



**Simple Random Walks in the plane :**  
**An energy based approach**

**Part III : Different Initial Fortunes**

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August, 2018

**Abstract**

We consider the same  $\pm 1$  game, with now different initial bounded fortunes. A player gets ruined and the game stops if his fortune reduces to 0.

Our main result is an explicit expression of the probability of any situation at time  $n$ , in terms of the initial fortunes  $F_A$  and  $F_B$  (Theorem 5). We deduce the probability that  $A$  wins the game, and the probability that the game continues after a given time  $n$ .

As we did in Part II, our approach relies upon a representation of the game in terms of "energy propagation", which allows us to use arguments from operator theory and special functions (more specifically, Chebycheff polynomials of first and second kind).

## I. Presentation

The basic settings are presented in Part I. Recall that we consider a simple random walk in the plane: a game, with two players. It is defined by a r.v.  $X$  taking the values  $\pm 1$  with probability  $\frac{1}{2}$ . The player  $A$  wins if  $X = 1$  and then he receives 1 Euro from the player  $B$ , and conversely if  $X = -1$ . Each player has an initial fortune, denoted respectively by  $F_A$  and  $F_B$ . The game stops if any of the players sees his fortune equal to zero. As we did earlier, we will assume that both fortunes take only odd values:  $F_A = 2a + 1$ ,  $F_B = 2b + 1$ , where  $a, b$  are positive integers. We assume for simplicity that  $a > b$ .

## II. A preliminary remark

We have two possible descriptions of the game:

### 1. Description 1

Each player has his fortune  $F_A, F_B$  and the game starts at 0. It finishes if one of the players gets ruined. More precisely, we set  $S_0 = 0$ , and for  $N \geq 1$ ,  $S_N = \sum_{n=1}^N X_n$ . If  $S_N = -F_A$ , the player  $A$  is ruined and the game stops ; if  $S_N = F_B$  the player  $B$  is ruined and the game also stops. So, mathematically speaking, the random walks starts at 0 and the game finishes when it touches one of the two barriers  $y = -F_A, y = F_B$ . In this description, the starting point is 0 and both barriers are dissymmetric.

### 2. Description 2

The barriers are set at the symmetric values  $\pm(2\xi + 1)$ , with  $\xi = \frac{a+b}{2}$  and the starting point is  $(0, 2y_0)$ , with  $y_0 = \frac{a-b}{2}$ .

Quite clearly, both descriptions are equivalent, mathematically speaking. Indeed, for  $A$  to win, in the first description, the random walk has to climb  $F_B$  steps, and in the second description it has to climb  $2\xi - 2y_0$  steps. For  $B$  to win, in the first description, the random walk has to go down  $F_A$  steps and in the second description it has to go down  $2\xi - 2y_0$  steps.

So we will work with description 2, which allows us to use the framework developed previously. The barriers are symmetrically set at  $\pm(2\xi + 1)$  and the starting point is at  $(0, 2y_0)$ .

### III. Notation

We refer to Part I. Instead of a random walk with multiple possible paths, we consider that we have the propagation of an energy, with the following rules:

- At time  $n = 0$ , the starting point  $(0, 2y_0)$  receives an energy equal to 1;
- At time  $n = 1$ , this energy is divided into two: each point  $(1, 2y_0 + 1)$  and  $(1, 2y_0 - 1)$  receives an energy equals to  $\frac{1}{2}$  and so on.

More generally, the energy of a point of coordinates  $(n, k)$  in the plane is equal to the probability that the random walk hits this point. It will be denoted by  $e(n, k)$ .

As we already did in Part I, we restrict ourselves to even values of the time  $(2n)$ . Also, we will need a "matrix-oriented" notation. Instead of the  $y$ -coordinate ranging between  $-\xi$  and  $+\xi$ , we will go downwards, starting at 1 and descending to  $2\xi + 1$ . More precisely, we set:

$$x(n, k) = e(2n, 2(\xi - k + 1)).$$

So  $x(n, 1) = e(2n, 2\xi)$  (near the top barrier) and  $x(n, 2\xi + 1) = e(2n, -2\xi)$  (near the bottom barrier). We have the propagation rules :

$$\begin{cases} x(n+1, 1) = \frac{1}{2}x(n, 1) + \frac{1}{4}x(n, 2) \\ x(n+1, k) = \frac{1}{4}x(n, k-1) + \frac{1}{2}x(n, k) + \frac{1}{4}x(n, k+1), k = 2, \dots, 2\xi \\ x(n+1, 2\xi + 1) = \frac{1}{4}x(n, 2\xi) + \frac{1}{2}x(n, 2\xi + 1) \end{cases} \quad (1)$$

They take into account the symmetric barriers  $y = \pm(2\xi + 1)$ : if a path hits any of the barriers, its energy is absorbed and disappears. The last non-zero values for  $x$  on each vertical are  $x(n, 1)$ ,  $x(n, 2\xi + 1)$ .

The initial value is  $x(0, y_0) = 1$ ,  $x(0, k) = 0$  if  $k \neq y_0$ . (2)

Let  $W_{2n}$  be the vertical for  $x = 2n$ , that is the set of all points  $A_{2n, 2k}$ ,  $k = -n, \dots, n$ .

## IV. Matrix representation

The system (1) may be represented as a matrix :

$$\begin{pmatrix} x(n+1,1) \\ x(n+1,2) \\ \dots \\ \dots \\ \dots \\ \dots \\ \dots \\ \dots \\ x(n+1,2^\xi) \\ x(n+1,2^{\xi+1}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \dots & \dots & \dots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \dots & \dots & \dots & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x(n,1) \\ x(n,2) \\ \dots \\ \dots \\ \dots \\ \dots \\ \dots \\ \dots \\ x(n,2^\xi) \\ x(n,2^{\xi+1}) \end{pmatrix} \quad (3)$$

This is real symmetric matrix, of size  $2^\xi + 1$ , denoted by  $M$ .

As we already said, the first coordinate is close to the upper barrier, the last coordinate close to the lower barrier.

The general approach is the same as in Part II, but the matrix is different. An important remark is that we do not need here a second change in coordinates, which makes things simpler.

**Lemma 1.** - *The matrix  $M$  is positive definite.*

### Proof of Lemma 1

We have to show that, for all non-zero column-vector  $X$  of size  $2^\xi + 1$ , we have  $X'MX > 0$ .

Let  $X = \begin{pmatrix} x_1 \\ \vdots \\ x_{2^{\xi+1}} \end{pmatrix}$ ; we have:

$$MX = \begin{pmatrix} \frac{1}{2}x_1 + \frac{1}{4}x_2 \\ \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3 \\ \vdots \\ \frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1} \\ \vdots \\ \frac{1}{4}x_{2\xi-1} + \frac{1}{2}x_{2\xi} + \frac{1}{4}x_{2\xi+1} \\ \frac{1}{4}x_{2\xi} + \frac{1}{2}x_{2\xi+1} \end{pmatrix}$$

and therefore:

$$\begin{aligned} X'MX &= \left(\frac{1}{2}x_1 + \frac{1}{4}x_2\right)x_1 + \left(\frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3\right)x_2 + \dots + \left(\frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1}\right)x_i + \dots + \\ &\quad + \left(\frac{1}{4}x_{2\xi-1} + \frac{1}{2}x_{2\xi} + \frac{1}{4}x_{2\xi+1}\right)x_{2\xi} + \left(\frac{1}{4}x_{2\xi} + \frac{1}{2}x_{2\xi+1}\right)x_{2\xi+1} \\ &= \frac{1}{4}x_1^2 + b + \frac{1}{4}x_{2\xi+1}^2 \end{aligned}$$

$$\text{with } b = \frac{1}{4}\left((x_1 + x_2)^2 + \dots + (x_i + x_{i-1})^2 + \dots + (x_{2\xi} + x_{2\xi+1})^2\right).$$

So clearly  $X'MX > 0$  if the  $x_i$  are not all equal to 0. This proves Lemma 1.

From this Lemma follows that all eigenvalues of  $M$  are real,  $> 0$ , and that  $M$  can be diagonalized in an orthogonal basis made of eigenvectors.

**Lemma 2.** - *All eigenvalues of  $M$  are  $< 1$ .*

### Proof of Lemma 2

Let us write the system of equations defining the eigenvalues and eigenvectors,  $MX = \lambda X$ .

$$\left\{ \begin{array}{l} \frac{1}{2}x_1 + \frac{1}{4}x_2 = \lambda x_1 \\ \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3 = \lambda x_2 \\ \vdots \\ \frac{1}{4}x_{i+1} + \frac{1}{2}x_i + \frac{1}{4}x_{i-1} = \lambda x_i \\ \vdots \\ \frac{1}{4}x_{2\xi-1} + \frac{1}{2}x_{2\xi} + \frac{1}{4}x_{2\xi+1} = \lambda x_{2\xi} \\ \frac{1}{4}x_{2\xi} + \frac{1}{2}x_{2\xi+1} = \lambda x_{2\xi+1} \end{array} \right. \quad (4)$$

It may be written:

$$\left\{ \begin{array}{l} x_2 = (4\lambda - 2)x_1 \\ x_3 = (4\lambda - 2)x_2 - x_1 \\ \vdots \\ x_{i+1} = (4\lambda - 2)x_i - x_{i-1} \\ \vdots \\ x_{2\xi+1} = (4\lambda - 2)x_{2\xi} - x_{2\xi-1} \\ x_{2\xi} = (4\lambda - 2)x_{2\xi+1} \end{array} \right. \quad (5)$$

We know that  $x_1 \neq 0$  (if  $x_1 = 0$ , all  $x_i = 0$ ), so we may take  $x_1 = 1$ . Assume  $\lambda \geq 1$ . From the equation  $x_2 = (4\lambda - 2)x_1$  we deduce  $x_2 > x_1 > 0$ . More generally, the equation  $x_{i+1} = (4\lambda - 2)x_i - x_{i-1}$  gives:

$$x_{i-1} - x_i = (4\lambda - 3)x_i - x_{i+1} \geq x_i - x_{i+1}$$

that is  $x_i - x_{i-1} \leq x_{i+1} - x_i$ . So the sequence of consecutive differences is increasing. Since

$x_2 > x_1$ , all differences are positive, the  $x_i$  are increasing and are  $> 0$ . Set  $S = \sum_{i=1}^{2\xi+1} x_i$ ; sum-

ming all equations, we get:

$$S - x_1 + x_{2\xi} = (4\lambda - 2)S - (S - x_{2\xi} - x_{2\xi+1}),$$

that is  $-x_1 - x_{2\xi+1} = 4(\lambda - 1)S$ . But this is a contradiction:  $\lambda > 1$ ,  $S > 0$  and  $x_1 > 0$ ,  $x_{2\xi+1} > 0$ .

Lemma 2 is proved.

From Lemmas 1 and 2 follows that all eigenvalues of  $M$  are strictly between 0 and 1.

**Proposition 3.** - For  $j = 1, \dots, 2\xi + 1$ , the  $j^{\text{th}}$  eigenvalue  $\lambda_j$  is:

$$\lambda_j = \cos^2 \frac{j\pi}{4(\xi + 1)}$$

and the  $j^{\text{th}}$  eigenvector has components :

$$V_j = (\sin(\mathcal{G}_j), \sin(2\mathcal{G}_j), \dots, \sin((2\xi + 1)\mathcal{G}_j))$$

with  $\mathcal{G}_j = \frac{j\pi}{2\xi + 2}$ ,  $j = 1, \dots, 2\xi + 1$ .

**Remark.** – We observe that this definition of  $\mathcal{G}_j$  is different from the one given in Part II.

### Proof of Proposition 3

We have  $x_1 \neq 0$  (otherwise all  $x_j$ 's are 0), so we may assume  $x_1 = 1$ . We set  $\mu = 2\lambda - 1$ , so  $-1 < \mu < 1$ . We set also  $y_0 = x_1$ ,  $y_1 = x_2, \dots, y_j = x_{j+1}, \dots, y_{2\xi} = x_{2\xi+1}$ . System (5) becomes:

$$\left\{ \begin{array}{l} y_0 = 1 \\ y_1 = 2\mu \\ y_2 = 2\mu y_1 - y_0 \\ \vdots \\ y_j = 2\mu y_{j-1} - y_{j-2} \\ \vdots \\ y_{2\xi} = 2\mu y_{2\xi-1} - y_{2\xi-2} \\ y_{2\xi-1} = 2\mu y_{2\xi} \end{array} \right. \quad (6)$$

Therefore,  $y_j = U_j(\mu)$  where  $U_j$  is the  $j^{\text{th}}$  Chebyshev's polynomial of second kind, for  $j = 0, \dots, 2\xi$ . The final equation in (6) may be written:

$$U_{2\xi-1}(\mu) = 2\mu U_{2\xi}(\mu) \quad (7)$$

that is, with  $\mu = \cos(\mathcal{G})$ :

$$\frac{\sin(2\xi\mathcal{G})}{\sin(\mathcal{G})} = 2\cos(\mathcal{G}) \frac{\sin((2\xi + 1)\mathcal{G})}{\sin(\mathcal{G})}.$$

We know that  $-1 < \cos \vartheta < 1$ , which implies  $\sin(\vartheta) \neq 0$ , so the above equation is equivalent to:

$$\sin(2\xi\vartheta) = 2\cos(\vartheta)\sin((2\xi+1)\vartheta) \quad (8)$$

We have :

$$\sin(2\xi\vartheta) - 2\cos(\vartheta)\sin((2\xi+1)\vartheta) = -\sin((2\xi+2)\vartheta)$$

Therefore, equation (8) is equivalent to:

$$\sin((2\xi+2)\vartheta) = 0 \quad (9)$$

that is  $\vartheta = \frac{j\pi}{2\xi+2}$ ,  $j \in \mathbb{Z}$ .

We are interested only in the values of  $\cos \vartheta$  ; moreover, the solutions  $j = 0$  and  $j = 2\xi + 2$  gives  $\vartheta = 0$ , which is impossible ( $-1 < \cos \vartheta < 1$ ). So we have the solutions:

$$\vartheta_j = \frac{j\pi}{2\xi+2}, \quad j = 1, \dots, 2\xi+1 \quad (10)$$

This gives, for  $j = 1, \dots, 2\xi+1$  :  $\mu_j = \cos(\vartheta_j) = \cos \frac{j\pi}{2(\xi+1)}$  and:

$$\lambda_j = \frac{\mu_j + 1}{2} = \frac{1}{2} \left( 1 + \cos \frac{j\pi}{2(\xi+1)} \right) = \cos^2 \frac{j\pi}{4(\xi+1)}.$$

The eigenvector in coordinates  $y_j$  defined by (6) gives  $y_j = U_j(\cos \vartheta) = \frac{\sin((j+1)\vartheta)}{\sin \vartheta}$ . After multiplication, we may take  $y_j = \sin((j+1)\vartheta)$ ,  $j = 0, \dots, 2\xi$ . Returning to the  $x$  coordinates, this gives:

$$x_1 = \sin \vartheta, x_2 = \sin(2\vartheta), \dots, x_j = \sin(j\vartheta), \dots, x_{2\xi+1} = \sin((2\xi+1)\vartheta)$$

So, the  $j^{\text{th}}$  eigenvector is:

$$V_j = \left( \sin(\vartheta_j), \sin(2\vartheta_j), \dots, \sin((2\xi+1)\vartheta_j) \right) \quad (11)$$

where  $\vartheta_j$  is given by (10). This finishes the proof of Proposition 3.



We already made the following remarks in Part II ; they are still valid here:

- The first eigenvector,  $V_1$ , has all its components real and  $> 0$ , but all other eigenvectors have some negative component;
- It follows from the general theory of symmetric matrices, positive defined, that any two eigenvectors  $V_{j_1}, V_{j_2}$  are mutually orthogonal, that is:

$$\sum_{l=1}^{2\xi+1} \sin(l\vartheta_{j_1}) \sin(l\vartheta_{j_2}) = 0,$$

where here  $\vartheta_{j_1} = \frac{j_1\pi}{2\xi+2}$ ,  $\vartheta_{j_2} = \frac{j_2\pi}{2\xi+2}$ .

We now compute the  $l_1$  – norm and the  $l_2$  – norm of the eigenvectors

### 3. Norms of the eigenvectors

**Proposition 4.** – *All eigenvectors have the same  $l_1$  – norm:*

$$|V_j|_1 = \frac{\sin\left(\frac{\pi}{2\xi+2}\right)}{1 - \cos\left(\frac{\pi}{2\xi+2}\right)} = \frac{\cos\left(\frac{1}{2}\frac{\pi}{2\xi+2}\right)}{\sin\left(\frac{1}{2}\frac{\pi}{2\xi+2}\right)}$$

*All eigenvectors have the same quadratic norm:*

$$|V_j|_2^2 = \xi + 1$$

for  $j = 1, \dots, 2\xi + 1$ .

#### Proof of Proposition 4

All of them are, up to changes of signs, reordering of the terms of  $V_1$ . Indeed, when  $j$  changes, the  $2\xi+1$  numbers  $\sin(k\vartheta_j)$  ( $k = 1, \dots, 2\xi+1$ ) are reordering of the  $\xi$  numbers  $\sin(k\vartheta_1)$ , except for the sign, which may become minus (this does not affect the norms).

So, let us compute  $|V_1|_1$ . Apply the matrix  $M$  to the eigenvector  $V_1$ : by definition, we get  $MV_1 = \lambda_1 V_1$  and the loss of energy is  $(1 - \lambda_1)|V_1|_1$ . But this loss of energy comes from the first and the last coordinates only:

- The first coordinate gives a loss of  $\frac{1}{4} \sin(\vartheta_1) = \frac{1}{4} \sin \frac{\pi}{2\xi+2}$  ;
- The last coordinate gives a loss of  $\frac{1}{4} \sin((2\xi+1)\vartheta_1) = \frac{1}{4} \sin\left(\frac{2\xi+1}{2\xi+2} \pi\right) = \frac{1}{4} \sin \frac{\pi}{2\xi+2}$ ,  
which is the same.

So the total loss is  $\frac{1}{2} \sin \frac{\pi}{2\xi+2}$ . So we get  $(1-\lambda_1)|V_{11}| = \frac{1}{2} \sin \frac{\pi}{2\xi+2}$ .

We know that  $\lambda_1 = \cos^2 \frac{\pi}{4(\xi+1)}$ , so  $1-\lambda_1 = \sin^2 \frac{\pi}{4(\xi+1)} = \frac{1}{2} \left(1 - \cos \frac{\pi}{2\xi+2}\right)$ , which gives :

$$|V_{11}| = \frac{\sin\left(\frac{\pi}{2\xi+2}\right)}{1 - \cos\left(\frac{\pi}{2\xi+2}\right)}$$

and this result is valid for all eigenvectors.

Let us now compute  $|V_{12}|^2$ . We use the identity:

$$\sum_{k=1}^{2\xi+1} \sin^2(kt) = \xi + \frac{3}{4} - \frac{1}{4} \frac{\sin((4\xi+3)t)}{\sin(t)}$$

which gives :

$$|V_{12}|^2 = \sum_{k=1}^{2\xi+1} \sin^2\left(\frac{k\pi}{2\xi+2}\right) = \xi + 1$$

This finishes the proof of Proposition 4.

We have completely characterized the matrix, its eigenvalues and eigenvectors. Now, we can proceed to the evaluation of the energy at each step.

#### 4. The energy at each step

Recall that both fortunes take odd values:  $F_A = 2a+1$ ,  $F_B = 2b+1$ , that  $\xi = \frac{a+b}{2}$  and that,

in usual coordinates, the starting point is  $(0, 2y_0)$ , with  $y_0 = \frac{a-b}{2}$ . We need to proceed to matrix coordinates; we set  $u = \xi - y_0 + 1$ , so  $u$  may take any value between 1 and  $2\xi+1$ .

The value  $u = 1$  corresponds to  $y_0 = \xi$  and the value  $u = 2\xi + 1$  to  $y_0 = -\xi$ . We notice that

$$u = \frac{1}{4}(F_A + F_B - 2) + \frac{1}{4}(F_B - F_A) + 1 = \frac{1}{2}F_B + \frac{1}{2}.$$

The parameter  $u$  will be called "Initial Fortune Characteristic" (in short IFC). If  $u = \xi + 1$ , both players are equal, if  $u = 1, \dots, \xi$ , the player  $A$  has the initial advantage ; if  $u = \xi + 2, \dots, 2\xi + 1$ , the player  $B$  has the initial advantage.

**Theorem 5.** - *Let  $u$ ,  $1 \leq u \leq 2\xi + 1$ , be the IFC. At each step, the energy is:*

$$x(n, k) = \frac{1}{\xi + 1} \sum_{j=1}^{2\xi+1} \sin(u \mathcal{G}_j) \sin(k \mathcal{G}_j) \cos^{2n} \left( \frac{\mathcal{G}_j}{2} \right)$$

$$\text{with } \mathcal{G}_j = \frac{j\pi}{2\xi + 2}, \quad j = 1, \dots, 2\xi + 1.$$

### Proof of Theorem 5

We start with the initial vector  $X_0$  defined by  $X_0(u) = 1$ ,  $X_0(k) = 0$  if  $k \neq u$ .

We decompose this vector on the basis of eigenvectors. We write  $X_0 = \sum_{j=1}^{2\xi+1} \alpha_j V_j$ .

Since the eigenvectors are orthogonal, the coefficients  $\alpha_j$  may be computed simply:

$$\alpha_j = \frac{\langle X_0, V_j \rangle}{|V_j|_2^2} = \frac{\sin(u \mathcal{G}_j)}{\xi + 1}$$

using Proposition 3. Then, at the  $n^{\text{th}}$  step (time  $2n$ ), the vector  $X_n$  is :

$$X_n = M^n X_0 = M^n \sum_{j=1}^{2\xi+1} \alpha_j V_j = \sum_{j=1}^{2\xi+1} \alpha_j M^n V_j = \sum_{j=1}^{2\xi+1} \alpha_j \lambda_j^n V_j$$

which gives for the  $k^{\text{th}}$  coordinate:

$$x(n, k) = \frac{1}{\xi + 1} \sum_{j=1}^{2\xi+1} \sin(u \mathcal{G}_j) \sin(k \mathcal{G}_j) \cos^{2n} \left( \frac{\mathcal{G}_j}{2} \right)$$

which proves Theorem 5.

One can give an equivalent formulation of Theorem 5, using binomial coefficients:

**Theorem 6.** – For all  $u, k, n$ , the energy is:

$$x(n, k) = \frac{1}{2^{2n}} \sum_m \left( \binom{2n}{n+k-u+4m(\xi+1)} - \binom{2n}{n-k-u+4m(\xi+1)} \right)$$

(recall that we use the convention  $\binom{n}{m} = 0$  if  $m < 0$  or if  $m > n$ )

**Proof of Theorem 6**

We use the linearization formula :

$$\cos^{2n} \left( \frac{\vartheta_j}{2} \right) = \frac{1}{2^{2n}} \sum_{l=0}^{2n} \binom{2n}{l} \cos((n-l)\vartheta_j)$$

so we write, from Theorem 5:

$$\begin{aligned} 2^{2n} (\xi+1) x(u, n, k) &= \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \sin(u\vartheta_j) \sin(k\vartheta_j) \cos((n-l)\vartheta_j) \\ &= \frac{1}{2} \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \cos((u-k)\vartheta_j) \cos((n-l)\vartheta_j) - \\ &\quad - \frac{1}{2} \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \cos((u+k)\vartheta_j) \cos((n-l)\vartheta_j) \end{aligned}$$

and thus :

$$\begin{aligned} 2^{2n+2} (\xi+1) x(u, n, k) &= \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \cos((u-k-n+l)\vartheta_j) + \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \cos((u-k+n-l)\vartheta_j) \\ &\quad - \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \cos((u+k-n+l)\vartheta_j) - \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \cos((u+k+n-l)\vartheta_j) \end{aligned}$$

So we have four terms and  $2^{2n+2} (\xi+1) x(u, n, k) = T_1 + T_2 - T_3 - T_4$

But  $T_1 = T_2$ . Indeed, one may exchange the roles of  $u$  and  $k$  : the number of paths going from point  $A$  to point  $B$ , not touching the barriers, is equal to the number of paths going from  $B$  to  $A$ .

Also,  $T_3 = T_4$ . Indeed, the summation upon  $l$  runs from 0 to  $2n$ , so both  $-n+l$  and  $n-l$  run from  $-n$  to  $n$ . So we have:

$$2^{2n+1}(\xi+1)x(u,n,k) = T_1 - T_3$$

$$\text{Let us study the first term : } T_1 = \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \cos((u-k-n+l)\vartheta_j)$$

We use the following formulas:

$$\sum_{j=1}^{2\xi+1} \cos(k\vartheta_j) = -\frac{1}{2}((-1)^k + 1) \text{ if } k \text{ is not an even multiple of } 2\xi+2 \quad (1)$$

$$\sum_{j=1}^{2\xi+1} \cos(k\vartheta_j) = 2\xi+1 \text{ if } k \text{ is an even multiple of } 2\xi+2 \text{ (including 0).} \quad (2)$$

We have to study if  $u-k-n+l = 2m(2\xi+2)$ ,  $m$  integer. This is equivalent to:

$$l = n+k-u+2m(2\xi+2)$$

Since  $0 \leq l \leq 2n$ , we need:

$$0 \leq n+k-u+2m(2\xi+2), \quad \frac{-n-k+u}{4\xi+4} \leq m$$

and:

$$n+k-u+2m(2\xi+2) \leq 2n, \quad m \leq \frac{n-k+u}{4\xi+4}$$

Let  $m_1, m_2$  be these two bounds. Let  $L_1 = \{l_m, l_m = n+k-u+2m(2\xi+2); m_1 \leq m \leq m_2\}$ .

We have:

$$\begin{aligned} T_1 &= \sum_{\substack{l=0 \\ l \notin L_1}}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \cos((u-k-n+l)\vartheta_j) + \sum_{m=m_1}^{m_2} \binom{2n}{l_m} \sum_{j=1}^{2\xi+1} \cos((u-k-n+l_m)\vartheta_j) \\ &= -\frac{1}{2} \sum_{\substack{l=0 \\ l \notin L_1}}^{2n} \binom{2n}{l} ((-1)^{u-k-n+l} + 1) + (2\xi+1) \sum_{m=m_1}^{m_2} \binom{2n}{n+k-u+4m(\xi+1)} \\ &= -\frac{1}{2} (-1)^{u-k} \sum_{\substack{l=0 \\ l \notin L_1}}^{2n} \binom{2n}{l} (-1)^{n-l} - \frac{1}{2} \sum_{\substack{l=0 \\ l \notin L_1}}^{2n} \binom{2n}{l} + (2\xi+1) \sum_{m=m_1}^{m_2} \binom{2n}{n+k-u+4m(\xi+1)} \end{aligned}$$

In order to simplify the notation, we set  $\Delta_1 = \sum_{m=m_1}^{m_2} \binom{2n}{n+k-u+4m(\xi+1)}$ . We have :

$$\sum_{\substack{l=0 \\ l \notin L_1}}^{2n} \binom{2n}{l} (-1)^{n-l} = \sum_{l=0}^{2n} \binom{2n}{l} (-1)^{n-l} - \sum_{m=m_1}^{m_2} \binom{2n}{l_m} (-1)^{n-l_m}$$

$$\sum_{l=0}^{2n} \binom{2n}{l} (-1)^{n-l} = 0$$

$$\sum_{m=m_1}^{m_2} \binom{2n}{l_m} (-1)^{n-l_m} = (-1)^{k-u} \Delta_1$$

So we get:

$$\sum_{m=m_1}^{m_2} \binom{2n}{l_m} (-1)^{n-l_m} = (-1)^{k-u} \Delta_1$$

$$\sum_{\substack{l=0 \\ l \notin L_1}}^{2n} \binom{2n}{l} (-1)^{n-l} = -(-1)^{k-u} \Delta_1$$

The same way:

$$\sum_{\substack{l=0 \\ l \notin L_1}}^{2n} \binom{2n}{l} = \sum_{l=0}^{2n} \binom{2n}{l} - \sum_{m=m_1}^{m_2} \binom{2n}{n+k-u+4m(\xi+1)} = 2^{2n} - \Delta_1$$

$$T_1 = \frac{1}{2} \Delta_1 - \frac{1}{2} (2^{2n} - \Delta_1) + (2\xi+1) \Delta_1 = -2^{2n-1} + (2\xi+2) \Delta_1$$

We now study the term  $T_3 = \sum_{l=0}^{2n} \binom{2n}{l} \sum_{j=1}^{2\xi+1} \cos((u+k-n+l) \mathcal{G}_j)$ . The computations are identical, except that  $k$  is replaced by  $-k$ .

We introduce

$$m_3 = \left\lceil \frac{-n+k+u}{4\xi+4} \right\rceil \text{ (ceiling : smallest integer greater than or equal to a number) and}$$

$$m_4 = \text{int} \left( \frac{n+k+u}{4\xi+4} \right)$$

Let  $L_3 = \{l_m, l_m = n-k-u+4m(\xi+1); m_3 \leq m \leq m_4\}$ . The same computation as above gives:

$$T_3 = -2^{2n-1} + (2\xi+2) \Delta_3$$

$$\text{with } \Delta_3 = \sum_{m=m_3}^{m_4} \binom{2n}{n-k-u+4m(\xi+1)}$$

Finally:

$$2^{2n+1}(\xi+1)x(u,n,k) = T_1 - T_3 = -2^{2n-1} + (2\xi+2)\Delta_1 - (-2^{2n-1} + (2\xi+2)\Delta_3) = 2(\xi+1)(\Delta_1 - \Delta_3)$$

and:

$$x(u,n,k) = \frac{\Delta_1 - \Delta_3}{2^n}$$

We observe that in  $\Delta_1$ , the quantity  $n+k-u+4m(\xi+1)$  runs from the closest integer to 0 to the closest integer to  $2n$ , so we may simply write :

$$\Delta_1 = \sum_m \binom{2n}{n+k-u+4m(\xi+1)}$$

$$\Delta_3 = \sum_m \binom{2n}{n-k-u+4m(\xi+1)}$$

This proves Theorem 6.

## 5. Advantage for each player

Let  $u$ ,  $1 \leq u \leq 2\xi+1$ , be the IFC. We define:

$P_A(n-)$  : the probability that  $A$  wins at a time  $< n$

$P_A(n+)$  : the probability that  $A$  wins at a time  $\geq n$

$P_B(n-)$  : the probability that  $B$  wins at a time  $< n$

$P_B(n+)$  : the probability that  $B$  wins at a time  $\geq n$

$E(n)$  : energy at time  $n$ , equal to the probability that the game is not finished at time  $n$ .

**Proposition 7.** – For any  $n$  :

$$P_A(n-) = \frac{1}{4} \sum_{m=0}^{n-1} x(m,1), \quad P_A(n+) = \frac{1}{4} \sum_{m=n}^{+\infty} x(m,1)$$

$$P_B(n-) = \frac{1}{4} \sum_{m=0}^{n-1} x(m,2\xi+1), \quad P_B(n+) = \frac{1}{4} \sum_{m=n}^{+\infty} x(m,2\xi+1)$$

$$E(n) = \sum_{k=1}^{2\xi+1} x(n,k)$$

### Proof of Proposition 7

Indeed, for  $A$  to win, the game must reach the upper barrier before time  $n$ . But, at each step, the probability to do this is  $1/4$  of the probability to be at the level 1; the same holds for  $B$ . The probability that the game is not finished at time  $n$  is simply the sum of the  $x(n, k)$  upon all values of  $k$  on the corresponding vertical. This proves Proposition 7.

We investigate now the overall issue of the game: what is the probability that  $A$  wins? We will consider later the probability to win before a given time.

First, we observe that, as a consequence of Theorem 5, the probability that the games continues up to time  $n$  tends to zero with  $n$ : the game cannot continue indefinitely. Let us denote by  $P_A$  the probability that  $A$  wins, and similarly  $P_B$  for  $B$ , so  $P_A + P_B = 1$ .

**Theorem 8.** - *Let  $u$ ,  $1 \leq u \leq 2\xi + 1$ , be the IFC. We have:*

$$P_A = 1 - \frac{u}{2\xi + 2} = \frac{F_A + 1}{F_A + F_B + 2}$$

and :

$$P_B = \frac{u}{2\xi + 2} = \frac{F_B + 1}{F_A + F_B + 2}$$

### Proof of Theorem 8

The probability that  $A$  wins is necessarily linear in  $u$ . Indeed, if  $u = \xi + 1$  (both fortunes are equal), the probability is  $\frac{1}{2}$ .

If  $u = \frac{\xi + 1}{2}$ , two things may happen, each with probability  $\frac{1}{2}$ : either the random walk touches the upper barrier ( $A$  wins) or it touches the middle line (equal fortunes), and in the latter case we are back to the previous case. So the probability for  $A$  to win when  $u = \frac{\xi + 1}{2}$  is  $\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ ; repeating this dichotomy argument shows that the probability that  $A$  wins is linear in  $u$ .

For  $u = 2\xi + 2$ , both quantities are equal to 0. For  $u = 1$ , by Proposition 7,  $P_A = \frac{1}{4} \sum_{m=0}^{\infty} x(m, 1)$  and by Theorem 5:



$$\begin{aligned}
P_A &= \frac{1}{4} \frac{1}{\xi+1} \sum_{j=1}^{2\xi+1} \sin^2(\mathcal{G}_j) \sum_{n=0}^{+\infty} \cos^{2n}\left(\frac{\mathcal{G}_j}{2}\right) = \frac{1}{4} \frac{1}{\xi+1} \sum_{j=1}^{2\xi+1} \frac{\sin^2(\mathcal{G}_j)}{1 - \cos^2\left(\frac{\mathcal{G}_j}{2}\right)} \\
&= \frac{1}{\xi+1} \sum_{j=1}^{2\xi+1} \cos^2\left(\frac{\mathcal{G}_j}{2}\right) = \frac{2\xi+1}{2\xi+2}
\end{aligned}$$

which is also the value of  $1 - \frac{u}{2\xi+2}$  for  $u=1$  ; this proves Theorem 8.

We now give quantitative estimates for  $P_A(n-)$ ,  $P_A(n+)$ , and the same for  $B$ . Obviously,  $P_A(n-) + P_B(n-) + E(n) = 1$  for all  $n$  ; all these probabilities depend on  $u$ , initial fortune, and on  $\xi$ , value of the barriers.

Obviously also,  $P_A = \lim_{n \rightarrow +\infty} P_A(n-)$ .

**Proposition 9.** – *We have, for any  $n$  :*

$$P_A(n+) = \frac{1}{2\xi+2} \sum_{j=1}^{2\xi+1} \frac{\sin(u \mathcal{G}_j)}{\sin\left(\frac{\mathcal{G}_j}{2}\right)} \cos^{2n+1}\left(\frac{\mathcal{G}_j}{2}\right)$$

Let  $U_k$  be the family of Chebyshev's polynomials of second kind. We have the equivalent formulation:

$$P_A(n+) = \frac{1}{2\xi+2} \sum_{j=1}^{2\xi+1} U_{2u-1}(y_j) y_j^{2n+1}, \text{ with } y_j = \cos\left(\frac{\mathcal{G}_j}{2}\right)$$

### Proof of Proposition 9

We compute the probability that  $A$  wins at a time  $\geq n$ , which happens if the random walk touches the upper barrier. Using Theorem 5, and Proposition 7, we have:

$$P_A(n+) = \frac{1}{4} \frac{1}{\xi+1} \sum_{j=1}^{2\xi+1} \sin(u \mathcal{G}_j) \sin(\mathcal{G}_j) \sum_{m=n}^{+\infty} \cos^{2m}\left(\frac{\mathcal{G}_j}{2}\right)$$

Using the identity  $\sum_{m=n}^{+\infty} \cos^{2m}\left(\frac{\mathcal{G}_j}{2}\right) = \frac{\cos^{2n}\left(\frac{\mathcal{G}_j}{2}\right)}{1 - \cos^2\left(\frac{\mathcal{G}_j}{2}\right)} = \frac{\cos^{2n}\left(\frac{\mathcal{G}_j}{2}\right)}{\sin^2\left(\frac{\mathcal{G}_j}{2}\right)}$ , we get:

$$P_A(n+) = \frac{1}{4} \frac{1}{\xi+1} \sum_{j=1}^{2\xi+1} \frac{\sin(u \mathcal{G}_j) \sin(\mathcal{G}_j)}{\sin^2\left(\frac{\mathcal{G}_j}{2}\right)} \cos^{2n}\left(\frac{\mathcal{G}_j}{2}\right)$$

Replacing  $\sin \mathcal{G}_j = 2 \sin \frac{\mathcal{G}_j}{2} \cos \frac{\mathcal{G}_j}{2}$  proves the first part of Proposition 9. In order to prove the second part, recall that the  $k^{\text{th}}$  Chebyshev' polynomial of the second kind  $U_k(x)$  satisfies:

$$U_k(\cos(\mathcal{G})) = \frac{\sin((k+1)\mathcal{G})}{\sin \mathcal{G}}.$$

**Corollary 10.** – For all  $n$  :

$$P_A(n-) = 1 - \frac{u}{2\xi+2} - \frac{1}{2\xi+2} \sum_{j=1}^{2\xi+1} \frac{\sin(u \mathcal{G}_j)}{\sin\left(\frac{\mathcal{G}_j}{2}\right)} \cos^{2n+1}\left(\frac{\mathcal{G}_j}{2}\right)$$

This is an obvious consequence of the previous Propositions. The probability that  $A$  wins before time  $n$  is of course increasing with  $n$ . We now study the probability that the game continues after step  $n$ .

## 6. Probability that the game continues

**Proposition 11.** – We have, for any  $u, n$ :

$$E(n) = \frac{1}{\xi+1} \sum_{l=0}^{\xi} \frac{\sin(u \mathcal{G}_{2l+1})}{\sin\left(\frac{\mathcal{G}_{2l+1}}{2}\right)} \cos^{2n+1}\left(\frac{\mathcal{G}_{2l+1}}{2}\right)$$

For  $l = 0, \dots, \xi$ , we set  $z_l = \cos\left(\frac{\mathcal{G}_{2l+1}}{2}\right)$ . Let  $U_k$  be the family of Chebyshev's polynomials of second kind. We have the identity:

$$E(n) = \frac{1}{\xi+1} \sum_{l=0}^{\xi} U_{2u-1}(z_l) z_l^{2n+1}$$

### Proof of Proposition 11

Using Theorem 5, let us compute the probability that the game is not finished at time  $n$ . We have :

$$E(n) = \sum_{k=1}^{2\xi+1} x(n, k) = \frac{1}{\xi+1} \sum_{j=1}^{2\xi+1} \sin(u \mathcal{G}_j) \cos^{2n}\left(\frac{\mathcal{G}_j}{2}\right) \sum_{k=1}^{2\xi+1} \sin(k \mathcal{G}_j)$$

The identity:

$$\sum_{k=1}^{2\xi+1} \sin(k\mathcal{G}_j) = \frac{1-(-1)^j \cos \frac{\mathcal{G}_j}{2}}{2} \frac{\cos \frac{\mathcal{G}_j}{2}}{\sin \frac{\mathcal{G}_j}{2}}$$

gives:

$$E(n) = \frac{1}{\xi+1} \sum_{j=1}^{2\xi+1} \frac{1-(-1)^j}{2} \frac{\sin(u\mathcal{G}_j)}{\sin \frac{\mathcal{G}_j}{2}} \cos^{2n+1} \left( \frac{\mathcal{G}_j}{2} \right)$$

But  $\frac{1-(-1)^j}{2} = 0$  if  $j$  is even,  $=1$  if  $j$  is odd. Set  $j = 2l+1$ ; when  $j = 1, \dots, 2\xi+1$ , we have

$l = 0, \dots, \xi$ . We obtain:

$$E(n) = \frac{1}{\xi+1} \sum_{l=0}^{\xi} \frac{\sin(u\mathcal{G}_{2l+1})}{\sin \left( \frac{\mathcal{G}_{2l+1}}{2} \right)} \cos^{2n+1} \left( \frac{\mathcal{G}_{2l+1}}{2} \right)$$

So, with  $z_l = \cos \left( \frac{\mathcal{G}_{2l+1}}{2} \right)$ , we have:

$$E(n) = \frac{1}{\xi+1} \sum_{l=0}^{\xi} U_{2u-1}(z_l) z_l^{2n+1}$$

This proves Proposition 11.

We observe that  $E(n)$  is not modified if  $u$  is replaced by  $2\xi+2-u$  (exchanging the roles of  $A$  and  $B$ ). Indeed:

$$\sin((2\xi+2-u)\mathcal{G}_{2l+1}) = \sin(\pi - u\mathcal{G}_{2l+1}) = \sin(u\mathcal{G}_{2l+1}).$$

## 7. Examples

We now turn to some numerical examples, in order to understand the shapes of the various profiles ; they may look surprising at first sight. Let us assume first  $\xi = 7$ . We consider 3 cases :  $u = 1$  (high initial fortune for  $A$ ),  $u = \xi + 1$  (equal fortunes for  $A$  and  $B$ ),  $u = 2\xi + 1$  (low initial fortune for  $A$ ).

Case 1 :  $u = 1$

By Theorem 8, the probability that  $A$  wins the game is  $P_A = \frac{2\xi + 1}{2\xi + 2} = \frac{15}{16} = 0.9375$ .

For the probabilities to win before a certain time, we find :

$n = 10$	$P_A(10-) \approx 0.66$
$n = 20$	$P_A(20-) \approx 0.755$
$n = 50$	$P_A(50-) \approx 0.842$
$n = 100$	$P_A(100-) \approx 0.888$

For the probabilities to continue until a certain time, we find :

$n = 10$	$E(10) \approx 0.336$
$n = 20$	$E(20) \approx 0.245$
$n = 50$	$E(50) \approx 0.156$
$n = 100$	$E(100) \approx 0.094$

Case 2 :  $u = \xi + 1$

By Theorem 8, the probability that  $A$  wins the game is  $P_A = \frac{\xi + 1}{2\xi + 2} = \frac{1}{2}$ .

For the probabilities to win before a certain time, we find :

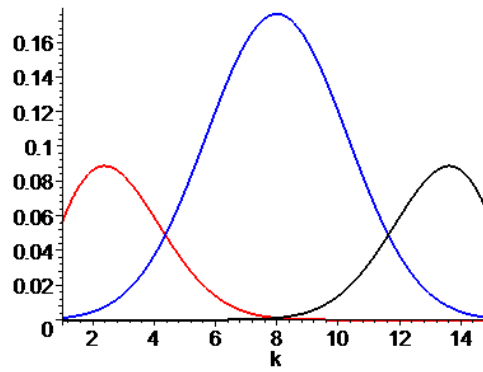
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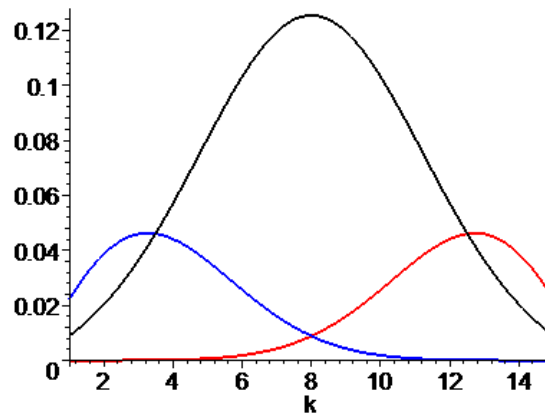
Let us draw explicitly the various profiles of energy, depending on the initial fortune.

For  $n = 10$ :

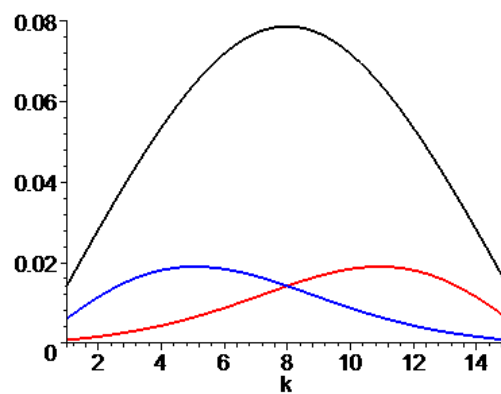


In black: profile for  $u = 1$  (strong  $A$ ), in blue : profile for even strengths, in red, profile for weak  $A$ .

For  $n = 20$  :

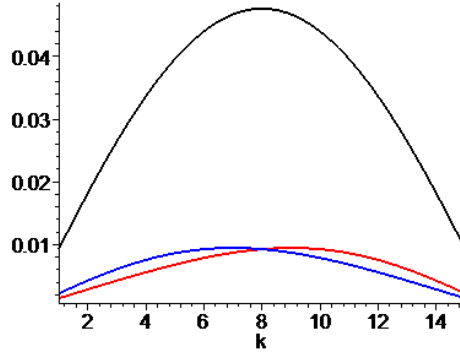


For  $n = 50$  :



We see here that the profiles for  $A$  initially strong and  $A$  initially weak tend to coincide.

For  $n = 100$  :



and here, for large  $n$ , they are almost identical, and well below the profile for  $A$  initially identical to  $B$ . This looks strange at first sight, but in fact, if initially  $A$  is very strong, for instance  $u = 1$ , the only paths which will reach the time 100 are the ones which meet the middle line, for which  $A = B$ . In other words, if  $A$  is very rich compared to  $B$ , he should win rather early, otherwise, very likely, both fortunes will be equal.

Let us state this as a Proposition

**Proposition 12.** - *When  $n \rightarrow +\infty$ , the profiles for any  $u$  tend to coincide.*

### Proof of Proposition 12

Let us prove it for the two most different profiles:  $u = 1$  and  $u = 2\xi + 1$ . In the first case, the energy profile is:

$$x_1(n, k) = \frac{1}{\xi + 1} \sum_{j=1}^{2\xi+1} \sin(\vartheta_j) \sin(k\vartheta_j) \lambda_j^n$$

and in the second case:

$$x_{2\xi+1}(n, k) = \frac{1}{\xi + 1} \sum_{j=1}^{2\xi+1} \sin((2\xi + 1)\vartheta_j) \sin(k\vartheta_j) \lambda_j^n$$

So the difference is:

$$\begin{aligned} d(n, k) &= \frac{1}{\xi + 1} \sum_{j=1}^{2\xi+1} \left( \sin(\vartheta_j) - \sin((2\xi + 1)\vartheta_j) \right) \sin(k\vartheta_j) \lambda_j^n \\ &= \frac{2}{\xi + 1} \sum_{j=1}^{2\xi+1} \sin(\xi\vartheta_j) \cos((\xi + 1)\vartheta_j) \sin(k\vartheta_j) \lambda_j^n \end{aligned}$$

But  $(\xi + 1)\vartheta_j = \frac{j\pi}{2}$ , so  $\cos((\xi + 1)\vartheta_j) = 0$  if  $j$  is odd, and  $= (-1)^{j/2}$  if  $j = 2j_1$ . Therefore:

$$d(n,k) = \frac{2}{\xi+1} \sum_{j_1=1}^{\xi} (-1)^{j_1} \sin(\xi \mathcal{G}_{2^{j_1}}) \sin(k \mathcal{G}_{2^{j_1}}) \lambda_{2^{j_1}}^n$$

We have  $\xi \mathcal{G}_{2^{j_1}} = \frac{\xi 2^{j_1} \pi}{2^{\xi+2}} = \frac{\xi 2^{j_1} \pi}{2^{\xi+1}} = j_1 \pi - \frac{j_1 \pi}{\xi+1}$  ; therefore:

$$\sin(\xi \mathcal{G}_{2^{j_1}}) = \sin\left(j_1 \pi - \frac{j_1 \pi}{\xi+1}\right) = (-1)^{j_1-1} \sin\left(\frac{j_1 \pi}{\xi+1}\right)$$

and finally:

$$d(n,k) = \frac{-2}{\xi+1} \sum_{j_1=1}^{\xi} (-1)^{j_1} \sin\left(\frac{j_1 \pi}{\xi+1}\right) \sin\left(k \frac{j_1 \pi}{\xi+1}\right) \lambda_{2^{j_1}}^n .$$

The first term in the sum involves  $\lambda_2^n$ , whereas the expressions of  $x_1$  and  $x_{2^{\xi+1}}$  involve  $\lambda_1^n$ ; the difference between both will tend to 0, when  $n \rightarrow +\infty$ , more quickly than each term separately. This proves our Proposition.

We observe that  $P_A(n+)$  is exponentially decreasing with  $n$ . Its shape, depending on  $u$ , for fixed  $\xi, n$ , is also rather surprising :

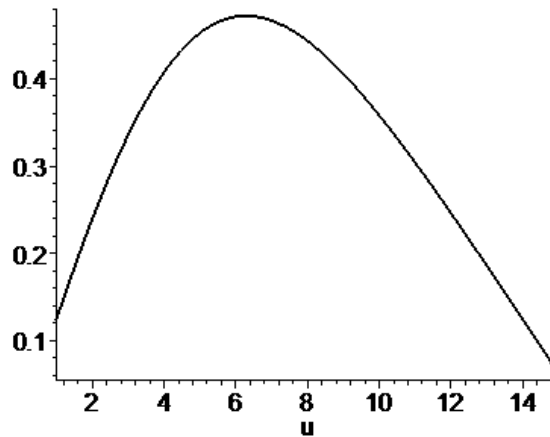


Fig. : the graph of  $P_A(n+)$ , for  $\xi = 7$ ,  $n = 35$ , as a function of  $u$ .

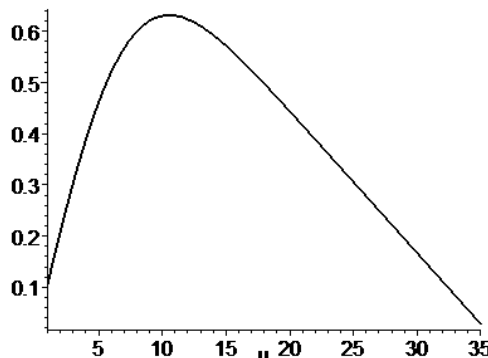


Fig. : the graph of  $P_A(n+)$ , for  $\xi = 17$ ,  $n = 70$ , as a function of  $u$ .

The low values for  $u$  small (when  $A$  has the largest initial fortune) come from the fact that, under these conditions,  $A$  should win at an early stage.

### Large fortunes

Let us now consider the case of large, but different, initial fortunes:  $F_A = 2\,000$ ,  $F_B = 1\,000$ . Let us compute the probabilities for  $n = 100\,000, 500\,000, 1\,000\,000$ . In each case, we compute the probability that  $A$  (resp.  $B$ ) wins before  $n$  and after  $n$  :

$n = 100\,000$	Proba wins before n	Proba wins after n
A	0.025	0.641
B	0.000122	0.333

$n = 500\,000$	Proba wins before n	Proba wins after n
A	0.317	0.350
B	0.0454	0.288

$n = 1\,000\,000$	Proba wins before n	Proba wins after n
A	0.479	0.188
B	0.153	0.181

These results are quite interesting. They show that, if one waits long enough, the probabilities to win the game after  $n$  tend to be equal. In the case of different fortunes (here,  $A$  is twice as rich as  $B$ , so his probability to win is roughly  $2/3$ ),  $A$  should win at an early stage, otherwise the game becomes more balanced.