1}{2}\frac{(2i-1)^2}{(i-1)(i+2)}\frac{1}{n} \le \frac{2}{n}$$

For $i < j,\ w_{i,j} < 0$ and we have the upper estimate:

$$|w_{i,j}| \le \frac{1}{2}\frac{(2i-1)^2}{j(j-1)-i(i-1)}\frac{1}{n}$$

This is an increasing function of i, and therefore:

$$|w_{i,j}| \le |w_{j-1,j}| \le \frac{1}{2}\frac{(2j-3)^2}{j(j-1)-(j-1)(j-3)}\frac{1}{n} = \frac{1}{6}\frac{(2j-3)^2}{(j-1)}\frac{1}{n} \le \frac{1}{6}\frac{(2n-3)^2}{(n-1)}\frac{1}{n} < 1.$$

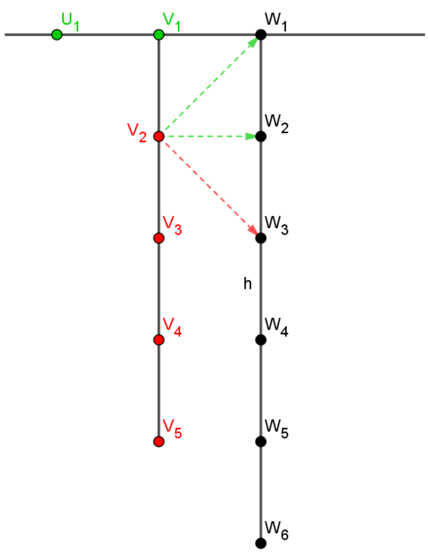
If V_i carries the energy v_i, the vector W_j will carry the energy $v_i \times w_{i,j}$ coming from V_i.

The first approximation is:

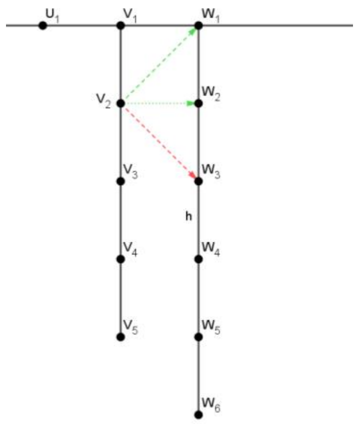
$$w_{i,j} \simeq \frac{-1}{2i(i-1)n}\frac{1}{2}\frac{(2i-1)^2}{i(i-1)-j(j-1)}\frac{1}{n} = \frac{1}{4i(i-1)}\frac{(2i-1)^2}{j(j-1)-i(i-1)}\frac{1}{n^2}$$

– Sign of transfer coefficients

Let us summarize what we found; this is the sign of energies at step n ; green: positive energies, red: negative energies.



and this is the sign of transfer coefficients (green is positive, red is negative):



4. **Introducing the attenuation during each period**

If the n^{th} period has length l_n, each eigenvector V_i is replaced, at the end of the period, by $\lambda_{n,i}^{l_n} V_i$, where $\lambda_{n,i} = \cos^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)$, by definition of an eigenvector. At the end of the n^{th} period, the energy carried by V_i is now $\lambda_{n,i}^{l_n} v_i = \cos^{2l_n}\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right) v_i$.

5. **Organization of the proof**

Let us now give an overall presentation of the idea of the proof. We start with an energy 1 on U_1, dimension $n-1$. Assume that we can prove that, in dimension $n+1$, the energy carried by W_1 is $\leq 1 - C\frac{l_n}{n^2}$ with a constant C independent from n and that:

– up to some j_0, all energies carried by $W_2, ..., W_{j_0}$ are negative;

– the energies carried by $W_{j_0+1}, ..., W_{n+1}$ are positive, but small, namely they satisfy:

$\sum_{j=j_0+1}^{n+1} w_j < c\frac{Log(n)}{n^2}$ for some constant c independent of n.

Then, we forget about the negative energies, and penalize ourselves: we bring the positive energies of $W_{j_0+1}, ..., W_{n+1}$ to W_1 ; this way, the total energy at step $n+1$ will be $\leq 1 - C\frac{l_n}{n^2}$, carried by the first eigenvector, W_1. Iterating the argument, at step $2N$, the energy will satisfy:

$$E_{2N} \leq \prod_{n=1}^{N} \delta_n$$

where δ_n is the transfer coefficient from $n-1$ to $n+1$.

We will have:

$$E_{2N} \leq \prod_{n=1}^{N} \left(1 - C\frac{l_n}{n^2}\right)$$

and:

$$Log\left(E_{2N}\right) \leq \sum_{n=1}^{N} Log\left(1 - C\frac{l_n}{n^2}\right) \sim -C\sum_{n=1}^{N} \frac{l_n}{n^2} \to -\infty \text{ when } N \to +\infty \text{ and } E_{2N} \to 0.$$

In order to achieve this program, we now need to compute precisely the energy carried by each vector W_j.

6. Energy carried by each W_j

We now compute the energy actually carried by each vector W_j, taking the attenuation into account. It is the sum of all energies sent by each V_i; therefore, with $l = l_n$:

$$w_j = \frac{8}{2n+1}\frac{8}{2n+3}\sum_{i=1}^{n} \frac{\sin^2\left(\frac{1}{2n-1}\frac{\pi}{2}\right)\cos^{2+2l}\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\cos\left(\frac{2i-1}{2n+1}\pi\right) - \cos\left(\frac{\pi}{2n-1}\right)} \frac{\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)\cos^2\left(\frac{2j-1}{2n+3}\frac{\pi}{2}\right)}{\cos\left(\frac{2j-1}{2n+3}\pi\right) - \cos\left(\frac{2i-1}{2n+1}\pi\right)}$$

which can be written:

$$w_j = \frac{8}{2n+1}\frac{8}{2n+3}\sin^2\left(\frac{1}{2n-1}\frac{\pi}{2}\right)\cos^2\left(\frac{2j-1}{2n+3}\frac{\pi}{2}\right) \times$$
$$\times \sum_{i=1}^{n} \frac{\cos^{2+2l}\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\cos\left(\frac{2i-1}{2n+1}\pi\right) - \cos\left(\frac{\pi}{2n-1}\right)} \frac{\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\cos\left(\frac{2j-1}{2n+3}\pi\right) - \cos\left(\frac{2i-1}{2n+1}\pi\right)}$$

or:

$$w_j = \frac{4}{2n+1}\frac{4}{2n+3}\sin^2\left(\frac{1}{2n-1}\frac{\pi}{2}\right) \times$$
$$\times \sum_{i=1}^{n} \sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right) \frac{1+\cos\left(\frac{2i-1}{2n+1}\pi\right)}{\cos\left(\frac{2i-1}{2n+1}\pi\right) - \cos\left(\frac{\pi}{2n-1}\right)} \frac{1+\cos\left(\frac{2j-1}{2n+3}\pi\right)}{\cos\left(\frac{2j-1}{2n+3}\pi\right) - \cos\left(\frac{2i-1}{2n+1}\pi\right)} \cos^{2l}\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)$$

First, we study the energy carried by the first vector, W_1:

Proposition 5. – *When* $n \to +\infty$, *we have:*

$$w_1 \sim 1 - \frac{\pi^2}{16nLog(n)} - r_n, \ \ with \ \ 0 < r_n < \frac{2\pi^2}{n^2}.$$

Proof of Proposition 5

We have:

$$w_1 = \frac{4}{2n+1}\frac{4}{2n+3}\sin^2\left(\frac{1}{2n-1}\frac{\pi}{2}\right)\times$$
$$\times\sum_{i=1}^{n} \sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)\frac{1+\cos\left(\frac{2i-1}{2n+1}\pi\right)}{\cos\left(\frac{2i-1}{2n+1}\pi\right)-\cos\left(\frac{\pi}{2n-1}\right)}\frac{1+\cos\left(\frac{1}{2n+3}\pi\right)}{\cos\left(\frac{1}{2n+3}\pi\right)-\cos\left(\frac{2i-1}{2n+1}\pi\right)}\cos^{2l}\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)$$

This energy is the sum of two terms: the term for $i=1$, which is positive (contribution of V_1) and the sum for $i \geq 2$, contributions of V_j, $j = 2,...,n$, which are all negative. So we write $w_1 = a + s$, with:

$$a = \frac{4}{2n+1}\frac{4}{2n+3}\sin^2\left(\frac{1}{2n-1}\frac{\pi}{2}\right)\sin^2\left(\frac{1}{2n+1}\frac{\pi}{2}\right)\times$$
$$\times\frac{1+\cos\left(\frac{1}{2n+1}\pi\right)}{\cos\left(\frac{1}{2n+1}\pi\right)-\cos\left(\frac{\pi}{2n-1}\right)}\frac{1+\cos\left(\frac{1}{2n+3}\pi\right)}{\cos\left(\frac{1}{2n+3}\pi\right)-\cos\left(\frac{1}{2n+1}\pi\right)}\cos^{2l}\left(\frac{1}{2n+1}\frac{\pi}{2}\right)$$

$$s = \frac{4}{2n+1}\frac{4}{2n+3}\sin^2\left(\frac{1}{2n-1}\frac{\pi}{2}\right)\times$$
$$\times\sum_{i=2}^{n} \sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)\frac{1+\cos\left(\frac{2i-1}{2n+1}\pi\right)}{\cos\left(\frac{2i-1}{2n+1}\pi\right)-\cos\left(\frac{\pi}{2n-1}\right)}\frac{1+\cos\left(\frac{1}{2n+3}\pi\right)}{\cos\left(\frac{1}{2n+3}\pi\right)-\cos\left(\frac{2i-1}{2n+1}\pi\right)}\cos^{2l}\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)$$

We first compute an estimate for a:

$$a=\frac{4}{2n+1}\frac{4}{2n+3}\sin^2\left(\frac{1}{2n-1}\frac{\pi}{2}\right)\sin^2\left(\frac{1}{2n+1}\frac{\pi}{2}\right)\times$$

$$\times\frac{1+\cos\left(\frac{1}{2n+1}\pi\right)}{\cos\left(\frac{1}{2n+1}\pi\right)-\cos\left(\frac{\pi}{2n-1}\right)}\frac{1+\cos\left(\frac{1}{2n+3}\pi\right)}{\cos\left(\frac{1}{2n+3}\pi\right)-\cos\left(\frac{1}{2n+1}\pi\right)}\cos^{2l}\left(\frac{1}{2n+1}\frac{\pi}{2}\right)$$

$$\sim\frac{\pi^4}{64n^6}\frac{1+\cos\left(\frac{1}{2n+1}\pi\right)}{\cos\left(\frac{1}{2n+1}\pi\right)-\cos\left(\frac{\pi}{2n-1}\right)}\frac{1+\cos\left(\frac{1}{2n+3}\pi\right)}{\cos\left(\frac{1}{2n+3}\pi\right)-\cos\left(\frac{1}{2n+1}\pi\right)}\cos^{2l}\left(\frac{1}{2n+1}\frac{\pi}{2}\right)$$

But:

$$\cos\left(\frac{\pi}{2n+1}\right)-\cos\left(\frac{\pi}{2n-1}\right)=2\sin\left(\frac{\pi}{4n^2-1}\right)\sin\left(\frac{2\pi n}{4n^2-1}\right)\sim2\left(\frac{\pi}{4n^2}\right)\left(\frac{2\pi n}{4n^2}\right)=\frac{\pi^2}{4n^3}$$

$$\cos\left(\frac{\pi}{2n+3}\right)-\cos\left(\frac{\pi}{2n+1}\right)=2\sin\left(\frac{\pi}{(2n+1)(2n+3)}\right)\sin\left(\frac{2\pi(n+1)}{(2n+1)(2n+3)}\right)\sim2\left(\frac{\pi}{4n^2}\right)\left(\frac{2\pi n}{4n^2}\right)=\frac{\pi^2}{4n^3}$$

and therefore:

$$a\sim\frac{\pi^4}{64n^6}\frac{1+\cos\left(\frac{1}{2n}\pi\right)}{\frac{\pi^2}{4n^3}}\frac{1+\cos\left(\frac{1}{2n}\pi\right)}{\frac{\pi^2}{4n^3}}\cos^{2l}\left(\frac{1}{4n}\pi\right)=\left(\frac{1+\cos\left(\frac{1}{2n}\pi\right)}{2}\right)^2\cos^{2l}\left(\frac{1}{4n}\pi\right)=\cos^{4+2l}\left(\frac{\pi}{4n}\right)$$

We observe that the sequence $\frac{l_n}{n^2}\to0,$ when $n\to+\infty.$ Indeed, we investigate barriers which are above $b(x)=\sqrt{x},$ for which this convergence already holds.

The link between the cosine above and l_n is made by the following Lemma:

Lemma 6. – *The divergence of the series with general term* $\frac{l_n}{n^2}$ *is equivalent to the divergence of the series with general term* $\alpha_n=1-\cos^{4+4l_n}\left(\frac{\pi}{4n}\right)$. *More precisely, for any* $\varepsilon>0$, *there exists* $n_0>1$ *such that, if* $n>n_0$*:*

$$1-\frac{(1+\varepsilon)\pi^2}{8}\frac{l_n}{n^2}\leq\cos^{4+4l_n}\left(\frac{\pi}{4n}\right)\leq1-\frac{(1-\varepsilon)\pi^2}{8}\frac{l_n}{n^2}$$

Proof of Lemma 6

We observe that $f(k) = -\frac{2k^2}{\pi^2} Log\left(\cos\left(\frac{\pi}{k}\right)\right)$ is a positive, decreasing function of k, which tends to 1 when $k \to +\infty$; let $\varepsilon > 0$, we have, for n large enough:

$$1 \leq -\frac{32n^2}{\pi^2} Log\left(\cos\left(\frac{\pi}{4n}\right)\right) \leq 1+\varepsilon$$

which gives:

$$-\frac{\pi^2}{8n^2}(1+l_n) \geq (4+4l_n) Log\left(\cos\left(\frac{\pi}{4n}\right)\right) \geq -(1+\varepsilon)\frac{\pi^2}{8n^2}(1+l_n)$$

and:

$$1-\exp\left\{-\frac{\pi^2}{8n^2}(1+l_n)\right\} \leq 1-\exp\left\{(4+4l_n) Log\left(\cos\left(\frac{\pi}{4n}\right)\right)\right\} \leq 1-\exp\left\{-(1+\varepsilon)\frac{\pi^2}{8n^2}(1+l_n)\right\}$$

that is:

$$1-\exp\left\{-\frac{\pi^2}{8n^2}(1+l_n)\right\} \leq 1-\cos^{4+4l_n}\left(\frac{\pi}{4n}\right) \leq 1-\exp\left\{-(1+\varepsilon)\frac{\pi^2}{8n^2}(1+l_n)\right\}$$

We have $1-\varepsilon < \frac{1-\exp(-x)}{x} < 1$ for $x > 0$ small enough, that is:

$$(1-\varepsilon)x < 1-\exp(-x) < x$$

from which we deduce:

$$(1-\varepsilon)\frac{\pi^2}{8n^2}(1+l_n) \leq 1-\cos^{4+4l_n}\left(\frac{\pi}{4n}\right) \leq (1+\varepsilon)\frac{\pi^2}{8n^2}(1+l_n)$$

which proves Lemma 6.

Therefore, $a \sim \cos^{4+4l_n}\left(\frac{\pi}{4n}\right) \sim 1 - \frac{\pi^2}{8}\frac{l_n}{n^2}$.

We now turn to an evaluation of s. All terms in s are negative. We have:

$$|s| \sim \frac{\pi^2}{n^4}\cos^2\left(\frac{1}{2n+3}\frac{\pi}{2}\right)\sum_{i=2}^{n} \frac{\cos^{2+2l}\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2n-1}\right)-\cos\left(\frac{2i-1}{2n+1}\pi\right)} \frac{\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\cos\left(\frac{1}{2n+3}\pi\right)-\cos\left(\frac{2i-1}{2n+1}\pi\right)}$$

which gives:

$$|s| \leq \frac{\pi^2}{n^4}\sum_{i=2}^{n} \frac{\cos^{2+2l}\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2n-1}\right)-\cos\left(\frac{2i-1}{2n+1}\pi\right)} \frac{\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\cos\left(\frac{1}{2n+3}\pi\right)-\cos\left(\frac{2i-1}{2n+1}\pi\right)}$$

Lemma 7. – *For all* $n \geq 4$ *and* $i \geq 2$, *we have:*

$$\frac{\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2n-1}\right)-\cos\left(\frac{2i-1}{2n+1}\pi\right)} \leq 1.$$

Proof of Lemma 7

We write:

$$\frac{\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2n-1}\right)-\cos\left(\frac{2i-1}{2n+1}\pi\right)} = \frac{\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2n-1}\right)-1+2\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}$$

The function $\frac{x}{\alpha+2x}$, $\alpha<0$, is decreasing in x ; so the maximum is attained for $i=2$; its value is:

$$\frac{\sin^2\left(\frac{3}{2n+1}\frac{\pi}{2}\right)}{\cos\left(\frac{\pi}{2n-1}\right)-\cos\left(\frac{3\pi}{2n+1}\right)}$$

We need to show that:

$$\sin^2\left(\frac{3}{2n+1}\frac{\pi}{2}\right) \leq \cos\left(\frac{\pi}{2n-1}\right)-\cos\left(\frac{3\pi}{2n+1}\right)$$

But:

$$\cos\left(\frac{\pi}{2n-1}\right)-\cos\left(\frac{3\pi}{2n+1}\right)=2\sin\left(\frac{1}{2}\frac{3\pi}{2n+1}-\frac{1}{2}\frac{\pi}{2n-1}\right)\sin\left(\frac{1}{2}\frac{3\pi}{2n+1}+\frac{1}{2}\frac{\pi}{2n-1}\right)$$

and:

$$\sin\left(\frac{1}{2}\frac{3\pi}{2n+1}+\frac{1}{2}\frac{\pi}{2n-1}\right)\geq\sin\left(\frac{1}{2}\frac{3\pi}{2n+1}\right)$$

We need to show that:

$$2\sin\left(\frac{1}{2}\frac{3\pi}{2n+1}-\frac{1}{2}\frac{\pi}{2n-1}\right)\geq\sin\left(\frac{1}{2}\frac{3\pi}{2n+1}\right)$$

But the function $\frac{\sin(x)}{x}$ is decreasing ; for $x<\frac{\pi}{4}$, we have $\frac{\sin(x)}{x}\geq\frac{\sin\left(\frac{\pi}{4}\right)}{\frac{\pi}{4}}=\frac{2\sqrt{2}}{\pi}$, therefore:

$$2\sin\left(\frac{1}{2}\frac{3\pi}{2n+1}-\frac{1}{2}\frac{\pi}{2n-1}\right)\geq\frac{4\sqrt{2}}{\pi}\left(\frac{1}{2}\frac{3\pi}{2n+1}-\frac{1}{2}\frac{\pi}{2n-1}\right)\geq\frac{1}{2}\frac{3\pi}{2n+1}$$ as soon as

$$\left(\frac{4\sqrt{2}}{\pi}-1\right)\frac{3\pi}{4\sqrt{2}}\geq\frac{2n+1}{2n-1}$$

Since $\left(\frac{4\sqrt{2}}{\pi}-1\right)\frac{3\pi}{4\sqrt{2}}\geq 1.33$, this happens when $n\geq 4$. This proves Lemma 7, since: $\frac{1}{2}\frac{3\pi}{2n+1}\geq\sin\left(\frac{1}{2}\frac{3\pi}{2n+1}\right)$. We return to the proof of Proposition 5.

From Lemma 7 follows that:

$$|s|\leq\frac{\pi^2}{n^4}\sum_{i=2}^{n}\frac{\cos^{2+2l}\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\cos\left(\frac{1}{2n+3}\pi\right)-\cos\left(\frac{2i-1}{2n+1}\pi\right)}$$

But:

$$\cos\left(\frac{1}{2n+3}\pi\right)-\cos\left(\frac{2i-1}{2n+1}\pi\right)=2\sin\left(\frac{2i-1}{2n+1}\frac{\pi}{2}-\frac{1}{2n+3}\frac{\pi}{2}\right)\sin\left(\frac{2i-1}{2n+1}\frac{\pi}{2}+\frac{1}{2n+3}\frac{\pi}{2}\right)$$

which gives:

$$|s| \le \frac{\pi^2}{2n^4} \sum_{i=2}^{n} \frac{\cos^{2+2l}\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\sin\left(\frac{2i-1}{2n+1}\frac{\pi}{2} - \frac{1}{2n+3}\frac{\pi}{2}\right)\sin\left(\frac{2i-1}{2n+1}\frac{\pi}{2} + \frac{1}{2n+3}\frac{\pi}{2}\right)}$$

We have:

$$\begin{aligned}\sin\left(\frac{2i-1}{2n+1}\frac{\pi}{2} - \frac{1}{2n+3}\frac{\pi}{2}\right) &= \sin\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)\cos\left(\frac{1}{2n+3}\frac{\pi}{2}\right) - \cos\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)\sin\left(\frac{1}{2n+3}\frac{\pi}{2}\right) \\ &\ge \sin\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)\cos\left(\frac{1}{2n+3}\frac{\pi}{2}\right) - \cos\left(\frac{3}{2n+1}\frac{\pi}{2}\right)\sin\left(\frac{1}{2n+3}\frac{\pi}{2}\right)\end{aligned}$$

We have $\sin\left(\frac{1}{2n+3}\frac{\pi}{2}\right) \le \frac{1}{3}\sin\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)$: indeed, the right-hand side takes its minimum for $i=2$ and this minimum is $\sin\left(\frac{3}{2n+1}\frac{\pi}{2}\right) \approx \frac{3\pi}{4n}$; on the left hand side, $\sin\left(\frac{1}{2n+3}\frac{\pi}{2}\right) \approx \frac{\pi}{4n}$.

Also:

$$\sin\left(\frac{2i-1}{2n+1}\frac{\pi}{2} - \frac{1}{2n+3}\frac{\pi}{2}\right) \ge \sin\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)\left(\cos\left(\frac{1}{2n+3}\frac{\pi}{2}\right) - \frac{1}{3}\cos\left(\frac{3}{2n+1}\frac{\pi}{2}\right)\right) \ge \frac{2}{3}\sin\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)$$

This gives:

$$|s| \le \frac{3\pi^2}{4n^4} \sum_{i=2}^{n} \frac{\cos^{2+2l}\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\sin\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)\sin\left(\frac{2i-1}{2n+1}\frac{\pi}{2} + \frac{1}{2n+3}\frac{\pi}{2}\right)} \le \frac{3\pi^2}{4n^4} \sum_{i=2}^{n} \frac{\cos^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}$$

We decompose this sum into two pieces: $i > n/2$ et $i \le n/2$, with:

$$S_1 = \frac{3\pi^2}{4n^4} \sum_{i=\frac{n}{2}+1}^{n} \frac{\cos^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}, \quad S_2 = \frac{3\pi^2}{4n^4} \sum_{i=2}^{n/2} \frac{\cos^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}$$

If $i \ge \frac{n}{2}+1$ $\frac{2i-1}{2n+1}\frac{\pi}{2} \ge \frac{\pi}{4}$, $\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right) \ge \sin^2\frac{\pi}{4} = \frac{1}{2}$

$$S_1 = \frac{3\pi^2}{4n^4} \sum_{i=\frac{n}{2}+1}^{n} \frac{\cos^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)} \le \frac{3\pi^2}{2n^4} \sum_{i=\frac{n}{2}+1}^{n} \cos^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right) \le \frac{3\pi^2}{2n^4}\frac{n}{2} = \frac{3\pi^2}{4n^3}$$

For S_2, $\frac{2i-1}{2n+1}\frac{\pi}{2} < \frac{\pi}{4}$, and therefore:

$$\tan\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right) > \frac{2i-1}{2n+1}\frac{\pi}{2},\ \sin\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right) > \frac{2i-1}{2n+1}\frac{\pi}{2}\cos\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right),$$

$$S_2 = \frac{3\pi^2}{4n^4} \sum_{i=2}^{n/2} \frac{\cos^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\sin^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)} \le \frac{3\pi^2}{4n^4} \sum_{i=2}^{n/2} \frac{\cos^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)}{\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)^2 \cos^2\left(\frac{2i-1}{2n+1}\frac{\pi}{2}\right)} = \frac{3}{n^4} \sum_{i=2}^{n/2} \frac{(2n+1)^2}{(2i-1)^2}$$

$$\le \frac{3}{n^4}(2n+1)^2 \frac{\pi^2}{6} \approx \frac{2\pi^2}{n^2}$$

Finally, we obtain asymptotically : $|s| \le \frac{2\pi^2}{n^2}$, and:

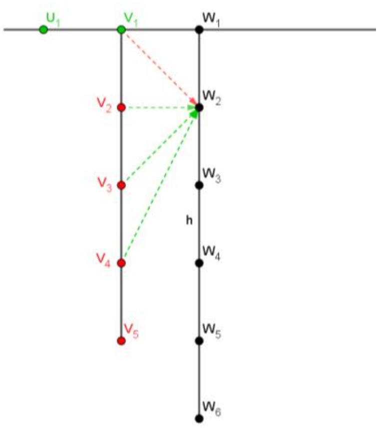
$w_1 \sim 1 - \frac{\pi^2}{16nLn(n)} - r_n$, with $0 < r_n < \frac{2\pi^2}{n^2}$

which proves Proposition 5.

We now study the energies carried by W_j, $j \ge 2$.

– Energy carried by W_2

It is a sum of terms:



All of them are negative.

$V_1 \to W_2$: contribution $v_1 w_{1,2} \sim \left(1+\frac{1}{2n}\right)\left(-\frac{1}{4n}\right) \sim -\frac{1}{4n}$

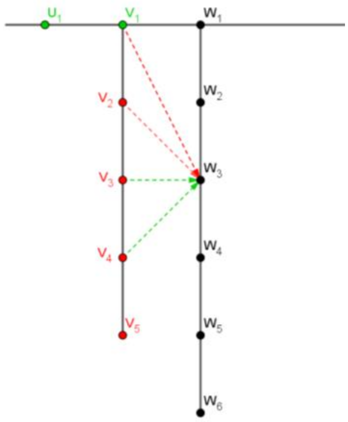
$V_2 \to W_2$: contribution $v_2 w_{1,2} \sim \left(-\frac{1}{4n}\right)\left(1+\frac{1}{2n}\right) \sim -\frac{1}{4n}$

$V_i \to W_2$ all $i > 2$; total contribution $\sum_{i=3}^{n} \frac{-1}{2i(i-1)n} \frac{(2i-1)^2}{i(i-1)-2} \frac{1}{2n} \approx \frac{-5}{8n^2}$

Therefore $w_2 \approx \frac{-1}{2n}$.

- Estimate upon $W_j \quad j \geq 3$:

It is a sum of terms and, as we see on the following picture (case of W_3), some of them are positive : $V_i \to W_j$ for $1 < i < j$, because $v_i < 0$ and the transfer coefficient $w_{i,j} < 0$. All others are negative.



$V_1 \to W_j$: contribution $v_1 w_{1,j} \sim \left(1+\frac{1}{2n}\right)\left(-\frac{1}{2j(j-1)n}\right) \sim -\frac{1}{2j(j-1)n}$

$V_i \to W_j$: $1 < i < j$, contribution:

$$\sum_{i=2}^{j-1} v_i w_{i,j} \sim -\frac{1}{2i(i-1)n} \frac{(2i-1)^2}{i(i-1)-j(j-1)} \frac{1}{2n} = \frac{1}{4i(i-1)} \frac{(2i-1)^2}{j(j-1)-i(i-1)} \frac{1}{n^2}$$

Since $(2i-1)^2 \leq 4i(i-1)$, we have :

$$\sum_{i=2}^{j-1} v_i w_{i,j} \leq \sum_{i=2}^{j-1} \frac{1}{j(j-1)-i(i-1)} \frac{1}{n^2}$$

$V_j \to W_j$: contribution $\frac{-1}{2j(j-1)} \frac{1}{n}\left(1+\frac{1}{2n}\right) \sim \frac{-1}{2j(j-1)} \frac{1}{n}$

$V_i \to W_j,\ i > j,$ total contribution:

$$\sum_{i=j+1}^{n} \frac{-1}{2i(i-1)}\frac{1}{n}\frac{1}{2}\frac{(2i-1)^2}{i(i-1)-j(j-1)}\frac{1}{n} = -\frac{1}{n^2}\sum_{i=j+1}^{n} \frac{1}{4i(i-1)}\frac{(2i-1)^2}{i(i-1)-j(j-1)}.$$

Lemma 8. – *If* $i \geq 2,$ *we have the estimate:*

$$1 \leq \frac{(2i-1)^2}{4i(i-1)} \leq \frac{9}{8}.$$

Proof of Lemma 8

Indeed, $\frac{(2i-1)^2}{4i(i-1)} = \frac{4i^2-4i+1}{4i^2-4i} = 1+\frac{1}{4i^2-4i} \leq \frac{9}{8}$ since $i \geq 2.$ This proves Lemma 8.

So, in all cases, we may consider that the total contribution of all V_i to a given W_j, for all $i \neq j$, is:

$$\sum_{i\neq j} v_i w_{i,j} = -\frac{1}{n^2}\sum_{i\neq j}^{n} \frac{1}{i(i-1)-j(j-1)} = \frac{1}{n^2}\sum_{i=2}^{j-1} \frac{1}{j(j-1)-i(i-1)} + \frac{1}{n^2}\sum_{i=j+1}^{n} \frac{1}{j(j-1)-i(i-1)}$$

The second sum exists only if $j < n.$

Proposition 9. - *If* $j < \frac{\sqrt{3n}}{2}$, *then the energy carried by each* W_j *is negative.*

Proof of Proposition 9

We have the decomposition:

$$\frac{1}{j(j-1)-i(i-1)} = \frac{1}{2j-1}\frac{1}{j+i-1} + \frac{1}{2j-1}\frac{1}{j-i}$$

Therefore:

$$S_1 = \frac{1}{n^2}\sum_{i=2}^{j-1} \frac{1}{j(j-1)-i(i-1)} = \frac{1}{2j-1}\frac{1}{n^2}\sum_{i=2}^{j-1} \frac{1}{j+i-1} + \frac{1}{2j-1}\frac{1}{n^2}\sum_{i=2}^{j-1} \frac{1}{j-i}$$

But:

$$\sum_{i=2}^{j-1}\frac{1}{j+i-1}=\sum_{k=1}^{2j-2}\frac{1}{k}-\sum_{k=1}^{j}\frac{1}{k}\sim Ln(2j-2)-Ln(j)$$

and:

$$\sum_{i=2}^{j-1}\frac{1}{j-i}=\sum_{i=1}^{j-2}\frac{1}{i}\sim Ln(j-2)$$

which gives:

$$S_1\sim\frac{1}{2j-1}\frac{1}{n^2}\left(Ln(2j-2)-Ln(j)\right)+\frac{1}{2j-1}\frac{1}{n^2}Ln(j-2)=\frac{1}{2j-1}\frac{1}{n^2}Ln\left(\frac{2(j-1)(j-2)}{j}\right)$$

which is always positive.

Similarly, if $j<n$:

$$S_2=\frac{1}{n^2}\sum_{i=j+1}^{n}\frac{1}{j(j-1)-i(i-1)}=\frac{1}{2j-1}\frac{1}{n^2}\sum_{i=j+1}^{n}\frac{1}{j+i-1}+\frac{1}{2j-1}\frac{1}{n^2}\sum_{i=j+1}^{n}\frac{1}{j-i}$$

which is always negative.

But:

$$\sum_{i=j+1}^{n}\frac{1}{j+i-1}=\sum_{k=1}^{n+j+1}\frac{1}{k}-\sum_{k=1}^{2j}\frac{1}{k}\sim Ln(n+j+1)-Ln(2j)$$

and:

$$\sum_{i=j+1}^{n}\frac{1}{j-i}=-\sum_{i=1}^{n-j}\frac{1}{i}\sim -Ln(n-j)$$

which gives:

$$S_2\sim\frac{1}{2j-1}\frac{1}{n^2}\left(Ln(n+j+1)-Ln(2j)\right)-\frac{1}{2j-1}\frac{1}{n^2}Ln(n-j)=\frac{1}{2j-1}\frac{1}{n^2}Ln\left(\frac{n+j+1}{2j(n-j)}\right)$$

And finally, if $j<n$:

$$\sum_{i\neq j}v_iw_{i,j}\sim\frac{1}{2j-1}\frac{1}{n^2}\left(Ln\left(\frac{2(j-1)(j-2)}{j}\right)+Ln\left(\frac{n+j+1}{2j(n-j)}\right)\right)=\frac{1}{2j-1}\frac{1}{n^2}Ln\left(\frac{(j-1)(j-2)(n+j+1)}{j^2(n-j)}\right)$$

Let us see when this quantity becomes positive. This is the case if and only if:

$$\frac{(j-1)(j-2)(n+j+1)}{j^2(n-j)} \geq 1$$

This is equivalent to:

$$(j-1)(j-2)(n+j+1) \geq j^2(n-j)$$

which simplifies to:

$$2j^3 - 2j^2 - 3jn - j + 2n + 2 \geq 0$$

Set :

$$y(j) = 2j^3 - 2j^2 - 3jn - j + 2n + 2$$

We have:

$$\frac{\partial y}{\partial j} = 4j^2 - 4j - 3n - 1$$

and:

$$\frac{\partial^2 y}{\partial j^2} = 8j - 4 > 0$$

Therefore, $\frac{\partial y}{\partial j}$ is increasing; it takes the value $7-3n<0$ for $j=2$ and the value $4n^2-7n-1>0$ for $j=n,$ so there is a unique j_0 such as $\frac{\partial y}{\partial j}=0$; this value is $j_0 = \frac{1}{2} + \frac{\sqrt{3n+2}}{2} \approx \frac{\sqrt{3n}}{2}$, and y is decreasing if $j<j_0$ and increasing if $j>j_0$. Since $y(2)=8-4n<0$, we have $y(j_0)<0$.

So we conclude that if $j<\frac{\sqrt{3n}}{2}$, $\sum_{i\neq j} v_i w_{i,j} < 0$, and this is true also for $\sum_{i=2}^{n} v_i w_{i,j}$, since the direct contribution $v_i w_{i,i}$ is negative. This proves Proposition 9.

From the estimate:

$$\frac{(j-1)(j-2)(n+j+1)}{j^2(n-j)} \leq \frac{2(n-2)(n-3)}{(n-1)^2} \leq 2$$

we deduce moreover:

$$\sum_{i=2}^{n} v_i w_{i,j} \le \frac{1}{2j-1}\frac{1}{n^2}Ln(2)$$

7. Example. – Total energy carried by W_{n+1}

All transfer coefficients are negative:

$$w_{i,n+1} = \frac{1}{2}\frac{(2i-1)^2}{i(i-1)-(n+1)n}\frac{1}{n}$$

Energy received from V_1 : $w_{1,n+1} = \frac{-1}{2(n+1)n^2} \approx \frac{-1}{2n^3}$

from V_i $2 \le i \le n$:

$$\frac{-1}{2i(i-1)n} \times \frac{1}{2}\frac{(2i-1)^2}{i(i-1)-(n+1)n}\frac{1}{n} = \frac{1}{n^2}\frac{1}{4i(i-1)}\frac{(2i-1)^2}{(n+1)n-i(i-1)} \approx \frac{1}{n^2}\frac{1}{(n+1)n-i(i-1)}$$

and thus $\sum_{i=2}^{n} w_i \approx \frac{1}{n^2}\frac{Ln(2n)}{2n} = \frac{Ln(2n)}{2n^3}$

So, the total energy received by W_{n+1} is $\sum_{i=1}^{n} w_i \approx \frac{Ln(2n)}{2n^3} - \frac{1}{2n^3} > 0$

8. Estimates upon the positive energy

Let us come back to the general situation. Let us denote by w_j^+ the positive energy carried by each W_j (each of them receives, from the V_i, both positive and negative contributions, and w_j^+ is the sum of the positive ones). Then, we have the estimate:

$$\sum_{j=2}^{n} w_j^+ \le \frac{1}{n^2}Ln(2)\sum_{j=2}^{n}\frac{1}{2j-1} \le \frac{1}{n^2}\frac{Ln(2)}{2}\sum_{j=1}^{n-1}\frac{1}{j} \approx \frac{Ln(2)}{2}\frac{Ln(n)}{n^2}$$

Since the series $\sum_{n=1}^{+\infty}\frac{Ln(n)}{n^2}$ is convergent, we can, at the end of each period, bring this positive energy back to W_1 without changing our conclusion: It fits inside the term r_n seen above (Proposition 5).

We may now convert a statement about the lengths l_n into a statement about the growth of the barrier. This is done by the following Proposition:

Proposition 10. - *Let $b(x)$ be a barrier, that is a positive, differentiable, strictly increasing function, tending to $+\infty$ when $x \to +\infty$. Then we have:*

$$\sum_{n=1}^{N} \frac{l_n}{n^2} \approx 2 \int_{1}^{t_{N+1}} \frac{dx}{b^2(x)}$$

Proof of Proposition 10

Let $\beta = b^{-1}$ be the inverse function of the function b (this inverse exists since b is strictly increasing). It is also positive, differentiable and strictly increasing. We have, by definition $t_n = \beta(n)$ and therefore:

$$l_n = t_{n+1} - t_n = \beta(2n+3) - \beta(2n+1) \approx 2\beta'(2n+1) = \frac{2}{b'(t_n)}.$$

So we may write:

$$\sum_{n=1}^{+\infty} \frac{l_n}{n^2} \approx \sum_{n=1}^{+\infty} \frac{2}{n^2 b'(t_n)} \approx 2 \int_{1}^{+\infty} \frac{1}{y^2 b'(\beta(y))} dy$$

Set $y = b(x)$, $dy = b'(x)dx$. The above integral becomes:

$$\int_{1}^{+\infty} \frac{1}{y^2 b'(\beta(y))} dy = \int_{b^{-1}(1)}^{+\infty} \frac{b'(x)}{b^2(x) b'(x)} dx = \int_{b^{-1}(1)}^{+\infty} \frac{dx}{b^2(x)}$$

which proves Proposition 10.

As an example, with the barrier $b(x) = \sqrt{x}$, we have $b(t_{n+1}) = 2n+3$, that is $t_{n+1} = (2n+3)^2$ and :

$$\int_{1}^{t_{N+1}} \frac{dx}{b^2(x)} = \int_{1}^{t_{N+1}} \frac{dx}{x} = 2Ln(2N+3)$$

III. Lower estimate

We will show that if the integral converges at infinity, the game may continue indefinitely: the remaining energy does not tend to zero. More precisely, we will show that:

$$E_N \geq c \exp\left(-\frac{\pi^2}{8}\sum_{n=1}^{N}\frac{l_n}{n^2}\right) \approx c\exp\left(-\frac{\pi^2}{8}\int_1^{t_{N+1}}\frac{dx}{b^2(x)}\right)$$

This part is much simpler than the previous one.

First of all, we have seen (Proposition 10 above) that the convergence of the integral is equivalent to the convergence of the series $\sum_{n=1}^{+\infty}\frac{l_n}{n^2} < +\infty$. The energy left after N periods will be $E_{t_N} = \exp\left(-\sum_{n=1}^{N}\frac{l_n}{n^2}\right)$ if we can prove that this energy is carried, during each period, by the first eigenvector of the corresponding matrix.

Proposition 11. - *Assume that, at the end of each period, we replace the first eigenvector V of this period by the first eigenvector W of the next period, with same normalization. Then the energy will disappear more easily with W than with V. So, if we perform this replacement at each step and, at the end, get a non-zero energy, it means that the whole game produces a non-zero energy.*

Proof of Proposition 11

Let V be the first eigenvector during the n^{th} period, normalized in l_1 norm:

$$V = 2\tan\frac{\vartheta_1}{2}\left(\sin(n\vartheta_1),...,\sin(\vartheta_1)\right),\ \vartheta_1 = \frac{\pi}{2n+1}$$

When we start the $(n+1)^{st}$ period, it becomes $V = 2\tan\frac{\vartheta_1}{2}\left(\sin(n\vartheta_1),...,\sin(\vartheta_1),0\right)$.

The first eigenvector of the $(n+1)^{st}$ period is $W = 2\tan\frac{\vartheta_2}{2}\left(\sin((n+1)\vartheta_2),...,\sin(\vartheta_2)\right)$, with $\vartheta_2 = \frac{\pi}{2n+3}$.

We will prove that the "tail" of W contains more energy than the tail of V; in simpler terms, W is closer to the barrier. More precisely, for any $k \leq n+1$, let us define the tails made of the last k terms; here they are, written in opposite order compared to V, W above:

$$V_k = 2\tan\frac{\vartheta_1}{2}\left(0,\sin(\vartheta_1),...,\sin((k-1)\vartheta_1)\right)$$
$$W_k = 2\tan\frac{\vartheta_2}{2}\left(\sin(\vartheta_2),...,\sin(k\vartheta_2)\right)$$

The comparison of the tails is made by the following lemma:

Lemma 12. – *For any* k, $k = 1, \ldots, n+1$, $|W_k|_1 \geq |V_k|_1$.

Proof of Lemma 12

We use the identity:

$$\sum_{j=1}^{k} \sin(j\vartheta) = \frac{\sin(k\vartheta) - \sin((k+1)\vartheta) + \sin(\vartheta)}{2(1-\cos(\vartheta))} \tag{1}$$

We have to show that:

$$\tan\left(\frac{\vartheta_1}{2}\right)\frac{\sin((k-1)\vartheta_1) - \sin(k\vartheta_1) + \sin(\vartheta_1)}{1-\cos(\vartheta_1)} \leq \tan\left(\frac{\vartheta_2}{2}\right)\frac{\sin(k\vartheta_2) - \sin((k+1)\vartheta_2) + \sin(\vartheta_2)}{1-\cos(\vartheta_2)} \tag{2}$$

Using the identity $\frac{\tan\frac{t}{2}}{1-\cos(t)} = \frac{1}{\sin(t)}$, (2) is equivalent to:

$$\frac{\sin((k-1)\vartheta_1) - \sin(k\vartheta_1) + \sin(\vartheta_1)}{\sin(\vartheta_1)} \leq \frac{\sin(k\vartheta_2) - \sin((k+1)\vartheta_2) + \sin(\vartheta_2)}{\sin(\vartheta_2)} \tag{3}$$

Or:

$$\frac{\sin((k-1)\vartheta_1) - \sin(k\vartheta_1)}{\sin(\vartheta_1)} \leq \frac{\sin(k\vartheta_2) - \sin((k+1)\vartheta_2)}{\sin(\vartheta_2)} \tag{4}$$

Using the identity $\sin(p) - \sin(q) = 2\cos\left(\frac{p+q}{2}\right)\sin\left(\frac{p-q}{2}\right)$, (4) becomes:

$$\cos((2k-1)\vartheta_1) \geq \cos((2k+1)\vartheta_2) \tag{5}$$

But the angles in (5) are smaller than π, so the cosine is decreasing. Therefore, (5) is equivalent to:

$$(2k-1)\vartheta_1 \leq (2k+1)\vartheta_2 \tag{6}$$

That is:

$$\frac{(2k-1)\pi}{2n+1} \le \frac{(2k+1)\pi}{2n+3} \tag{7}$$

which itself is equivalent to $k \le n+1$, which is satisfied. This proves Lemma 12. Now, Proposition 11 follows from Corollary 4b, Part II.

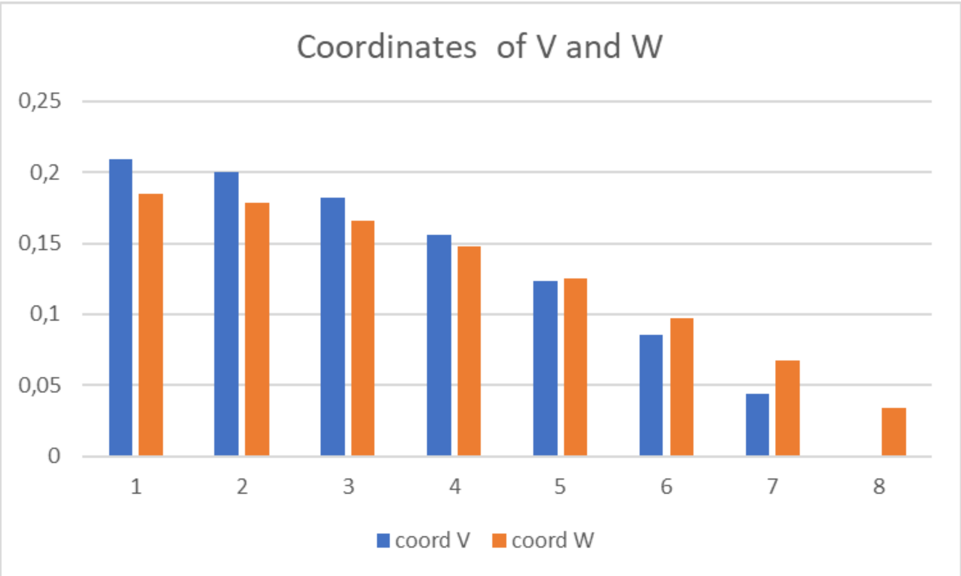
Remark. - All $s(W_k)-s(V_k)$ are >0, but the sequence is not increasing. Here are the values for $n=7$:

$0.034, 0.057, 0.069, 0.070, 0.062, 0.046, \ 0.025$.

The final difference is 0, since both vectors have sum equal to 1.

Here are the coordinates, for $n=7$, in matrix notation (first coordinate on the axis, last coordinate near the barrier) :

coord V	coord W
0,20905693	0,18453672
0,19992014	0,17825254
0,18204588	0,16589819
0,15621534	0,14789437
0,12355744	0,12485419
0,08549949	0,09756225
0,0437048	0,06694794
0	0,0340538



Let us now finish the proof of Theorem 2.

If we proceed only with the first eigenvector during each period, starting with $V_{1,1}$, the energy becomes $\lambda_{1,1}^{l_1}$, then $\lambda_{1,1}^{l_1}\lambda_{2,1}^{l_2}$, then $\lambda_{1,1}^{l_1}\lambda_{2,1}^{l_2}\cdots\lambda_{N,1}^{l_n}$, so:

$$E_{t_N} \geq c\lambda_{1,1}^{l_1}\lambda_{2,1}^{l_2}\cdots\lambda_{N,1}^{l_n}$$

with a constant c which depends only on the first step (converting the energy at the origin into an energy carried by the first eigenvector on the first stage).

Therefore:

$$E_N \geq c\exp\left(\sum_{n=1}^{N} Log\left(\lambda_{n,1}^{l_n}\right)\right) = c\exp\left(\sum_{n=1}^{N} l_n Log\left(\lambda_{n,1}\right)\right) \approx c\exp\left(-\frac{\pi^2}{8}\sum_{n=1}^{N}\frac{l_n}{n^2}\right)$$

which proves Theorem 2.

The energy profile during the n^{th} period is approximately proportional to the first eigenvector, and this approximation is more and more accurate when $n \to +\infty$. This means that, during the n^{th} period, the energy profile (in the X variable) is proportional to $V_n = \left(\sin\left(n\vartheta\right),\ldots,\sin\left(\vartheta\right)\right)$, with $\vartheta = \dfrac{\pi}{2n+1}$.

As an example, we may take $b\left(x\right) = \sqrt{x}Ln(x)$

Then $\displaystyle\int \frac{dx}{b^2\left(x\right)} = -\frac{1}{Ln\left(x\right)}$ and we obtain the estimate:

$$E_N \geq c\exp\left\{\frac{-\pi^2}{8}\left(\frac{1}{Ln\left(2\right)} - \frac{1}{Ln\left(t_{N+1}\right)}\right)\right\}.$$

In this case, the probability that the game stops at any time N does not tend to 0. There is a non-zero probability that the game continues indefinitely.