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Simple Random Walks in the plane:

An energy-based approach

Part III: Different Initial Fortunes

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Abstract

We consider the same ± 1 game, with now different initial bounded fortunes. A player gets ruined and the game stops if his fortune reduces to 0.

Our main result is an explicit expression of the probability of any situation at time n, in terms of the initial fortunes F_A and F_B (Theorem 5). We deduce the probability that A wins the game, and the probability that the game continues after a given time n.

As we did in Part II, our approach relies upon a representation of the game in terms of "energy propagation", which allows us to use arguments from operator theory and special functions (more specifically, Chebycheff polynomials of first and second kind).

I. Presentation

The basic settings are presented in Part I. Recall that we consider a simple random walk in the plane: a game, with two players. It is represented by a r.v. X taking the values ± 1 with probability $\frac{1}{2}$. The player A wins if $X = 1$ and then he receives 1 Euro from the player B, and conversely if $X = -1$. Each player has an initial fortune, denoted respectively by F_A and F_B. The game stops if any of the players sees his fortune equal to zero. We will assume here that both fortunes take only even values: $F_A = 2a$, $F_B = 2b$, where a, b are positive integers. We assume for simplicity that $a > b$.

II. A preliminary remark

We have two possible descriptions of the game:

1. Description 1

Each player has his fortune F_A, F_B and the game starts at 0. It finishes if one of the players gets ruined. More precisely, we set $S_0 = 0$, and for $N \geq 1$, $S_N = \sum_{n=1}^{N} X_n$. If $S_N = -F_A$, the player A is ruined and the game stops; if $S_N = F_B$ the player B is ruined and the game also stops. So, mathematically speaking, the random walks starts at 0 and the game finishes when it touches one of the two barriers $y = -F_A$, $y = F_B$. In this description, the starting point is 0 and both barriers are dissymmetric.

2. Description 2

The barriers are set at the symmetric values $\pm(2\xi + 2)$, with $\xi = \frac{a+b}{2} - 1$ and the starting point is $(0, 2y_0)$, with $y_0 = \frac{a-b}{2}$.

Quite clearly, both descriptions are equivalent, mathematically speaking. Indeed, for A to win, in the first description, the random walk has to climb $2b$ steps, and in the second description it has to climb $2\xi + 2 - 2y_0$ steps. For B to win, in the first description, the random walk has to go down $2a$ steps and in the second description it has to go down $2\xi + 2 + 2y_0$ steps.

So, we will work with description 2, which allows us to use the framework developed previously. The barriers are symmetrically set at $\pm(2\xi + 2)$ and the starting point is at $(0, 2y_0)$.

III. Notation

We refer to Part I. Instead of a random walk with multiple possible paths, we consider that we have the propagation of an energy, with the following rules:

- At time $n=0,$ the starting point $(0,2y_0)$ receives an energy equal to 1;
- At time $n=1,$ this energy is divided into two: each point $(1,2y_0+1)$ and $(1,2y_0-1)$ receives an energy equals to $\frac{1}{2}$ and so on.

More generally, the energy of a point of coordinates (n,k) in the plane is equal to the probability that the random walk hits this point. It will be denoted by $e(n,k)$.

As we already did in Part I, we restrict ourselves to even values of the time $(2n)$. Also, we will need a "matrix-oriented" notation. Instead of the $y-$coordinate ranging between $-\xi$ and $+\xi,$ we will go downwards, starting at 1 and descending to $2\xi+1.$ More precisely, we set:

$$x(n,k)=e(2n,2(\xi-k+1)).$$

So $x(n,1)=e(2n,2\xi)$ (near the top barrier) and $x(n,2\xi+1)=e(2n,-2\xi)$ (near the bottom barrier). We have the propagation rules:

$$\begin{cases} x(n+1,1)=\frac{1}{2}x(n,1)+\frac{1}{4}x(n,2) \\ x(n+1,k)=\frac{1}{4}x(n,k-1)+\frac{1}{2}x(n,k)+\frac{1}{4}x(n,k+1), k=2,...,2\xi \\ x(n+1,2\xi+1)=\frac{1}{4}x(n,2\xi)+\frac{1}{2}x(n,2\xi+1) \end{cases} \qquad (1)$$

They take into account the symmetric barriers $y=\pm(2\xi+2)$: if a path hits any of the barriers, its energy is absorbed and disappears. The last non-zero values for x on each vertical are $x(n,1), x(n,2\xi+1)$.

The initial value is $x(0,y_0)=1, x(0,k)=0$ if $k\neq y_0$. (2)

Let W_{2n} be the vertical for $x=2n,$ that is the set of all points $A_{2n,2k}$, $k=-n,...,n.$

IV. Matrix representation

The system (1) may be represented as a matrix:

$$\begin{pmatrix} x(n+1,1) \\ x(n+1,2) \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ x(n+1,2\xi) \\ x(n+1,2\xi+1) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \dots & \dots & \dots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \dots & \dots & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \dots & \dots & \dots & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x(n,1) \\ x(n,2) \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ \cdots \\ x(n,2\xi) \\ x(n,2\xi+1) \end{pmatrix} \qquad (3)$$

This is a real symmetric matrix, of size $2\xi+1$, denoted by M.

As we already said, the first coordinate is close to the upper barrier, the last coordinate close to the lower barrier.

The general approach is the same as in Part II, but the matrix is different. An important remark is that we do not need here a second change in coordinates, which makes things simpler.

Lemma 1. - *The matrix M is positive definite.*

Proof of Lemma 1

We have to show that, for all non-zero column-vector X of size $2\xi+1$, we have $X^t M X > 0$.

Let $X = \begin{pmatrix} x_1 \\ \vdots \\ x_{2\xi+1} \end{pmatrix}$; we have:

$$MX = \begin{pmatrix} \frac{1}{2}x_1 + \frac{1}{4}x_2 \\ \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3 \\ \vdots \\ \frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1} \\ \vdots \\ \frac{1}{4}x_{2\xi-1} + \frac{1}{2}x_{2\xi} + \frac{1}{4}x_{2\xi+1} \\ \frac{1}{4}x_{2\xi} + \frac{1}{2}x_{2\xi+1} \end{pmatrix}$$

and therefore:

$$\begin{aligned} X^t MX &= \left(\frac{1}{2}x_1 + \frac{1}{4}x_2\right)x_1 + \left(\frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3\right)x_2 + \ldots + \left(\frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1}\right)x_i + \ldots + \\ &\quad + \left(\frac{1}{4}x_{2\xi-1} + \frac{1}{2}x_{2\xi} + \frac{1}{4}x_{2\xi+1}\right)x_{2\xi} + \left(\frac{1}{4}x_{2\xi} + \frac{1}{2}x_{2\xi+1}\right)x_{2\xi+1} \\ &= \frac{1}{4}x_1^2 + b + \frac{1}{4}x_{2\xi+1}^2 \end{aligned}$$

with $b = \frac{1}{4}\left(\left(x_1 + x_2\right)^2 + \ldots + \left(x_i + x_{i-1}\right)^2 + \ldots + \left(x_{2\xi} + x_{2\xi+1}\right)^2\right).$

So clearly $X^t MX > 0$ if they x_i are not all equal to 0. This proves Lemma 1.

From this Lemma follows that all eigenvalues of M are real, >0, and that M can be diagonalized in an orthogonal basis made of eigenvectors.

Lemma 2. - *All eigenvalues of* M *are* <1.

Proof of Lemma 2

Let us write the system of equations defining the eigenvalues and eigenvectors, $MX = \lambda X$.

$$
\begin{cases}
\frac{1}{2}x_1 + \frac{1}{4}x_2 = \lambda x_1 \\
\frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3 = \lambda x_2 \\
\vdots \\
\frac{1}{4}x_{i+1} + \frac{1}{2}x_i + \frac{1}{4}x_{i-1} = \lambda x_i \\
\vdots \\
\frac{1}{4}x_{2\xi-1} + \frac{1}{2}x_{2\xi} + \frac{1}{4}x_{2\xi+1} = \lambda x_{2\xi} \\
\frac{1}{4}x_{2\xi} + \frac{1}{2}x_{2\xi+1} = \lambda x_{2\xi+1}
\end{cases} \tag{4}
$$

It may be written:

$$
\begin{cases}
x_2 = (4\lambda - 2)x_1 \\
x_3 = (4\lambda - 2)x_2 - x_1 \\
\vdots \\
x_{i+1} = (4\lambda - 2)x_i - x_{i-1} \\
\vdots \\
x_{2\xi+1} = (4\lambda - 2)x_{2\xi} - x_{2\xi-1} \\
x_{2\xi} = (4\lambda - 2)x_{2\xi+1}
\end{cases} \tag{5}
$$

We know that $x_1 \neq 0$ (if $x_1 = 0,$ all $x_i = 0$), so we may take $x_1 = 1.$ Assume $\lambda \geq 1$. From the equation $x_2 = (4\lambda - 2)x_1$ we deduce $x_2 > x_1 > 0$. More generally, the equation $x_{i+1} = (4\lambda - 2)x_i - x_{i-1}$ gives:

$$x_{i-1} - x_i = (4\lambda - 3)x_i - x_{i+1} \geq x_i - x_{i+1}$$

that is $x_i - x_{i-1} \leq x_{i+1} - x_i$. So the sequence of consecutive differences is increasing. Since $x_2 > x_1$, all differences are positive, the x_i are increasing and are $> 0.$ Set $S = \sum_{i=1}^{2\xi+1} x_i$; summing all equations, we get:

$$S - x_1 + x_{2\xi} = (4\lambda - 2)S - \left(S - x_{2\xi} - x_{2\xi+1}\right),$$

that is $-x_1 - x_{2\xi+1} = 4(\lambda - 1)S$. But this is a contradiction: $\lambda > 1$, $S > 0$ and $x_1 > 0,$ $x_{2\xi+1} > 0.$ Lemma 2 is proved.

From Lemmas 1 and 2 follows that all eigenvalues of M are strictly between 0 and 1.

Proposition 3. - *For* $j = 1,...,2\xi + 1$*, the* j^{th} *eigenvalue* λ_j *is:*

$$\lambda_j = \cos^2 \frac{j\pi}{4(\xi + 1)}$$

and the j^{th} *eigenvector has components:*

$$V_j = \left(\sin(\vartheta_j), \sin(2\vartheta_j), ..., \sin((2\xi + 1)\vartheta_j)\right)$$

with $\vartheta_j = \frac{j\pi}{2\xi + 2}$, $j = 1,...,2\xi + 1.$

Remark. – We observe that this definition of ϑ_j is different from the one given in Part II.

Proof of Proposition 3

We have $x_1 \neq 0$ (otherwise all $x_j's$ are 0), so we may assume $x_1 = 1$. We set $\mu = 2\lambda - 1$, so $-1 < \mu < 1$. We set also $y_0 = x_1, y_1 = x_2, ..., y_j = x_{j+1}, ..., y_{2\xi} = x_{2\xi+1}$. System (5) becomes:

$$\begin{cases} y_0 = 1 \\ y_1 = 2\mu \\ y_2 = 2\mu y_1 - y_0 \\ \vdots \\ y_j = 2\mu y_{j-1} - y_{j-2} \\ \vdots \\ y_{2\xi} = 2\mu y_{2\xi-1} - y_{2\xi-2} \\ y_{2\xi-1} = 2\mu y_{2\xi} \end{cases} \tag{6}$$

Therefore, $y_j = U_j(\mu)$ where U_j is the j^{th} Chebyshev's polynomial of second kind, for $j = 0,...,2\xi.$ The final equation in (6) may be written:

$$U_{2\xi-1}(\mu) = 2\mu U_{2\xi}(\mu) \tag{7}$$

that is, with $\mu = \cos(\vartheta)$:

$$\frac{\sin(2\xi\vartheta)}{\sin(\vartheta)} = 2\cos(\vartheta)\frac{\sin((2\xi + 1)\vartheta)}{\sin(\vartheta)}.$$

We know that $-1<\cos\vartheta<1$, which implies $\sin(\vartheta)\neq 0$, so the above equation is equivalent to:

$$\sin(2\xi\vartheta)=2\cos(\vartheta)\sin((2\xi+1)\vartheta) \tag{8}$$

We have:

$$\sin(2\xi\vartheta)-2\cos(\vartheta)\sin((2\xi+1)\vartheta)=-\sin((2\xi+2)\vartheta).$$

Therefore, equation (8) is equivalent to:

$$\sin((2\xi+2)\vartheta)=0 \tag{9}$$

that is $\vartheta=\dfrac{j\pi}{2\xi+2}$, $j\in Z$.

We are interested only in the values of $\cos\vartheta$; moreover, the solutions $j=0$ and $j=2\xi+2$ give $\vartheta=0$, which is impossible ($-1<\cos\vartheta<1$). So, we have the solutions:

$$\vartheta_j=\frac{j\pi}{2\xi+2},\quad j=1,\dots,2\xi+1 \tag{10}$$

This gives, for $j=1,\dots,2\xi+1$: $\mu_j=\cos(\vartheta_j)=\cos\dfrac{j\pi}{2(\xi+1)}$ and:

$$\lambda_j=\frac{\mu_j+1}{2}=\frac{1}{2}\left(1+\cos\frac{j\pi}{2(\xi+1)}\right)=\cos^2\frac{j\pi}{4(\xi+1)}.$$

The eigenvector in coordinates y_j defined by (6) gives $y_j=U_j(\cos\vartheta)=\dfrac{\sin((j+1)\vartheta)}{\sin\vartheta}$. After multiplication, we may take $y_j=\sin((j+1)\vartheta)$, $j=0,\dots,2\xi$. Returning to the x coordinates, this gives:

$$x_1=\sin\vartheta,\ x_2=\sin(2\vartheta),\dots,x_j=\sin(j\vartheta),\dots,x_{2\xi+1}=\sin((2\xi+1)\vartheta).$$

So, the j^{th} eigenvector is:

$$V_j=\left(\sin(\vartheta_j),\sin(2\vartheta_j),\dots,\sin((2\xi+1)\vartheta_j)\right) \tag{11}$$

where ϑ_j is given by (10). This finishes the proof of Proposition 3.

We already made the following remarks in Part II; they are still valid here:

- The first eigenvector, V_1, has all its components real and >0, but all other eigenvectors have some negative component;

- It follows from the general theory of symmetric matrices, positive defined, that any two eigenvectors V_{j_1}, V_{j_2} are mutually orthogonal, that is:

$$\sum_{l=1}^{2\xi+1} \sin\left(l\vartheta_{j_1}\right)\sin\left(l\vartheta_{j_2}\right) = 0,$$

where here $\vartheta_{j_1} = \frac{j_1\pi}{2\xi+2}, \vartheta_{j_2} = \frac{j_2\pi}{2\xi+2}$.

We now compute the l_1- norm and the l_2- norm of the eigenvectors

3. Norms of the eigenvectors

Proposition 4. – *All eigenvectors have the same quadratic norm:*

$$\left|V_j\right|_2^2 = \xi + 1$$

for $j = 1,\ldots,2\xi+1.$

The j^{th} *eigenvector carries the energy:*

$$s\left(V_j\right) = \frac{1}{4}\frac{1+(-1)^{j-1}}{\sin\frac{j\pi}{4(\xi+1)}}.$$

Proof of Proposition 4

Let us first compute $\left|V_j\right|_2^2$. We use the identity:

$$\sum_{k=1}^{2\xi+1} \sin^2(kt) = \xi + \frac{3}{4} - \frac{1}{4}\frac{\sin\left((4\xi+3)t\right)}{\sin(t)}$$

which gives:

$$\left|V_j\right|_2^2 = \sum_{k=1}^{2\xi+1} \sin^2\left(\frac{k\vartheta_j}{2\xi+2}\right) = \xi + \frac{3}{4} - \frac{1}{4}\frac{\sin\left((4\xi+3)\vartheta_j\right)}{\sin\left(\vartheta_j\right)}$$

But:

$$\frac{\sin\left(\left(4\xi+3\right)\vartheta_j\right)}{\sin\left(\vartheta_j\right)}=\frac{\sin\left(\left(4\xi+3\right)\frac{j\pi}{2\xi+2}\right)}{\sin\left(\frac{j\pi}{2\xi+2}\right)}=-1$$

since:

$$\left(4\xi+3\right)\frac{j\pi}{2\xi+2}=2j\pi-\frac{j\pi}{2\xi+2}$$

Let us now compute $s\left(V_j\right)$. Apply the matrix M to the eigenvector V_j: by definition, we get $MV_j=\lambda_j V_j$ and the loss of energy is $\left(1-\lambda_j\right)s\left(V_j\right)$. But this loss of energy comes from the first and the last coordinates only:

– The first coordinate gives a loss of $\frac{1}{4}\sin\left(\vartheta_j\right)=\frac{1}{4}\sin\frac{j\pi}{2\xi+2}$;

– The last coordinate gives a loss of :

$$\frac{1}{4}\sin\left(\left(2\xi+1\right)\vartheta_j\right)=\frac{1}{4}\sin\left(\frac{2\xi+1}{2\xi+2}j\pi\right)=\frac{1}{4}\sin\left(j\pi-\frac{j\pi}{2\xi+2}\right)=\frac{1}{4}\left(-1\right)^{j-1}\sin\left(\frac{j\pi}{2\xi+2}\right).$$

So, the total loss is $\frac{1}{4}\left(1+\left(-1\right)^{j-1}\right)\sin\left(\frac{j\pi}{2\xi+2}\right)$.

So, we get $\left(1-\lambda_j\right)s\left(V_j\right)=\frac{1}{4}\left(1+\left(-1\right)^{j-1}\right)\sin\left(\frac{j\pi}{2\xi+2}\right)$.

We know that $\lambda_j=\cos^2\frac{j\pi}{4\left(\xi+1\right)}$, so $1-\lambda_j=\sin^2\frac{j\pi}{4\left(\xi+1\right)}$, which gives:

$$s\left(V_j\right)=\frac{\frac{1}{4}\left(1+\left(-1\right)^{j-1}\right)\sin\left(\frac{j\pi}{2\xi+2}\right)}{\sin^2\frac{j\pi}{4\left(\xi+1\right)}}=\frac{1}{4}\frac{1+\left(-1\right)^{j-1}}{\sin\frac{j\pi}{4\left(\xi+1\right)}}$$

which proves our claim. We observe that $s\left(V_j\right)$ are always ≥ 0 ; the non-zero ones are decreasing.

This finishes the proof of Proposition 4.
We have completely characterized the matrix, its eigenvalues and eigenvectors. Now, we can proceed to the evaluation of the energy at each step.

4. The energy at each step

Recall that both fortunes take even values: $F_A = 2a,\ F_B = 2b,$ that $\xi = \dfrac{a+b}{2} - 1$ and that, in usual coordinates, the starting point is $(0, 2y_0)$, with $y_0 = \dfrac{a-b}{2}$. We need to proceed to matrix coordinates; we set $u = \xi - y_0 + 1,$ so, when y_0 changes, u may take any value between 1 and $2\xi + 1$. The value $u = 1$ corresponds to $y_0 = \xi$ and the value $u = 2\xi + 1$ to $y_0 = -\xi$. We notice that $u = \left(\dfrac{a+b}{2} - 1\right) - \left(\dfrac{a-b}{2}\right) + 1 = b$.

The parameter u will be called "Initial Fortune Characteristic" (in short IFC). If $u = \xi + 1$, both players are equal, if $u = 1, \ldots, \xi,$ the player A has the initial advantage; if $u = \xi + 2, \ldots, 2\xi + 1$, the player B has the initial advantage.

Theorem 5. - *Let* $u,\ 1 \le u \le 2\xi + 1,$ *be the IFC. At each step, the energy is:*

$$x(n,k) = \frac{1}{\xi + 1} \sum_{j=1}^{2\xi+1} \sin(u\,\vartheta_j)\sin(k\vartheta_j)\cos^{2n}\left(\frac{\vartheta_j}{2}\right)$$

with $\vartheta_j = \dfrac{j\pi}{2\xi + 2}$, $j = 1, \ldots, 2\xi + 1$.

Proof of Theorem 5

We start with the initial vector X_0 defined by $X_0(u) = 1,\ X_0(k) = 0$ if $k \ne u.$

We decompose this vector on the basis of eigenvectors. We write $X_0 = \sum_{j=1}^{2\xi+1} \alpha_j V_j.$

Since the eigenvectors are orthogonal, the coefficients α_j may be computed simply:

$$\alpha_j = \frac{< X_0, V_j >}{\left|V_j\right|_2^2} = \frac{\sin(u\,\vartheta_j)}{\xi + 1}$$

using Proposition 3. Then, at the n^{th} step (time $2n$), the vector X_n is :

$$X_n = M^n X_0 = M^n \sum_{j=1}^{2\xi+1} \alpha_j V_j = \sum_{j=1}^{2\xi+1} \alpha_j M^n V_j = \sum_{j=1}^{2\xi+1} \alpha_j \lambda_j^n V_j$$

which gives for the k^{th} coordinate:

$$x(n,k) = \frac{1}{\xi+1} \sum_{j=1}^{2\xi+1} \sin(u\,\vartheta_j)\sin(k\vartheta_j)\cos^{2n}\left(\frac{\vartheta_j}{2}\right)$$

which proves Theorem 5.

One can give an equivalent formulation of Theorem 5, using binomial coefficients:

Theorem 6. – *For all* $u,k,n,$ *the energy is:*

$$x(n,k) = \frac{1}{2^{2n}} \sum_m \left(\binom{2n}{n+k-u+4m(\xi+1)} - \binom{2n}{n-k-u+4m(\xi+1)} \right)$$

(recall that we use the convention $\binom{n}{m} = 0$ if $m < 0$ or if $m > n$)

Proof of Theorem 6

We use the linearization formula:

$$\cos^{2n}\left(\frac{\vartheta_j}{2}\right) = \frac{1}{2^{2n}} \sum_{l=0}^{2n} \binom{2n}{l} \cos((n-l)\vartheta_j).$$

So, we write, from Theorem 5:

$$\begin{aligned}
2^{2n}(\xi+1)\,x(u,n,k) &= \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \sin(u\,\vartheta_j)\sin(k\vartheta_j)\cos((n-l)\vartheta_j) \\
&= \frac{1}{2} \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \cos((u-k)\,\vartheta_j)\cos((n-l)\vartheta_j) - \\
&\quad - \frac{1}{2} \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \cos((u+k)\,\vartheta_j)\cos((n-l)\vartheta_j)
\end{aligned}$$

and thus:

$$\begin{aligned}
2^{2n+2}(\xi+1)\,x(u,n,k) &= \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \cos((u-k-n+l)\,\vartheta_j) + \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \cos((u-k+n-l)\,\vartheta_j) \\
&\quad - \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \cos((u+k-n+l)\,\vartheta_j) - \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \cos((u+k+n-l)\,\vartheta_j)
\end{aligned}$$

So, we have four terms and $2^{2n+2}(\xi+1)\,x(u,n,k) = T_1 + T_2 - T_3 - T_4$.

But $T_1 = T_2$. Indeed, one may exchange the roles of u and k :the number of paths going from point A to point B, not touching the barriers, is equal to the number of paths going from B to A.

Also, $T_3 = T_4$. Indeed, the summation upon l runs from 0 to $2n$, so both $-n+l$ and $n-l$ run from $-n$ to n. So, we have:

$$2^{2n+1}(\xi+1)\,x(u,n,k) = T_1 - T_3.$$

Let us study the first term: $T_1 = \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \cos\left(\left(u-k-n+l\right)\vartheta_j\right)$. We use the following formulas:

$$\sum_{j=1}^{2\xi+1} \cos\left(k\vartheta_j\right) = -\frac{1}{2}\left(\left(-1\right)^k + 1\right) \text{ if } k \text{ is not an even multiple of } 2\xi+2 \quad (1)$$

$$\sum_{j=1}^{2\xi+1} \cos\left(k\vartheta_j\right) = 2\xi+1 \text{ if } k \text{ is an even multiple of } 2\xi+2 \text{ (including 0)}. \quad (2)$$

We have to study if $u-k-n+l = 2m\left(2\xi+2\right)$, m integer. This is equivalent to:

$$l = n+k-u+2m\left(2\xi+2\right).$$

Since $0 \le l \le 2n$, we need:

$$0 \le n+k-u+2m\left(2\xi+2\right),\ \frac{-n-k+u}{4\xi+4} \le m$$

and:

$$n+k-u+2m\left(2\xi+2\right) \le 2n,\ m \le \frac{n-k+u}{4\xi+4}.$$

Let m_1, m_2 be these two bounds. Let $L_1 = \left\{l_m, l_m = n+k-u+2m\left(2\xi+2\right);\, m_1 \le m \le m_2\right\}$.

We have:

$$T_1=\sum_{\substack{l=0\\ l\notin L_1}}^{2n}\binom{2n}{l}\sum_{j=1}^{2\xi+1}\cos\left(\left(u-k-n+l\right)\vartheta_j\right)+\sum_{m=m_1}^{m_2}\binom{2n}{l_m}\sum_{j=1}^{2\xi+1}\cos\left(\left(u-k-n+l_m\right)\vartheta_j\right)$$

$$=-\frac{1}{2}\sum_{\substack{l=0\\ l\notin L_1}}^{2n}\binom{2n}{l}\left(\left(-1\right)^{u-k-n+l}+1\right)+\left(2\xi+1\right)\sum_{m=m_1}^{m_2}\binom{2n}{n+k-u+4m\left(\xi+1\right)}$$

$$=-\frac{1}{2}\left(-1\right)^{u-k}\sum_{\substack{l=0\\ l\notin L_1}}^{2n}\binom{2n}{l}\left(-1\right)^{n-l}-\frac{1}{2}\sum_{\substack{l=0\\ l\notin L_1}}^{2n}\binom{2n}{l}+\left(2\xi+1\right)\sum_{m=m_1}^{m_2}\binom{2n}{n+k-u+4m\left(\xi+1\right)}$$

In order to simplify the notation, we set $\Delta_1=\sum_{m=m_1}^{m_2}\binom{2n}{n+k-u+4m\left(\xi+1\right)}$. We have:

$$\sum_{\substack{l=0\\ l\notin L_1}}^{2n}\binom{2n}{l}\left(-1\right)^{n-l}=\sum_{l=0}^{2n}\binom{2n}{l}\left(-1\right)^{n-l}-\sum_{m=m_1}^{m_2}\binom{2n}{l_m}\left(-1\right)^{n-l_m}$$

$$\sum_{l=0}^{2n}\binom{2n}{l}\left(-1\right)^{n-l}=0$$

$$\sum_{m=m_1}^{m_2}\binom{2n}{l_m}\left(-1\right)^{n-l_m}=\left(-1\right)^{k-u}\Delta_1$$

So, we get:

$$\sum_{m=m_1}^{m_2}\binom{2n}{l_m}\left(-1\right)^{n-l_m}=\left(-1\right)^{k-u}\Delta_1$$

$$\sum_{\substack{l=0\\ l\notin L_1}}^{2n}\binom{2n}{l}\left(-1\right)^{n-l}=-\left(-1\right)^{k-u}\Delta_1$$

The same way:

$$\sum_{\substack{l=0\\ l\notin L_1}}^{2n}\binom{2n}{l}=\sum_{l=0}^{2n}\binom{2n}{l}-\sum_{m=m_1}^{m_2}\binom{2n}{n+k-u+4m\left(\xi+1\right)}=2^{2n}-\Delta_1$$

$$T_1=\frac{1}{2}\Delta_1-\frac{1}{2}\left(2^{2n}-\Delta_1\right)+\left(2\xi+1\right)\Delta_1=-2^{2n-1}+\left(2\xi+2\right)\Delta_1$$

We now study the term $T_3=\sum_{l=0}^{2n}\binom{2n}{l}\sum_{j=1}^{2\xi+1}\cos\left(\left(u+k-n+l\right)\vartheta_j\right)$. The computations are identical, except that k is replaced by $-k$.

We introduce:

$m_3 = \left\lceil \frac{-n+k+u}{4\xi+4} \right\rceil$ (ceiling: smallest integer greater than or equal to a number) and:

$m_4 = \text{int}\left(\frac{n+k+u}{4\xi+4}\right).$

Let $L_3 = \{l_m, l_m = n-k-u+4m(\xi+1)\,;\, m_3 \le m \le m_4\}$. The same computation as above gives:

$$T_3 = -2^{2n-1} + (2\xi+2)\Delta_3$$

with $\Delta_3 = \sum_{m=m_3}^{m_4} \binom{2n}{n-k-u+4m(\xi+1)}$. Finally:

$$2^{2n+1}(\xi+1)x(u,n,k) = T_1 - T_3 = -2^{2n-1} + (2\xi+2)\Delta_1 - \left(-2^{2n-1} + (2\xi+2)\Delta_3\right) = 2(\xi+1)(\Delta_1 - \Delta_3)$$

and:

$$x(u,n,k) = \frac{\Delta_1 - \Delta_3}{2^n}.$$

We observe that, in Δ_1, the quantity $n+k-u+4m(\xi+1)$ runs from the closest integer to 0 to the closest integer to $2n$, so we may simply write:

$$\Delta_1 = \sum_m \binom{2n}{n+k-u+4m(\xi+1)}$$

$$\Delta_3 = \sum_m \binom{2n}{n-k-u+4m(\xi+1)}$$

This proves Theorem 6.

5. Advantage for each player

Let u, $1 \le u \le 2\xi+1$, be the IFC. We define:

$P_A(n-)$: the probability that A wins at a time $< n$;
$P_A(n+)$: the probability that A wins at a time $\ge n$;
$P_B(n-)$: the probability that B wins at a time $< n$;
$P_B(n+)$: the probability that B wins at a time $\ge n$;

$E(n)$: energy at time $n,$ equal to the probability that the game is not finished at time $n.$

Proposition 7. – *For any* n *:*

$$P_A(n-)=\frac{1}{4}\sum_{m=0}^{n-1} x(m,1),\ P_A(n+)=\frac{1}{4}\sum_{m=n}^{+\infty} x(m,1)$$
$$P_B(n-)=\frac{1}{4}\sum_{m=0}^{n-1} x(m,2\xi+1),\ P_B(n+)=\frac{1}{4}\sum_{m=n}^{+\infty} x(m,2\xi+1)$$
$$E(n)=\sum_{k=1}^{2\xi+1} x(n,k)$$

Proof of Proposition 7

Indeed, for A to win, the game must reach the upper barrier before time $n.$ But, at each step, the probability to do this is $1/4$ of the probability to be at the level 1; the same holds for $B.$ The probability that the game is not finished at time n is simply the sum of the $x(n,k)$ upon all values of k on the corresponding vertical. This proves Proposition 7.

We now investigate the overall issue of the game: what is the probability that A wins? We will consider later the probability to win before a given time.

First, we observe that, as a consequence of Theorem 5, the probability that the games continues up to time n tends to zero with n: the game cannot continue indefinitely. Let us denote by P_A the probability that A wins, and similarly P_B for $B,$ so $P_A+P_B=1.$

Theorem 8. - *Let* $u,$ $1\le u\le 2\xi+1,$ *be the IFC. We have:*

$$P_A=1-\frac{u}{2\xi+2}=\frac{F_A+1}{F_A+F_B+2}$$

and:

$$P_B=\frac{u}{2\xi+2}=\frac{F_B+1}{F_A+F_B+2}.$$

Proof of Theorem 8

The probability that A wins is necessarily linear in $u.$ Indeed, if $u=\xi+1$ (both fortunes are equal), the probability is $\frac{1}{2}.$

If $u=\frac{\xi+1}{2}$, two things may happen, each with probability $\frac{1}{2}$: either the random walk touches the upper barrier (A wins) or it touches the middle line (equal fortunes), and in the latter case we are back to the previous case. So the probability for A to win when $u=\frac{\xi+1}{2}$ is $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$; repeating this dichotomy argument shows that the probability that A wins is linear in u.

For $u=2\xi+2$, both quantities are equal to 0. For $u=1$, by Proposition 7, $P_A=\frac{1}{4}\sum_{m=0}^{\infty} x(m,1)$
and by Theorem 5:

$$P_A=\frac{1}{4}\frac{1}{\xi+1}\sum_{j=1}^{2\xi+1}\sin^2(\vartheta_j)\sum_{n=0}^{+\infty}\cos^{2n}\left(\frac{\vartheta_j}{2}\right)=\frac{1}{4}\frac{1}{\xi+1}\sum_{j=1}^{2\xi+1}\frac{\sin^2(\vartheta_j)}{1-\cos^2\left(\frac{\vartheta_j}{2}\right)}$$

$$=\frac{1}{\xi+1}\sum_{j=1}^{2\xi+1}\cos^2\left(\frac{\vartheta_j}{2}\right)=\frac{2\xi+1}{2\xi+2}$$

which is also the value of $1-\frac{u}{2\xi+2}$ for $u=1$; this proves Theorem 8.

We now give quantitative estimates for $P_A(n-), P_A(n+)$, and the same for B. Obviously, $P_A(n-)+P_B(n-)+E(n)=1$ for all n; all these probabilities depend on u, initial fortune, and on ξ, value of the barriers.

Obviously also, $P_A=\lim_{n\to+\infty}P_A(n-)$.

Proposition 9. – *We have, for any* n *:*

$$P_A(n+)=\frac{1}{2\xi+2}\sum_{j=1}^{2\xi+1}\frac{\sin(u\,\vartheta_j)}{\sin\left(\frac{\vartheta_j}{2}\right)}\cos^{2n+1}\left(\frac{\vartheta_j}{2}\right).$$

Let U_k *be the family of Chebyshev's polynomials of second kind. We have the equivalent formulation:*

$$P_A(n+)=\frac{1}{2\xi+2}\sum_{j=1}^{2\xi+1}U_{2u-1}(y_j)\,y_j^{2n+1},\ \textit{with}\ y_j=\cos\left(\frac{\vartheta_j}{2}\right).$$

Proof of Proposition 9

We compute the probability that A wins at a time $\geq n$, which happens if the random walk touches the upper barrier. Using Theorem 5, and Proposition 7, we have:

$$P_A(n+)=\frac{1}{4}\frac{1}{\xi+1}\sum_{j=1}^{2\xi+1}\sin(u\,\vartheta_j)\sin(\vartheta_j)\sum_{m=n}^{+\infty}\cos^{2m}\left(\frac{\vartheta_j}{2}\right)$$

Using the identity $\sum_{m=n}^{+\infty}\cos^{2m}\left(\frac{\vartheta_j}{2}\right)=\frac{\cos^{2n}\left(\frac{\vartheta_j}{2}\right)}{1-\cos^2\left(\frac{\vartheta_j}{2}\right)}=\frac{\cos^{2n}\left(\frac{\vartheta_j}{2}\right)}{\sin^2\left(\frac{\vartheta_j}{2}\right)}$, we get:

$$P_A(n+)=\frac{1}{4}\frac{1}{\xi+1}\sum_{j=1}^{2\xi+1}\frac{\sin(u\,\vartheta_j)\sin(\vartheta_j)}{\sin^2\left(\frac{\vartheta_j}{2}\right)}\cos^{2n}\left(\frac{\vartheta_j}{2}\right).$$

Replacing $\sin\vartheta_j=2\sin\frac{\vartheta_j}{2}\cos\frac{\vartheta_j}{2}$ proves the first part of Proposition 9. In order to prove the second part, recall that the k^{th} Chebyshev' polynomial of the second kind $U_k(x)$ satisfies:

$$U_k(\cos(\vartheta))=\frac{\sin((k+1)\vartheta)}{\sin\vartheta}.$$

Corollary 10. – *For all n:*

$$P_A(n-)=1-\frac{u}{2\xi+2}-\frac{1}{2\xi+2}\sum_{j=1}^{2\xi+1}\frac{\sin(u\,\vartheta_j)}{\sin\left(\frac{\vartheta_j}{2}\right)}\cos^{2n+1}\left(\frac{\vartheta_j}{2}\right)$$

This is an obvious consequence of the previous Propositions. The probability that A wins before time n is of course increasing with n. We now study the probability that the game continues after step n.

6. Probability that the game continues

Proposition 11. –*We have, for any u,n:*

$$E(n)=\frac{1}{\xi+1}\sum_{l=0}^{\xi}\frac{\sin(u\,\vartheta_{2l+1})}{\sin\left(\frac{\vartheta_{2l+1}}{2}\right)}\cos^{2n+1}\left(\frac{\vartheta_{2l+1}}{2}\right)$$

For $l=0,\dots,\xi$, we set $z_l=\cos\left(\frac{\vartheta_{2l+1}}{2}\right)$. Let U_k be the family of Chebyshev's polynomials of second kind. We have the identity:

$$E(n)=\frac{1}{\xi+1}\sum_{l=0}^{\xi}U_{2u-1}(z_l)z_l^{2n+1}$$

Proof of Proposition 11

Using Theorem 5, let us compute the probability that the game is not finished at time n. We have:

$$E(n)=\sum_{k=1}^{2\xi+1}x(n,k)=\frac{1}{\xi+1}\sum_{j=1}^{2\xi+1}\sin(u\,\vartheta_j)\cos^{2n}\left(\frac{\vartheta_j}{2}\right)\sum_{k=1}^{2\xi+1}\sin(k\vartheta_j).$$

The identity:

$$\sum_{k=1}^{2\xi+1}\sin(k\vartheta_j)=\frac{1-(-1)^j}{2}\frac{\cos\frac{\vartheta_j}{2}}{\sin\frac{\vartheta_j}{2}}$$

gives:

$$E(n)=\frac{1}{\xi+1}\sum_{j=1}^{2\xi+1}\frac{1-(-1)^j}{2}\frac{\sin(u\,\vartheta_j)}{\sin\frac{\vartheta_j}{2}}\cos^{2n+1}\left(\frac{\vartheta_j}{2}\right)$$

But $\frac{1-(-1)^j}{2}=0$ if j is even, $=1$ if j is odd. Set $j=2l+1$; when $j=1,...,2\xi+1$, we have $l=0,...,\xi$. We obtain:

$$E(n)=\frac{1}{\xi+1}\sum_{l=0}^{\xi}\frac{\sin(u\,\vartheta_{2l+1})}{\sin\left(\frac{\vartheta_{2l+1}}{2}\right)}\cos^{2n+1}\left(\frac{\vartheta_{2l+1}}{2}\right).$$

So, with $z_l=\cos\left(\frac{\vartheta_{2l+1}}{2}\right)$, we have:

$$E(n)=\frac{1}{\xi+1}\sum_{l=0}^{\xi}U_{2u-1}(z_l)z_l^{2n+1}.$$

This proves Proposition 11.

We observe that $E(n)$ is not modified if u is replaced by $2\xi+2-u$ (exchanging the roles of A and B). Indeed:

$\sin\left((2\xi+2-u)\,\vartheta_{2l+1}\right)=\sin\left(\pi-u\vartheta_{2l+1}\right)=\sin\left(u\,\vartheta_{2l+1}\right).$

7. Examples

We now turn to some numerical examples, in order to understand the shapes of the various profiles; they may look surprising at first sight. Let us assume first $\xi=7$. We consider 3 cases: $u=1$ (high initial fortune for A), $u=\xi+1$ (equal fortunes for A and B), $u=2\xi+1$ (low initial fortune for A).

Case 1: $u=1$

By Theorem 8, the probability that A wins the game is $P_A=\frac{2\xi+1}{2\xi+2}=\frac{15}{16}=0.9375$.

For the probabilities to win before a certain time, we find:

$n=10$	$P_A(10-)\approx 0.66$
$n=20$	$P_A(20-)\approx 0.755$
$n=50$	$P_A(50-)\approx 0.842$
$n=100$	$P_A(100-)\approx 0.888$

For the probabilities to continue until a certain time, we find:

$n=10$	$E(10)\approx 0.336$
$n=20$	$E(20)\approx 0.245$
$n=50$	$E(50)\approx 0.156$
$n=100$	$E(100)\approx 0.094$

Case 2: $u=\xi+1$

By Theorem 8, the probability that A wins the game is $P_A=\frac{\xi+1}{2\xi+2}=\frac{1}{2}$.

For the probabilities to win before a certain time, we find:

$n=10$	$P_A(10-)\approx 0.66$
$n=20$	$P_A(20-)\approx 0.755$
$n=50$	$P_A(50-)\approx 0.842$
$n=100$	$P_A(100-)\approx 0.888$

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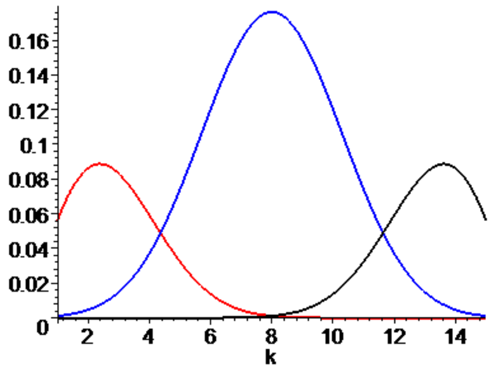
$n=50$	$E(50)\approx 0.156$
$n=100$	$E(100)\approx 0.094$

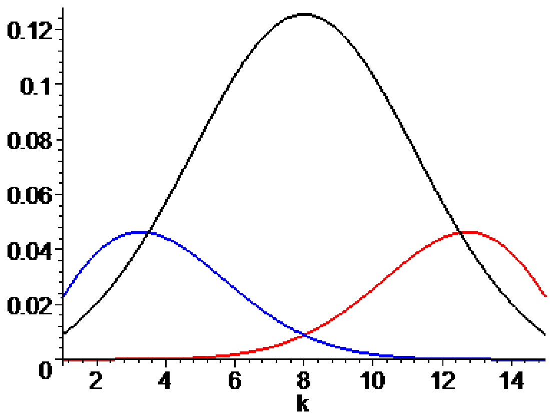
Let us draw explicitly the various profiles of energy, depending on the initial fortune.

For $n=10$:

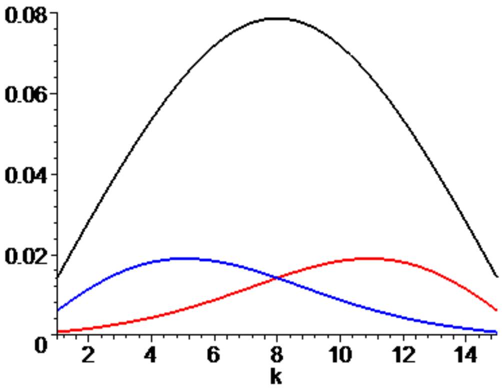


In black: profile for $u=1$ (strong A), in blue: profile for even strengths, in red, profile for weak A.

For $n=20$:

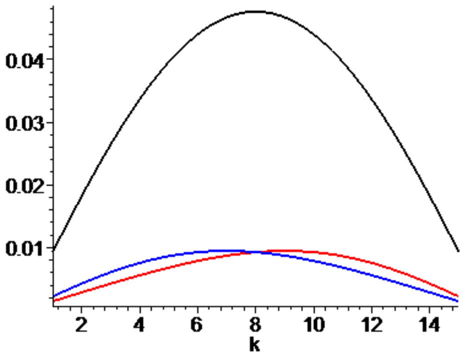


For $n=50$:



We see here that the profiles for A initially strong and A initially weak tend to coincide.

For $n = 100$:



and here, for large n, they are almost identical, and well below the profile for A initially identical to B. This looks strange at first sight, but in fact, if initially A is very strong, for instance $u = 1$, the only paths which will reach the time 100 are the ones which meet the middle line, for which $A = B$. In other words, if A is very rich compared to B, he should win rather early, otherwise, very likely, both fortunes will be equal.

Let us state this as a Proposition:

Proposition 12. - *When* $n \to +\infty$, *the profiles for any* u *tend to coincide.*

Proof of Proposition 12

Let us prove it for the two most different profiles: $u = 1$ and $u = 2\xi + 1$. In the first case, the energy profile is:

$$x_1(n,k) = \frac{1}{\xi+1} \sum_{j=1}^{2\xi+1} \sin(\vartheta_j) \sin(k\vartheta_j) \lambda_j^n$$

and in the second case:

$$x_{2\xi+1}(n,k) = \frac{1}{\xi+1} \sum_{j=1}^{2\xi+1} \sin((2\xi+1)\vartheta_j) \sin(k\vartheta_j) \lambda_j^n$$

So, the difference is:

$$d(n,k) = \frac{1}{\xi+1}\sum_{j=1}^{2\xi+1} \left(\sin(\vartheta_j) - \sin((2\xi+1)\vartheta_j)\right)\sin(k\vartheta_j)\lambda_j^n$$
$$= \frac{2}{\xi+1}\sum_{j=1}^{2\xi+1} \sin(\xi\vartheta_j)\cos((\xi+1)\vartheta_j)\sin(k\vartheta_j)\lambda_j^n$$

But $(\xi+1)\vartheta_j = \frac{j\pi}{2}$, so $\cos((\xi+1)\vartheta_j) = 0$ if j is odd, and $=(-1)^{j_1}$ if $j = 2j_1$. Therefore:

$$d(n,k) = \frac{2}{\xi+1}\sum_{j_1=1}^{\xi} (-1)^{j_1}\sin(\xi\vartheta_{2j_1})\sin(k\vartheta_{2j_1})\lambda_{2j_1}^n$$

We have $\xi\vartheta_{2j_1} = \frac{\xi 2 j_1\pi}{2\xi+2} = \frac{\xi j_1\pi}{\xi+1} = j_1\pi - \frac{j_1\pi}{\xi+1}$; therefore:

$$\sin(\xi\vartheta_{2j_1}) = \sin\left(j_1\pi - \frac{j_1\pi}{\xi+1}\right) = (-1)^{j_1-1}\sin\left(\frac{j_1\pi}{\xi+1}\right)$$

and finally:

$$d(n,k) = \frac{-2}{\xi+1}\sum_{j_1=1}^{\xi} (-1)^{j_1}\sin\left(\frac{j_1\pi}{\xi+1}\right)\sin\left(k\frac{j_1\pi}{\xi+1}\right)\lambda_{2j_1}^n.$$

The first term in the sum involves λ_2^n, whereas the expressions of x_1 and $x_{2\xi+1}$ involve λ_1^n; the difference between both will tend to 0, when $n \to +\infty$, more quickly than each term separately. This proves our Proposition.

We observe that $P_A(n+)$ is exponentially decreasing with n. Its shape, depending on u, for fixed ξ, n, is also rather surprising :

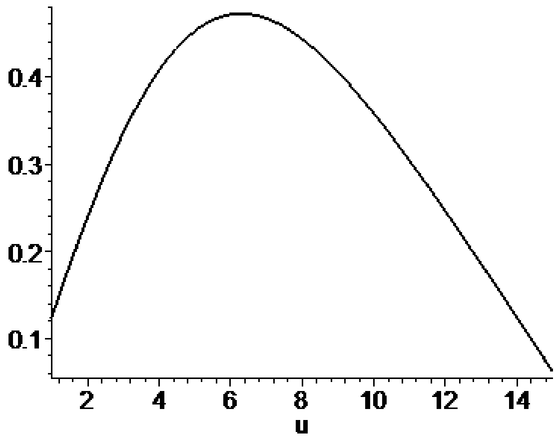


Fig.: the graph of $P_A(n+)$, *for* $\xi = 7$, $n = 35$, *as a function of* u.

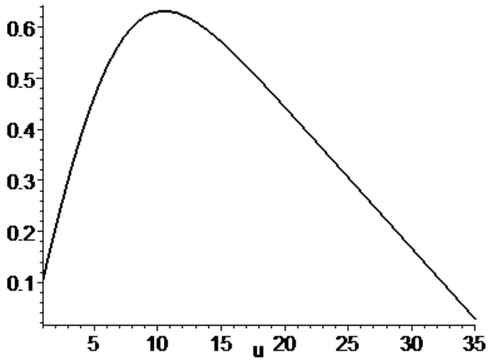


Fig.: the graph of $P_A(n+)$, *for* $\xi = 17$, $n = 70$, *as a function of* u.

The low values for u small (when A has the largest initial fortune) come from the fact that, under these conditions, A should win at an early stage.

Large fortunes

Let us now consider the case of large, but different, initial fortunes: $F_A = 2\,000$, $F_B = 1\,000$. Let us compute the probabilities for $n = 100\,000, 500\,000, 1\,000\,000$. In each case, we compute the probability that A (resp. B) wins before n and after n :

$n = 100\,000$	Proba wins before n	Proba wins after n
A	0.025	0.641
B	0.000122	0.333

$n = 500\,000$	Proba wins before n	Proba wins after n
A	0.317	0.350
B	0.0454	0.288

$n = 1\,000\,000$	Proba wins before n	Proba wins after n
A	0.479	0.188
B	0.153	0.181

These results are quite interesting. They show that, if one waits long enough, the probabilities to win the game after n tend to be equal. In the case of different fortunes (here, A is twice as rich as B, so his probability to win is roughly 2/3), A should win at an early stage, otherwise the game becomes more balanced.

8. Asymptotic profile for a general initial situation

Assume now that, according to what we saw in Part I, we start with a general initial distribution of energy on the y axis, denoted by u_i, $i = 1, \ldots, 2\xi + 1$. The question is: what do we get asymptotically, when $n \to +\infty$?

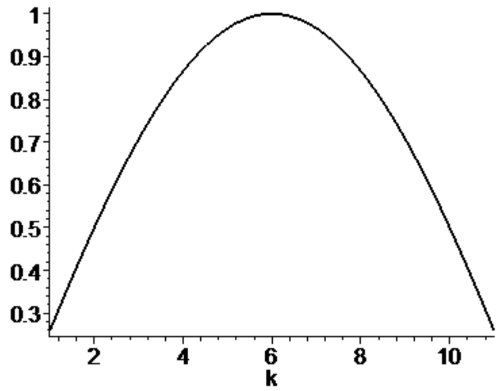
The answer to this question follows easily from Theorem 5 above:

Proposition 13. – *Asymptotically, when* $n \to +\infty$*, the energy at position* k *is equivalent to* $\sin(k\vartheta_1) C\lambda_1^n$*, with* $C = \frac{1}{\xi+1}\sum_{i=1}^{2\xi+1}\sin(u_i\vartheta_1)$ *and, as before,* $\lambda_1 = \cos^2\frac{\pi}{4\pi+4}$*,* $\vartheta_1 = \frac{\pi}{2\xi+2}$.

Proof of Proposition 13

This is an immediate consequence of Theorem 5 above, since asymptotically only the first term in the sum remains.

Here is an example, in the case $\xi = 5$:



Shape of the asymptotic energy profile, when $\xi = 5$

The interesting fact is that this shape is independent of the initial energy, up to the constant factor C defined above.

V. Counting the paths

Just as we did in Part II, let us count the paths which reach any vertical W_{2n} when the barrier is set at $2\xi+2$. The times under consideration are all even times ($2n$).

If we look at the 3VRP (Three-Value Random Process), its values (which exist only at even times) must be in the interval $[-2\xi, 2\xi]$; the 3VRP is confined in this interval. The RW, which has values at all times, is confined in the interval $[-2\xi-1, 2\xi+1]$. As we did in Part II, we may consider 4 types of paths:

- Those which touch the upper boundary, not the lower boundary (number N_1);
- Those which touch the lower boundary, not the upper boundary (number N_2);
- Those which touch neither the lower nor the upper boundaries (number N_3);
- Those which touch both boundaries (number N_4).

These four sets are disjoint, and their union is equal to the set of all confined strips.

If we consider only even times, counting is easy, because we work directly with the 3VRP. In order to determine N_c, we put the barriers at $\pm(2\xi+2)$, as we did here; let us denote by $N(-2\xi,2\xi)$ the number of paths confined in the strip $[-2\xi,2\xi]$; then $N(-2\xi,2\xi)=2^{2n}E_{2n}$, where E_{2n} is the energy, computed above, which reaches the vertical W_{2n}. More specifically, if M is the matrix defined in § IV above, the energy remaining at time $2n$ is $E_{2n}=\left|M^n X_0\right|_1$, where X_0 has "1" at the $\xi+1^{st}$ place, 0 elsewhere.

If we repeat the same argument with the upper barrier at 2ξ and the lower barrier at $-2\xi-2$ (starting point at 0), we obtain the number of strips confined in the range $[-2\xi,2\xi-2]$, $N(-2\xi,2\xi-2)$. Then the difference $N(-2\xi,2\xi)-N(-2\xi,2\xi-2)$ is the number N_1 of strips, confined in $[-2\xi,2\xi]$, which touch 2ξ at least once.

The same way, if we repeat the same argument with the upper barrier at $2\xi+2$ and the lower barrier at -2ξ (starting point at 0), we obtain the number of strips confined in the range $[-2\xi+2,2\xi]$, $N(-2\xi+2,2\xi)$. Then the difference $N(-2\xi,2\xi)-N(-2\xi+2,2\xi)$ is the number N_2 of strips, confined in $[-2\xi,2\xi]$, which touch -2ξ at least once.

If we shrink both barriers, putting them at $\pm 2\xi$, we obtain the number N_3 of strips which are confined in $[-2\xi+2,2\xi-2]$, that is which do not touch any of the boundaries.

And finally, the number of strips N_4 which touch both boundaries are obtained from the difference $N_4=N_c-(N_1+N_2+N_3)$.

For instance, when $n=8$, $\xi=2$, the matrix M has size 5 and $N_c=46\,732$; here is the repartition:

Total number of paths	46 732
Touch upper, not lower	4 365
Touch lower, not upper	4 365
do not touch upper nor lower	38 000
Touch both	2

In order to consider all times, not just even times, we would have to develop a theory, analogous to the one presented here, where the barriers would be put at odd numbers.

Remark. – One may wonder if the reflection principle might be of any use in counting the paths touching both the upper and lower barrier. This idea is described by the following figure:

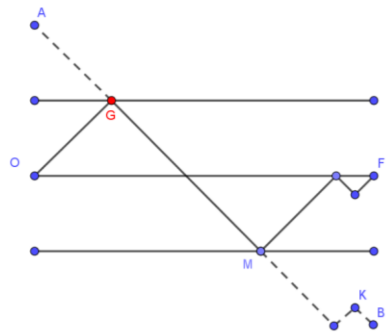


Fig.: paths touching both barriers and paths obtained by symmetry

Let us consider the paths from O to F, touching the upper barrier in G and the lower barrier in M. One can use a symmetry to build the path GA from GO and to build the path MB from MF. So, this way, one obtains a correspondence between the paths from O to F, touching the barriers at G, M respectively, and all paths from A to B, which are easy to count. Unfortunately, the paths from A to B are not necessarily restricted to the strip after the point G, as the following picture shows:

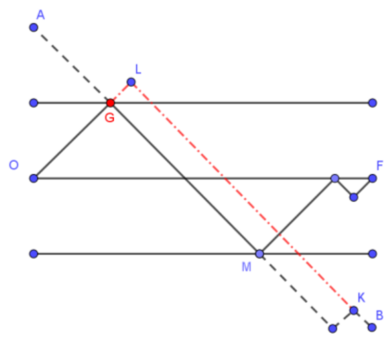


Fig.: difference between both sets of paths

So, in fact, there are more paths from A to B than paths from O to F, contained in the strip, touching both barriers.