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# Simple Random Walks in the plane:

# An energy based approach

# Part III : Variable Fortunes

by Bernard Beauzamy

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In this Third Part, we investigate the case of variable barriers : the barrier is represented by a function of time, here $f(n) = \pm\sqrt{n}$ . We restrict ourselves to the case of symmetric barriers, which means that the rules are the same for both players: if at some time $n$ the fortune of one of the players reaches the barrier, the game stops. The question is: what is the probability that the game continues after $N$ steps and, more precisely, what is the probability of the fortune of each player ?

We use the "energy based" approach, described in Parts 1 and 2.

## 1. Transition between two periods

Since now our barrier is not constant, we will represent it by a succession of constant segments. Let us describe this representation in detail in the case of the barrier $\pm\sqrt{n}$.

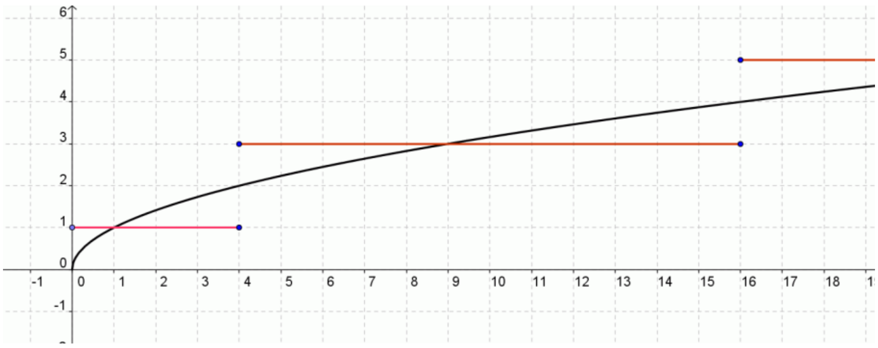


*Figure 1.1: discretisation of the barrier*

The changes will occur at times $4n^2$, $n = 0,1,...$. On the interval $4n^2, 4(n+1)^2$, the barrier is at $2n+1$ (recall from Part II that we want even values of time and odd values for the barrier). So, in the notation introduced in Part II, $\xi = n$ and this value is used on an interval of length $l_n = 4(n+1)^2 - 4n^2$, that is $l_n = 8n+4$.

Let $M_n$ be the associated matrix (see Part II) and let $\lambda_n$ be the largest eigenvalue of this matrix. From Part II, Theorem 3, follows that:

$$\lambda_n \sim 1 - \frac{\pi^2}{16n^2} \tag{1.1}$$

We have now to investigate the transition between two periods. The barrier was at $2\xi+1$ and moves to $2\xi+3$.

Let us first consider the transition on the energy, that is the variables $e(2n,2k)$ (see Part II).

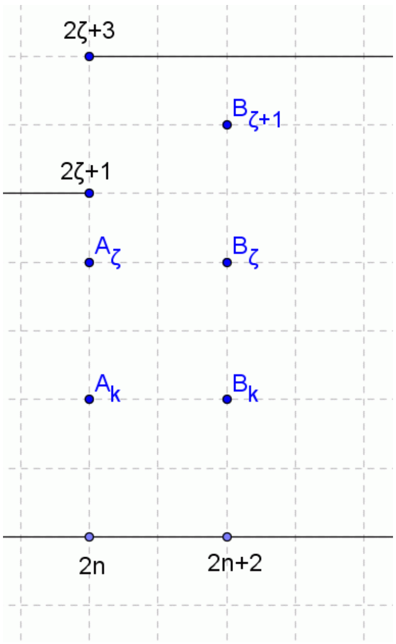

*Figure 1.2 : Notation for the transition*

It will be convenient to have a simple notation just for the transition. On the vertical corresponding to time $2n,$ we have $\xi+1$ points $A_0,...,A_\xi$; at time $2n+2,$ we have $\xi+2$ points $B_0,...,B_{\xi+1}$. We denote by $a_k$ the energy at the point $A_k$ and similarly $b_k$ for the $B_k$.

For the first $\xi$ points, we have the usual transition equations:

$$b_0=\frac{1}{2}\left(a_0+a_1\right) \quad (1.2)$$

$$b_k=\frac{1}{4}a_{k-1}+\frac{1}{2}a_k+\frac{1}{4}a_{k+1} \text{ for } k=1,...,\xi-1 \ (1.3)$$

The last two equations are different from the constant case; they are:

$$b_\xi=\frac{1}{4}a_{\xi-1}+\frac{1}{2}a_\xi \ (1.4)$$

$$b_{\xi+1}=\frac{1}{4}a_\xi \ (1.5)$$

If the barrier was constantly at $2\xi+1,$ instead of (1.4), we would have:

$$b_\xi=\frac{1}{4}a_{\xi-1}+\frac{1}{4}a_\xi \ (1.6)$$

and instead of (1.5) :

$$b_{\xi+1}=0 \ (1.7)$$

So, the fact that the barrier moves one step higher means more energy:

- for $b_\xi$, increase of $\frac{1}{4}a_\xi$
- for $b_{\xi+1}$, increase of $\frac{1}{4}a_\xi$

so a total increase of energy equal to $\frac{1}{2}a_\xi$.

Let us now turn to the variables $x(n,k)$ and describe the transition on these variables. Recall that, for $k=0,...,\xi-1$ and $n \geq 2$:

$$x_k = \frac{1}{2}(a_k + a_{k+1})$$

We have (see Part II):

$b_0 = x_0$ (1.8)

$b_k = \frac{1}{2}(x_{k-1} + x_k)$ for $k=1,...,\xi-1$ (1.9)

$b_\xi = \frac{1}{4}(a_{\xi-1} + a_\xi) + \frac{1}{4}(a_\xi + a_{\xi+1})$ with $a_{\xi+1} = 0$

which gives:

$b_\xi = \frac{1}{2}x_{\xi-1} + \frac{1}{2}x_\xi$ (1.10)

$b_{\xi+1} = \frac{1}{4}(a_\xi + a_{\xi+1}) = \frac{1}{2}x_\xi$ (1.11)

Let us define $y_k = \frac{1}{2}(b_k + b_{k+1})$, $k=0,...,\xi$.

We get:

$$y_0 = \frac{1}{2}\left(x_0 + \frac{1}{2}(x_0 + x_1)\right) = \frac{3}{4}x_0 + \frac{1}{4}x_1$$

$$y_k = \frac{1}{4}(x_{k-1} + 2x_k + x_{k+1}),\ k=1,...,\xi-1$$

$$y_\xi = \frac{1}{2}\left(\frac{1}{2}x_{\xi-1} + \frac{1}{2}x_\xi + \frac{1}{2}x_\xi\right) = \frac{1}{4}x_{\xi-1} + \frac{1}{2}x_\xi$$

The equations are the same as in the constant case; we simply have one more intermediate equation.

With the original notation, we have:

$$\begin{cases} x(n+1,0)=\dfrac{3}{4}x(n,0)+\dfrac{1}{4}x(n,1) \\ x(n+1,k)=\dfrac{1}{4}x(n,k-1)+\dfrac{1}{2}x(n,k)+\dfrac{1}{4}x(n,k+1), \text{for } k=1,\dots,\xi-1 \\ x(n+1,\xi)=\dfrac{1}{4}x(n,\xi-1)+\dfrac{1}{2}x(n,\xi) \end{cases} \tag{1.12}$$

We have proved:

**Proposition 1.1** - *On the variables* $x(n,k)$*, the fact that the barrier is shifted one step higher leads simply to a new intermediate equation in the transition equations.*

This is quite important in practice, because it means that the theory developed in Part II will apply, despite the changes of position for the barrier. We have simply to take into account the fact that the matrix $M_n$ will increase at each step, and the corresponding eigenvalues will change accordingly.

## 2. Changes in the eigenvalues and in the eigenvectors

We now work constantly on the variables $X_n = x(n,i)$. In dimension $\xi$, we know (see Part II) that the eigenvalues are of the form:

$$\lambda_j = \frac{1+\cos(\vartheta_j)}{2}, \ \vartheta_j = \frac{2j-1}{2\xi+1}\pi, \ j=1,\dots,\xi,$$

and the same expressions will remain in dimension $\xi+1$, with $\xi$ replaced by $\xi+1$.

In dimension $\xi$, the eigenvectors were:

$$V_{\xi,j} = \left(\sin(\xi\vartheta_j),\dots,\sin(\vartheta_j)\right), \ j=1,\dots,\xi$$

and in dimension $\xi+1$, they will be:

$$V_{\xi+1,j} = \left(\sin((\xi+1)\vartheta_j),\sin(\xi\vartheta_j),\dots,\sin(\vartheta_j)\right), \ j=1,\dots,\xi+1$$

The natural embedding from dimension $\xi$ to dimension $\xi+1$ (simply adding a zero as the last coordinate) preserves the eigenvectors. Indeed, we have:

**Proposition 2.1.** - *The image of the vector* $V_{\xi,j} = \left(\sin\left(\xi\vartheta_j\right),\ldots,\sin\left(\vartheta_j\right),0\right)$, *with* $\vartheta_j = \dfrac{2j-1}{2\xi+1}\pi$, *by the matrix* $M_{\xi+1}$ *is the vector:*

$$M_{\xi+1}V_{\xi,j} = \left(\lambda_{\xi,j}\sin\left(\xi\vartheta_j\right),\ldots,\lambda_{\xi,j}\sin\left(\vartheta_j\right),\frac{1}{4}\sin\left(\vartheta_j\right)\right)$$

and we have $\left|M_{\xi+1}V_{\xi,j}\right|_1 = \left|V_{\xi,j}\right|_1$.

**Proof of Proposition 2.1**

This is obvious for the first coordinates, so we have to look only at the last 3. The image of 0 (last coordinate) is $\frac{1}{4}\sin\left(\vartheta_j\right)$. The image of $\sin\left(2\vartheta_j\right)$ is $\lambda_{\xi,j}\sin\left(2\vartheta_j\right)$. The image of $\sin\left(\vartheta_j\right)$ is by definition $\frac{1}{4}\sin\left(2\vartheta_j\right)+\frac{1}{2}\sin\left(\vartheta_j\right)=\sin\left(\vartheta_j\right)\left(\frac{1+\cos\left(\vartheta_j\right)}{2}\right)=\lambda_{\xi,j}\sin\left(\vartheta_j\right)$. The assertion on the $l_1$ norm is obvious, since there is no loss of energy in the transition.

This proves Proposition 2.1.

The matrix $M$, at any stage, operates only on three coordinates (and only on 2, at the first and last coordinates). So, if these three coordinates are those of an eigenvector of a previous situation, the result is a multiplication by the corresponding eigenvalue.

## 3. Energy transition

Assume we have $l_1$ times the value $\xi_1$ for the barrier. After $l_1$ steps, the vector of energies will be $X_{l_1}$, with $\left|X_{l_1}\right|_2 \le a\left(l_1,\xi_1\right)\left|X_0\right|_2$, where $a\left(l_1,\xi_1\right)$ is the attenuation of quadratic energy during the episode of length $l_1$ where the barrier is at $\xi_1$. We have $a\left(l_1,\xi_1\right)\le\lambda_1$, where $\lambda_1$ is the largest eigenvalue of the matrix $M$ with size $\xi_1$.

Assume that, after the first $l_1$ steps, we have $l_2$ times the value $\xi_2$ for the barrier.
After $l_1+l_2$ steps, the vector of energies will be $X_{l_1+l_2}$ with $\left|X_{l_1+l_2}\right|_2 \le a\left(l_2,\xi_2\right)\left|X_{l_1}\right|_2$, where $a\left(l_2,\xi_2\right)$ is the attenuation of quadratic energy during the episode of length $l_2$ where the barrier is at $\xi_2$, and $a\left(l_2,\xi_2\right)\le\lambda_2$, where $\lambda_2$ is the largest eigenvalue of the matrix $M$ with size $\xi_2$.

Therefore:

$$\left|X_{l_1+l_2}\right|_2 \le a(l_1,\xi_1)\,a(l_2,\xi_2)\left|X_0\right|_2$$

Let us now consider a sequence of steps of respective lengths $l_1,...,l_N$. We get, at the end:

$$\left|X_{l_1+\cdots+l_N}\right|_2 \le a(l_1,\xi_1)\cdots a(l_N,\xi_N)\left|X_0\right|_2 \quad (3.1)$$

The quadratic energy will tend to 0, when $N \to +\infty$, as soon as:

$$a(l_1,\xi_1)\cdots a(l_N,\xi_N) \to 0$$

This will be satisfied as soon as:

$$S_N = \sum_{n=1}^{N} l_n Log(\lambda_n) \to -\infty \text{ when } N \to +\infty.$$

## 4. Estimates for a square root barrier

**Theorem 4.1** - *Consider the symmetric barrier* $y = \pm\sqrt{x}$. *At each step, we have the estimates:*

*Quadratic energy :* $E_2\left(W_{2(4N+4)^2+2}\right) \le \dfrac{1}{2}\sqrt{\dfrac{3}{2}}N^{-\frac{\pi^2}{2}}$

*Total energy :* $E_1\left(W_{2(4N+4)^2+2}\right) \le \dfrac{1}{2}\sqrt{\dfrac{3}{2}}N^{-\frac{\pi^2}{2}+\frac{1}{2}}$

**Proof of Theorem 4.1**

The length of the $n^{th}$ step is $l_n = 8n+4$ with $\xi_n = n$ ; during this step, we have seen the estimate (Lemma 10.1, Part II), for the largest eigenvalue:

$$\lambda_n \approx 1 - \frac{\pi^2}{16n^2}$$

which gives, using the estimate $Log(1-x) \le -x,\ 0 \le x \le 1$:

$$S_N = \sum_{n=1}^{N} l_n Log(\lambda_n) \approx \sum_{n=1}^{N}(8n+4)Log\left(1-\frac{\pi^2}{16n^2}\right) \le -\sum_{n=1}^{N}(8n+4)\frac{\pi^2}{16n^2} \approx -\frac{\pi^2}{2}\sum_{n=1}^{N}\frac{1}{n} \approx -\frac{\pi^2}{2}Log(N)$$

So this series diverges.

At the end of the $N^{th}$ step, we have:

$$\left|X_{(4N+4)^2}\right|_2 \leq |X_0|_2 \exp(S_N) = \frac{1}{2} N^{-\frac{\pi^2}{2}}$$

At the end of the $N^{th}$ step, the barrier is at $2N+1$ and we have $N$ points on the corresponding vertical. Therefore:

$$\left|X_{(4N+4)^2}\right|_1 \leq \frac{1}{2} N^{-\frac{\pi^2}{2}+\frac{1}{2}}$$

This quantity tends to 0 when $N$ increases.

In order to convert these estimates in terms of energy, we use the estimates (6.2) and (6.3) in Part II, that is:

$$E_1(W_{2n+2}) \leq \frac{3}{2}|X_n|_1 \; ; \; E_2(W_{2n+2}) \leq \sqrt{\frac{3}{2}}|X_n|_2$$

This proves Theorem 4.1.