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Simple Random Walks in the plane:

An energy based approach

Part III : Variable Fortunes

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In this Third Part, we investigate the case of variable barriers : the barrier is represented by a function of time; say for instance $b(n) = \pm\sqrt{n}$, to start with.

We consider symmetric barriers, which means that the rules are the same for both players: if at some time n the fortune of one of the players reaches the barrier, the game stops. The question is: what is the probability that the game continues after N steps and, more precisely, what is the probability of the fortune of each player (what we call "profile" of fortune) ?

The main theorem is as follows:

Theorem. - The probability P_N that the game continues after N steps tends to 0 when $N \to +\infty$ if and only if the integral $\int^{+\infty} \frac{dx}{b^2(x)}$ diverges at $+\infty$. More precisely, this probability satisfies the estimate:

$$E_N \le \exp\left(-\frac{\pi^2}{16} \int_1^{t_{N+1}} \frac{dx}{b^2(x)} \right)$$

where t_n is the unique number such that $b(t_n) = n$.

During the n^{th} period (see definition below), the profile of fortune is proportional to the vector:

$$\left(\sin(\vartheta), \sin(2\vartheta), ..., \sin(n\vartheta), \sin((n-1)\vartheta), ..., \sin(\vartheta) \right)$$

where $\vartheta = \dfrac{\pi}{2n+1}$.

The case of the barrier $f(n) = \pm\sqrt{n\, Log(n)}$ is of special interest, because it lies above Khinchine's safety curve $\varphi(n) = \sqrt{2n\, Log(Log(n))}$. Still, the main Theorem shows that the probability that the game continues after N steps tends to 0 when $N \to +\infty$, and gives quantitative estimates for this probability.

We use the "energy based" approach, described in Parts 1 and 2.

I. Transition between two periods

1. Energy propagation

For us, a "barrier" will be a positive function $b(x)$, defined on $x \geq 0$, differentiable, increasing, and satisfying $b(x) \to +\infty$ when $x \to +\infty$. A simple example is $b(x) = \sqrt{x}$.

We have a continuous barrier, but our game uses only integer values. So we have to convert our barrier into a succession of constant segments, with integer values.

Let us describe this representation in detail in the case of the barrier $\pm\sqrt{n}$.

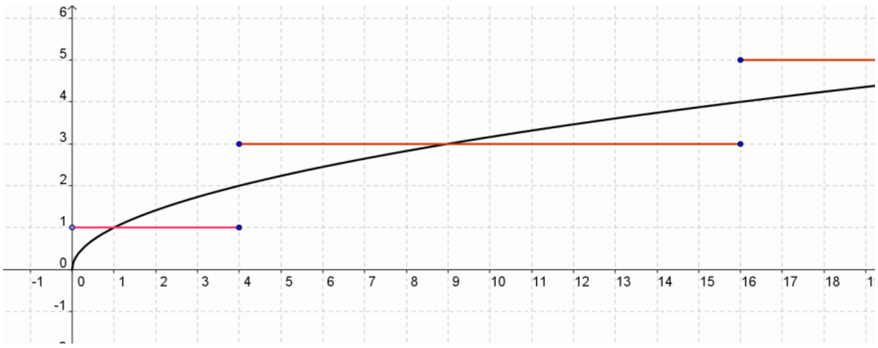


Figure 1.1: discretisation of the barrier

The changes will occur at times $4n^2$, $n = 0,1,\ldots$. On the interval $4n^2 \leq x < 4(n+1)^2$, the barrier is at $2n+1$ (recall from Part II that we want even values of time and odd values for the barrier). So, in the notation introduced in Part II, $\xi = n$ and this value is used on an interval of length $l_n = 4(n+1)^2 - 4n^2$, that is $l_n = 8n+4$.

The interval of time during which $\xi = n$ is called the n^{th} period. From now on, we forget about the continuous curve and remember only the segments.

We have now to investigate the transition between two periods. The barrier was at $2\xi + 1$ and moves to $2\xi + 3$.

Let us first consider the transition on the energy, that is the variables $e(2n, 2k)$ (see Part II).

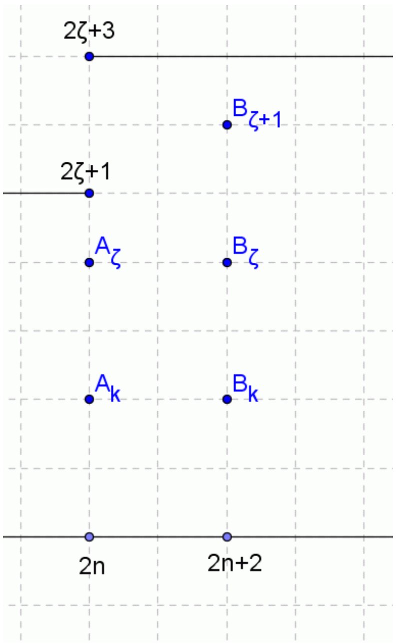


Figure 1.2 : Notation for the transition

It will be convenient to have a simple notation just for the transition. On the vertical corresponding to time $2n,$ we have $\xi+1$ points $A_0,...,A_\xi$; at time $2n+2,$ we have $\xi+2$ points $B_0,...,B_{\xi+1}$. We denote by a_k the energy at the point A_k and similarly b_k for the B_k.

For the first ξ points, we have the usual transition equations:

$$b_0 = \frac{1}{2}\left(a_0 + a_1\right) \qquad (1.1)$$

$$b_k = \frac{1}{4}a_{k-1} + \frac{1}{2}a_k + \frac{1}{4}a_{k+1} \text{ for } k = 1,...,\xi-1 \quad (1.2)$$

The last two equations are different from the constant case; they are:

$$b_\xi = \frac{1}{4}a_{\xi-1} + \frac{1}{2}a_\xi \qquad (1.3)$$

$$b_{\xi+1} = \frac{1}{4}a_\xi \qquad (1.4)$$

If the barrier was constantly at $2\xi+1,$ instead of (1.3), we would have:

$$b_\xi = \frac{1}{4}a_{\xi-1} + \frac{1}{4}a_\xi \qquad (1.5)$$

and instead of (1.4) :

$$b_{\xi+1} = 0 \qquad (1.6)$$

So, the fact that the barrier moves one step higher means more energy:

- for b_ξ, increase of $\frac{1}{4}a_\xi$
- for $b_{\xi+1}$, increase of $\frac{1}{4}a_\xi$

which represents a total increase of energy equal to $\frac{1}{2}a_\xi$.

Let us now turn to the variables $x(n,k)$ and describe the transition on these variables. Recall that, for $k=0,...,\xi-1$ and $n\geq 2$:

$$x_k=\frac{1}{2}(a_k+a_{k+1})$$

We have (see Part II):

$$b_0=x_0 \qquad (1.7)$$

$$b_k=\frac{1}{2}(x_{k-1}+x_k) \text{ for } k=1,...,\xi-1 \qquad (1.8)$$

$$b_\xi=\frac{1}{4}(a_{\xi-1}+a_\xi)+\frac{1}{4}(a_\xi+a_{\xi+1}) \text{ with } a_{\xi+1}=0$$

which gives:

$$b_\xi=\frac{1}{2}x_{\xi-1}+\frac{1}{2}x_\xi \qquad (1.9)$$

$$b_{\xi+1}=\frac{1}{4}(a_\xi+a_{\xi+1})=\frac{1}{2}x_\xi \qquad (1.10)$$

Let us define $y_k=\frac{1}{2}(b_k+b_{k+1})$, $k=0,...,\xi$.

We get:

$$y_0=\frac{1}{2}\left(x_0+\frac{1}{2}(x_0+x_1)\right)=\frac{3}{4}x_0+\frac{1}{4}x_1$$

$$y_k=\frac{1}{4}(x_{k-1}+2x_k+x_{k+1}),\ k=1,...,\xi-1$$

$$y_\xi=\frac{1}{2}\left(\frac{1}{2}x_{\xi-1}+\frac{1}{2}x_\xi+\frac{1}{2}x_\xi\right)=\frac{1}{4}x_{\xi-1}+\frac{1}{2}x_\xi$$

The equations are the same as in the constant case; we simply have one more intermediate equation.

With the original notation, we have:

$$\begin{cases} x(n+1,0)=\dfrac{3}{4}x(n,0)+\dfrac{1}{4}x(n,1) \\ x(n+1,k)=\dfrac{1}{4}x(n,k-1)+\dfrac{1}{2}x(n,k)+\dfrac{1}{4}x(n,k+1), \text{for } k=1,\dots,\xi-1 \\ x(n+1,\xi)=\dfrac{1}{4}x(n,\xi-1)+\dfrac{1}{2}x(n,\xi) \end{cases} \tag{1.11}$$

We have proved:

Proposition 1.1 - *On the variables* $x(n,k)$*, the fact that the barrier is shifted one step higher leads simply to a new intermediate equation in the transition equations.*

This is quite important in practice, because it means that the theory developed in Part II will apply, despite the changes of position for the barrier. We have simply to take into account the fact that the matrix M_n will increase by one dimension at the transition between two periods and the corresponding eigenvalues will change accordingly.

We note here that this result applies to any transition, where the barrier is shifted one step up, and does not depend on the particular function (here $\pm\sqrt{n}$).

2. Changes in the eigenvalues and in the eigenvectors

We now work constantly on the variables $X_n = x(n,i)$. In dimension ξ, we know (see Part II) that the eigenvalues are of the form:

$$\lambda_j = \frac{1+\cos(\vartheta_j)}{2}, \; \vartheta_j = \frac{2j-1}{2\xi+1}\pi, \; j=1,\dots,\xi,$$

and the same expressions will remain in dimension $\xi+1$, with ξ replaced by $\xi+1$.

In dimension ξ, the eigenvectors were:

$$V_{\xi,j} = \left(\sin(\xi\vartheta_j),\dots,\sin(\vartheta_j)\right), \; j=1,\dots,\xi$$

and in dimension $\xi+1$, they will be:

$$V_{\xi+1,j} = \left(\sin((\xi+1)\vartheta_j),\sin(\xi\vartheta_j),\dots,\sin(\vartheta_j)\right), \; j=1,\dots,\xi+1.$$

The natural embedding from dimension ξ to dimension $\xi+1$ (simply adding a zero as the last coordinate) preserves the eigenvectors. Indeed, we have:

Proposition 2.1. - *The image of the vector* $V_{\xi,j}=\left(\sin\left(\xi\vartheta_j\right),\ldots,\sin\left(\vartheta_j\right),0\right)$, *with* $\vartheta_j=\dfrac{2j-1}{2\xi+1}\pi$, *by the matrix* $M_{\xi+1}$ *is the vector:*

$$M_{\xi+1}V_{\xi,j}=\left(\lambda_{\xi,j}\sin\left(\xi\vartheta_j\right),\ldots,\lambda_{\xi,j}\sin\left(\vartheta_j\right),\frac{1}{4}\sin\left(\vartheta_j\right)\right)$$

and we have $\left|M_{\xi+1}V_{\xi,j}\right|_1=\left|V_{\xi,j}\right|_1$.

Proof of Proposition 2.1

This is obvious for the first coordinates, so we have to look only at the last 3. The image of 0 (last coordinate) is $\frac{1}{4}\sin\left(\vartheta_j\right)$. The image of $\sin\left(2\vartheta_j\right)$ is $\lambda_{\xi,j}\sin\left(2\vartheta_j\right)$. The image of $\sin\left(\vartheta_j\right)$ is by definition $\frac{1}{4}\sin\left(2\vartheta_j\right)+\frac{1}{2}\sin\left(\vartheta_j\right)=\sin\left(\vartheta_j\right)\left(\frac{1+\cos\left(\vartheta_j\right)}{2}\right)=\lambda_{\xi,j}\sin\left(\vartheta_j\right)$. The assertion on the l_1 norm is obvious, since there is no loss of energy in the transition.

This proves Proposition 2.1.

The matrix M, at any stage, operates only on three coordinates (and only on 2, at the first and last coordinates). So, if these three coordinates are those of an eigenvector of a previous situation, the result is a multiplication by the corresponding eigenvalue.

3. Energy transition

Assume we have l_1 times the value ξ_1 for the barrier. After l_1 steps, the vector of energies will be X_{l_1}, with $\left|X_{l_1}\right|_2\leq a\left(l_1,\xi_1\right)\left|X_0\right|_2$, where $a\left(l_1,\xi_1\right)$ is the attenuation of quadratic energy during the episode of length l_1 where the barrier is at ξ_1. We have $a\left(l_1,\xi_1\right)\leq\lambda_1$, where λ_1 is the largest eigenvalue of the matrix M with size ξ_1.

Assume that, after the first l_1 steps, we have l_2 times the value ξ_2 for the barrier. After l_1+l_2 steps, the vector of energies will be $X_{l_1+l_2}$ with $\left|X_{l_1+l_2}\right|_2\leq a\left(l_2,\xi_2\right)\left|X_{l_1}\right|_2$, where $a\left(l_2,\xi_2\right)$ is the attenuation of quadratic energy during the episode of length l_2 where the barrier is at ξ_2, and $a\left(l_2,\xi_2\right)\leq\lambda_2$, where λ_2 is the largest eigenvalue of the matrix M with size ξ_2.

Therefore:

$$\left|X_{l_1+l_2}\right|_2 \leq a\left(l_1,\xi_1\right) a\left(l_2,\xi_2\right)\left|X_0\right|_2$$

Let us now consider a sequence of periods of respective lengths $l_1,\ldots,l_N$. We get, at the end:

$$\left|X_{l_1+\cdots+l_N}\right|_2 \leq a\left(l_1,\xi_1\right)\cdots a\left(l_N,\xi_N\right)\left|X_0\right|_2 \tag{3.1}$$

The quadratic energy will tend to 0, when $N \to +\infty$, as soon as:

$$a\left(l_1,\xi_1\right)\cdots a\left(l_N,\xi_N\right) \to 0.$$

This will be satisfied as soon as:

$$S_N = \sum_{n=1}^{N} l_n Log\left(\lambda_n\right) \to -\infty \text{ when } N \to +\infty.$$

4. Estimates for a square root barrier

Theorem 4.1 - *Consider the symmetric barrier $b(x) = \pm\sqrt{x}$. During each period, we have the estimates:*

Quadratic energy : $E_2\left(W_{2(4N+4)^2+2}\right) \leq \frac{1}{2}\sqrt{\frac{3}{2}} N^{-\frac{\pi^2}{2}}$

Total energy : $E_1\left(W_{2(4N+4)^2+2}\right) \leq \frac{1}{2}\sqrt{\frac{3}{2}} N^{-\frac{\pi^2}{2}+\frac{1}{2}}$

Proof of Theorem 4.1

The duration of the n^{th} period is $l_n = 8n+4$ with $\xi_n = n$; during this period, we have seen the estimate (Lemma 10.1, Part II), for the largest eigenvalue:

$$\lambda_n \approx 1 - \frac{\pi^2}{16n^2}$$

which gives, using the inequality $Log(1-x) \leq -x,\ 0 \leq x \leq 1$:

$$S_N = \sum_{n=1}^{N} l_n Log\left(\lambda_n\right) \approx \sum_{n=1}^{N}(8n+4) Log\left(1-\frac{\pi^2}{16n^2}\right) \leq -\sum_{n=1}^{N}(8n+4)\frac{\pi^2}{16n^2} \approx -\frac{\pi^2}{2}\sum_{n=1}^{N}\frac{1}{n} \approx -\frac{\pi^2}{2} Log(N)$$

So this series diverges. At the end of the N^{th} period, we have:

$$\left|X_{(4N+4)^2}\right|_2 \leq |X_0|_2 \exp(S_N) = \frac{1}{2} N^{-\frac{\pi^2}{2}}$$

At the end of the N^{th} period, the barrier is at $2N+1$ and we have N points on the corresponding vertical. Therefore:

$$\left|X_{(4N+4)^2}\right|_1 \leq \frac{1}{2} N^{-\frac{\pi^2}{2}+\frac{1}{2}}$$

This quantity tends to 0 when N increases.

In order to convert these estimates in terms of energy, we use the estimates (6.2) and (6.3) in Part II, that is:

$$E_1(W_{2n+2}) \leq \frac{3}{2}|X_n|_1 \ ; \ E_2(W_{2n+2}) \leq \sqrt{\frac{3}{2}}|X_n|_2$$

This proves Theorem 4.1.

The proof is rather simple here, because we may work directly with the l_2 norm, and convert the result at the end to the l_1 norm. But such a rough argument does not work for other barriers, such as for instance $b(n) = \pm 8\sqrt{n}$; we now treat this case.

II. The case of the barrier $b(n) = \pm c\sqrt{n}$

In order to fix ideas, we will take the case of $b(n) = \pm 8\sqrt{n}$. As we did earlier, we work on the variable X_n rather than on the energy directly.

1. Transition times

Let $t_n = \left[\left(\frac{n}{8}\right)^2\right]$. On the interval $[t_n, t_{n+1}]$, the barrier takes the value $b_n = n$; the duration of this period is $l_n \approx \left(\frac{n+1}{8}\right)^2 - \left(\frac{n}{8}\right)^2 = \frac{n}{32}$.

On the original setting, this corresponds to a barrier at $2n+1$ on the interval $[2t_n, 2t_{n+1}]$.

From $n=0$ to $n=64$, the value of the barrier is 64 and this barrier is useless. On the interval from 64 to $\left[\left(\frac{65}{8}\right)^2\right] = 66$ the value of the barrier is 64, on $\left[\left(\frac{65}{8}\right)^2\right] \to \left[\left(\frac{66}{8}\right)^2\right]$, that is $66 \to 68$, the barrier is 65, and so on.

2. The previous estimates are insufficient

Let us first observe that the simple proof we saw for the barrier $\sqrt{x}$ does not work here. Indeed, with the previous notation:

$$\lambda_n \approx 1-\frac{\pi^2}{16n^2} \quad l_n = \frac{n}{32}$$

$$S_N = \sum_{n=1}^{N} l_n Log\left(\lambda_n\right) \approx \sum_{n=1}^{N} \frac{n}{32} Log\left(1-\frac{\pi^2}{16n^2}\right) \leq -\frac{\pi^2}{512}\sum_{n=1}^{N}\frac{1}{n} \approx -\frac{\pi^2}{512} Log\left(N\right)$$

At the end of the N^{th} step, we have:

$$\left|X_{\frac{N^2}{64}}\right|_2 \leq \left|X_0\right|_2 \exp\left(S_N\right) = \frac{1}{2} N^{-\frac{\pi^2}{512}}$$

This tends to 0, but very slowly, and for the energy we get the estimate:

$$\left|X_{(4N+4)^2}\right|_1 \leq \frac{1}{2} N^{-\frac{\pi^2}{512}+\frac{1}{2}}$$

which is useless. In the previous case, we had a sharp estimate for the l_2 - norm, which proved to be sufficient for the l_1 - norm. But this is not the case anymore for other barriers.

3. Initial step

We have the energy 1 at the origin (time 0).

We first introduce the barrier at altitude $\xi_1 = 64$, $n_1 = 64$.

At distance n_1, the energy 1 at O becomes X_1 with components:

$$x\left(n_1, i\right) = \frac{1}{2^{2n_1}}\binom{2n_1+1}{n_1+i} \tag{3.1}$$

The total amount of energy is still 1: there is no loss during the first n_1 steps.

Let M be the operator reflecting the propagation of energy. It operates first on a space of dimension $\xi_{1,}$ then $\xi_1+1,$ and so on.

Now, the vector X_1 does not have a satisfactory shape, in the sense that we have no information at all on the iterates $M^n X_1$; we have such an information only for the eigenvectors of the matrix M (and these eigenvectors depend on the dimension, of course). Therefore, we want to replace X_1 by the vector $V_{1,1}$, first eigenvector of the matrix M in dimension n_1. We know that:

$$V_{1,1} = \frac{2\left(\sin\left(\xi_1 \vartheta_{1,1}\right),...,\sin\left(\vartheta_{1,1}\right)\right)}{\tan\left(\xi_1 \vartheta_{1,1}\right)} \tag{3.2}$$

with $\xi_1 = n_1 = 64,\ \vartheta_{1,1} = \dfrac{\pi}{2\xi_1 + 1}$. This vector is normalized in l_1 norm : it has positive coefficients, with sum equal to 1.

The replacement of the vector X_1 by $V_{1,1}$ is done, using the following Lemma :

Lemma 3.1. - *For every* n, $\left|M^n X_1\right|_1 \leq c\left|M^n V_{n_1,1}\right|_1$ with $c = \dfrac{\binom{2n_1+1}{n_1+i}}{2^{2n_1}\cos\left(\vartheta_{1,1}\right)}$.

Proof of Lemma 3.1

The first coefficient of X_1 is $x\left(n_1,1\right) = \dfrac{1}{2^{2n_1+1}}\dbinom{2n_1+1}{n_1+1}$. We have $X_1(i) \leq c V_{n_1,1}(i)$ for every $i = 1,...,n_1$, with this choice of c. The operator M has positive coefficients, so it respects the order : during each period, we will have $\left|M^n X_1\right|_1 \leq c\left|M^n V_{n_1,1}\right|_1$, because the l_1 is simply the sum of all coefficients (the coefficients are positive). This proves Lemma 3.1.

4. Study of the first transition

We are now with an eigenvector, $V_{1,1}$, of the matrix M in dimension n_1. Unfortunately, this vector, when we pass to dimension n_1+1, does not become an eigenvector of the next matrix. More precisely (without normalisation),

$V_{1,1} = \left(\sin\left(n_1 \vartheta_{1,1}\right),...,\sin\left(\vartheta_{1,1}\right)\right)$ (n_1 coordinates)

becomes, after embedding :

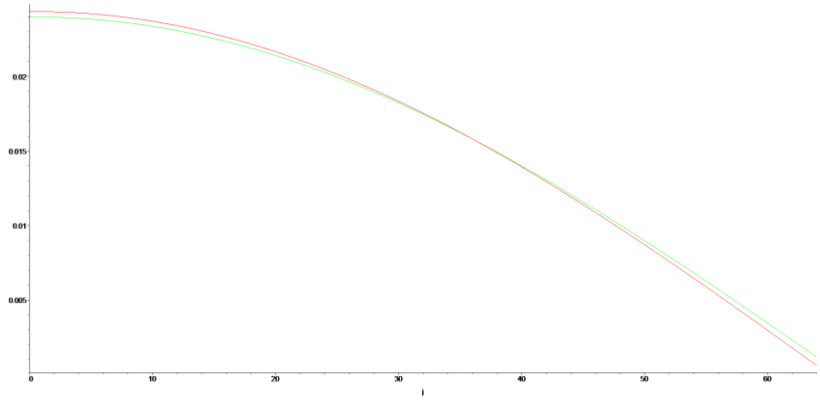
$V'_{1,1} = \left(\sin\left(n_1 \vartheta_{1,1}\right),...,\sin\left(\vartheta_{1,1}\right),0\right)$ (n_1+1 coordinates)

where as the first eigenvector of the new matrix is :

$V_{2,1} = \left(\sin\left((n_1+1)\vartheta_{2,1}\right), \ldots, \sin\left(\vartheta_{2,1}\right)\right)$ ($n_1 + 1$ coordinates)

with $\vartheta_{2,1} = \dfrac{\pi}{2\xi_1 + 3}$.

There is no simple connection between $V'_{1,1}$ and $V_{2,1}$: both are very close, as the following picture shows ($V'_{1,1}$ is in red and $V_{2,1}$ is in green) :



Both are very close, but we cannot simply say that we replace $V'_{1,1}$ by $V_{2,1}$, because $V_{2,1}$ is slightly closer to the barrier. We may compute the loss in this replacement, but the sum of such losses, over all transitions, in infinite, so such an approach, keeping only one eigenvector at each step, must be abandoned: we have to keep all eigenvectors.

First, what we do is to expand $V_{1,1}{}'$ on the basis of eingenvectors of the matrix M_2 in dimension ξ_2. We describe the general stage of the process.

5. General transition

We will go from dimension $\xi - 1$ to dimension ξ. In order to simplify the notation, we denote by V_i the eigenvectors in dimension $\xi - 1$ ($i = 1, \ldots, \xi - 1$) and by W_j the vectors in dimension ξ ($j = 1, \ldots, \xi$). Also in order to simplify the notation, we identify V_i and V'_i (one more coordinate, equal to 0).

We have the decomposition on the basis of eigenvectors in dimension ξ :

$$V_i = \sum_{j=1}^{\xi} \alpha_{i,j} W_j = \sum_{j=1}^{\xi} \frac{\langle V_i, W_j \rangle}{\left|W_j\right|_2^2} W_j = \frac{1}{\dfrac{\xi}{2} + \dfrac{1}{4}} \sum_{j=1}^{\xi} \langle V_i, W_j \rangle W_j \tag{5.1}$$

In this decomposition, some coefficients are negative, so they do not represent an "energy" in the previous sense of the word. Still, the above decomposition is valid, and represents an algebraic (Hilbert space) decomposition, in which the total energy is the sum of all components of all vectors. There is a conceptual difficulty here, because we have to leave the framework of "ordinary energy" (every component is positive), and to adopt the framework of "algebraic energy" (some components may be negative).

We introduce a notation for the sum of components of a vector :

$$s(V_i) = \sum_{l=1}^{\xi-1} V_i(l)$$

and the same for $s(W_j)$.

With this notation, the total energy at the beginning of the ξ^{th} period (just after the embedding $\xi-1 \to \xi$) is :

$$E_\xi = \sum_{i=1}^{\xi-1} s(V_i) \tag{5.2}$$

which may be written:

$$\begin{aligned} E_\xi &= \sum_{i=1}^{\xi-1} s(V_i) = \frac{1}{\frac{\xi}{2}+\frac{1}{4}} \sum_{i=1}^{\xi-1} s\left(\sum_{j=1}^{\xi} \langle V_i, W_j \rangle W_j \right) \\ &= \frac{1}{\frac{\xi}{2}+\frac{1}{4}} \sum_{i=1}^{\xi-1} \sum_{j=1}^{\xi} \langle V_i, W_j \rangle s(W_j) = \frac{1}{\frac{\xi}{2}+\frac{1}{4}} \left\langle \sum_{i=1}^{\xi-1} V_i, \sum_{j=1}^{\xi} s(W_j) W_j \right\rangle \end{aligned} \tag{5.3}$$

Let $l = l_\xi$ be the duration of the ξ^{th} period, and let λ_j, instead of $\lambda_{\xi,j}$ ($j=1,\dots,\xi$) be the eigenvalues during this period.

The total energy at the end of the ξ^{th} period is:

$$F_\xi = \frac{1}{\frac{\xi}{2}+\frac{1}{4}} \sum_{i=1}^{\xi-1} s\left(\sum_{j=1}^{\xi} \langle V_i, W_j \rangle \lambda_j^l W_j \right) \tag{5.4}$$

Indeed, during this period, each W_j is transformed into $\lambda_j^l W_j$. So we have:

$$F_\xi = \frac{1}{\frac{\xi}{2}+\frac{1}{4}} \sum_{i=1}^{\xi-1} s\left(\sum_{j=1}^{\xi} \left\langle V_i, W_j \right\rangle \lambda_j^l W_j \right)$$

$$= \frac{1}{\frac{\xi}{2}+\frac{1}{4}} \sum_{i=1}^{\xi-1} \sum_{j=1}^{\xi} \left\langle V_i, W_j \right\rangle \lambda_j^l s\left(W_j\right) = \frac{1}{\frac{\xi}{2}+\frac{1}{4}} \left\langle \sum_{i=1}^{\xi-1} V_i, \sum_{j=1}^{\xi} s\left(W_j\right) \lambda_j^l W_j \right\rangle$$

We want to compare F_ξ and E_ξ. The following Theorem answers our question and is the key point in our approach.

6. Sharp transition estimates

Theorem 6.1. - The total energy at the end of the ξ^{th} period, denoted by F_ξ, and the total energy at the beginning of the ξ^{th} period, denoted by E_ξ, are linked by the inequality:

$$F_\xi \le \lambda_1^l E_\xi$$

where $l = l_\xi$ is the duration of the ξ^{th} period, and $\lambda_1 = \lambda_{\xi,1}$ is the largest eigenvalue during this period.

Proof of Theorem 6.1

The statement is equivalent to:

$$\left\langle \sum_{i=1}^{\xi-1} V_i, \sum_{j=1}^{\xi} s\left(W_j\right) \lambda_j^l W_j \right\rangle \le \lambda_1^l \left\langle \sum_{i=1}^{\xi-1} V_i, \sum_{j=1}^{\xi} s\left(W_j\right) W_j \right\rangle \tag{6.1}$$

We will study all terms separately.

Lemma 6.2. - *All components of the vector* $\sum_{i=1}^{\xi-1} V_i$ *are positive.*

Proof of Lemma 6.2

In dimension $\xi-1$, the components of the i^{th} vecteur, V_i, are $\sin\left(\left(\xi-1\right)\vartheta_i\right),...,\sin\left(\vartheta_i\right)$, with $\vartheta_i = \vartheta_{\xi-1,i} = \dfrac{2i-1}{2\xi-1}\pi$.

The k^{th} component (starting from the right) of $\sum_{i=1}^{\xi-1} V_i$ is therefore:

$$C_k = \sum_{i=1}^{\xi-1} \sin\left(\frac{k\left(2i-1\right)}{2\xi-1}\pi \right)$$

We use the identity:

$$\sum_{i=1}^{\xi-1} \sin\left((2i-1)\alpha\right) = \frac{2\sin(\alpha)\left(1-\cos^2\left((\xi-1)\alpha\right)\right)}{1-\cos(2\alpha)}$$

in which we take $\alpha = \dfrac{k\pi}{2\xi-1}$; since $1 \le k \le \pi$, $0 < \alpha < \dfrac{\pi}{2}$, we have $\sin(\alpha) > 0$, and this proves Lemma 6.2.

We now compute the sum of the components of each eigenvector. By definition, it is:

$$s(W_j) = \sum_{k=1}^{\xi} \sin\left(\frac{k(2j-1)}{2\xi+1}\pi\right)$$

Lemma 6.3. - *For each* $j = 1,\ldots,\xi$, *we have:*

$$s(W_j) = \frac{\sin\vartheta_j}{2\left(1-\cos\vartheta_j\right)} = \frac{\tan\left(\xi\vartheta_j\right)}{2}, \text{ with } \vartheta_j = \vartheta_{\xi,j} = \frac{2j-1}{2\xi+1}\pi .$$

Proof of Lemma 6.3

We have the identity, for any ϑ and any ξ:

$$\sum_{k=1}^{\xi} \sin(k\vartheta) = \frac{\sin(\xi\vartheta) - \sin\left((\xi+1)\vartheta\right) + \sin(\vartheta)}{2\left(1-\cos(\vartheta)\right)} \tag{6.2}$$

The numerator is:

$$Num = \sin\left(\xi\frac{(2j-1)\pi}{2\xi+1}\right) - \sin\left((\xi+1)\frac{(2j-1)\pi}{2\xi+1}\right) + \sin\left(\frac{(2j-1)\pi}{2\xi+1}\right)$$

But in fact:

$$\sin\left(\xi\frac{(2j-1)\pi}{2\xi+1}\right) = \sin\left((\xi+1)\frac{(2j-1)\pi}{2\xi+1}\right) \tag{6.3}$$

Indeed, this follows from the equality:

$$\xi\frac{(2j-1)\pi}{2\xi+1} = \pi - (\xi+1)\frac{(2j-1)\pi}{2\xi+1} + 2k\pi \tag{6.4}$$

with $k = 2j - 2$.

So we get simply:

$$Num = \sin\left(\frac{(2j-1)\pi}{2\xi+1}\right)$$

We have finally:

$$s(W_j) = \frac{\sin\left(\frac{(2j-1)\pi}{2\xi+1}\right)}{2\left(1-\cos\left(\frac{(2j-1)\pi}{2\xi+1}\right)\right)} = \frac{1}{2}\frac{1}{\tan\frac{2j-1}{2\xi+1}\frac{\pi}{2}} = \frac{1}{2}\frac{1}{\tan\frac{\vartheta_j}{2}}$$

which proves Lemma 6.3.

It follows from Lemma 6.3 that, since $j \le \xi,\ 2j-1 \le 2\xi+1,$ the term $s(W_j)$ is positive.

We now study the vector $T = \sum_{j=1}^{\xi} s(W_j) W_j$.

The k^{th} component (starting from the right) is, using Lemma 6.3:

$$T_k = \sum_{j=1}^{\xi} s(W_j)\sin(k\vartheta_j) = \sum_{j=1}^{\xi} \frac{\sin(k\vartheta_j)}{2\tan\frac{\vartheta_j}{2}} = \sum_{j=1}^{\xi} \frac{\sin\left(k\frac{2j-1}{2\xi+1}\pi\right)}{2\tan\frac{2j-1}{2\xi+1}\frac{\pi}{2}}$$

We observe that the coefficient $\tan\left(\frac{\vartheta_j}{2}\right)$ is positive and increasing with j.

Proposition 6.4. - For each k, we have the identity :

$$T_k = \frac{2\xi-1}{4} > 0$$

Proof of Proposition 6.4

Set $\varphi_j = \frac{2j-1}{2\xi+1}\frac{\pi}{2} = \frac{\vartheta_j}{2}$. Then :

$$T_k = \frac{1}{2}\sum_{j=1}^{\xi} \frac{\sin(2k\varphi_j)}{\tan(\varphi_j)}$$

We use the identities:

$$\frac{\sin(2kx)}{\tan x}=\frac{1}{2}\left(\frac{\sin((2k+1)x)}{\sin x}+\frac{\sin((2k-1)x)}{\sin x}\right)$$

$$\frac{\sin((2k+1)x)}{\sin x}-\frac{\sin((2k-1)x)}{\sin x}=4\cos^2(kx)-2=2\left(\cos^2(kx)-1\right)=2\cos(2kx)$$

They give :

$$T_k=\frac{1}{4}\left(\sum_{j=1}^{\xi}\frac{\sin((2k+1)\varphi_j)}{\sin\varphi_j}+\sum_{j=1}^{\xi}\frac{\sin((2k-1)\varphi_j)}{\sin\varphi_j}\right)$$

Set:

$$W_k=\sum_{j=1}^{\xi}\frac{\sin((2k+1)\varphi_j)}{\sin\varphi_j}$$

Then $T_k=\frac{1}{4}(W_k+W_{k-1})$ and:

$$\frac{\sin((2k+1)\varphi_j)}{\sin\varphi_j}-\frac{\sin((2k-1)\varphi_j)}{\sin\varphi_j}=2\cos(2k\varphi_j)=2\cos\left(k\frac{2j-1}{2\xi+1}\pi\right)$$

We use the identity:

$$\sum_{j=1}^{\xi}\cos((2j-1)\vartheta)=\frac{\sin(2\xi\vartheta)}{2\sin(\vartheta)}$$

which gives:

$$W_k-W_{k-1}=\frac{\sin\left(2\xi\frac{k\pi}{2\xi+1}\right)}{\sin\left(\frac{k\pi}{2\xi+1}\right)}$$

But:

$$\sin\left(2\xi\frac{k\pi}{2\xi+1}\right)=(-1)^{k-1}\sin\left(\frac{k\pi}{2\xi+1}\right)$$

and therefore:

$W_k - W_{k-1} = (-1)^{k-1}$

Since $W_0 = \xi$, this gives:

$W_{2k} = \xi$, $W_{2k-1} = \xi - 1$

$$T_k = \frac{1}{4}(W_k + W_{k-1}) = \frac{2\xi - 1}{4},$$

which proves Proposition 6.4.

Let us now finish the proof of Theorem 6.1. We want to show that:

$$\left\langle \sum_{i=1}^{\xi-1} V_i, \sum_{j=1}^{\xi} s(W_j)\lambda_j^l W_j \right\rangle \le \lambda_1^l \left\langle \sum_{i=1}^{\xi-1} V_i, \sum_{j=1}^{\xi} s(W_j) W_j \right\rangle \tag{6.5}$$

The scalar product on the left hand-side is, by definition:

$$\begin{aligned}
\left\langle \sum_{i=1}^{\xi-1} V_i, \sum_{j=1}^{\xi} s(W_j)\lambda_j^l W_j \right\rangle &= \sum_{l=1}^{\xi} \sum_{i=1}^{\xi-1} V_i(l) \sum_{j=1}^{\xi} s(W_j)\lambda_j^l W_j(l) \\
&\le \lambda_1^l \sum_{l=1}^{\xi} \sum_{i=1}^{\xi-1} V_i(l) \sum_{j=1}^{\xi} s(W_j) W_j(l) \\
&= \lambda_1^l \left\langle \sum_{i=1}^{\xi-1} V_i, \sum_{j=1}^{\xi} s(W_j) W_j \right\rangle
\end{aligned}$$

since all terms are positive. This finishes the proof of Theorem 6.1.

7. Combining several periods

From Theorem 6.1 follows that the loss of total energy, during a period of altitude ξ for the barrier, and duration l, is $\le \lambda_1^l$, where λ_1 is the largest eigenvalue of the matrix M in dimension ξ.

Now, assume that the barrier is $f(n) = \pm 8\sqrt{n}$. As we saw, the duration of the n^{th} period is $l_n = \frac{n}{32}$. The first eigenvalue, $\lambda_{n,1}$, satisfies:

$$\lambda_n = \cos^2\left(\frac{1}{2n+1}\frac{\pi}{2}\right)$$

and therefore, since $Log(x) \le x - 1$ when $x > 0$:

$$Log\left(\lambda_n^{l_n}\right)=l_nLog\left(\cos^2\left(\frac{1}{2n+1}\frac{\pi}{2}\right)\right)\le -l_n\left(1-\cos^2\left(\frac{1}{2n+1}\frac{\pi}{2}\right)\right)=-l_n\sin^2\left(\frac{1}{2n+1}\frac{\pi}{2}\right)\approx -l_n\frac{\pi^2}{16n^2}$$

We have, with $\delta=\frac{\pi^2}{512}$:

$$\sum_{n=1}^{N} Log\left(\lambda_n^{l_n}\right)\approx -\delta\sum_{n=1}^{N}\frac{1}{n}\approx -\delta Log\left(N\right)$$

and therefore:

$\prod_{n=1}^{N}\lambda_n^{l_n}\approx\frac{1}{N^{\delta}}\to 0$ when $N\to+\infty$, which proves that the energy left at the N^{th} period tends to zero.

8. Quantitative statements

Let us summarize the results we obtained.

Theorem 8.1. - *Let* $b(x)=\pm 8\sqrt{x}$ *be the barrier. At the end of the period* $t_n=\left[\left(\frac{n}{8}\right)^2\right]$, *the total energy satisfies:*

$$E_{t_n}\le\frac{3}{2}n^{-\frac{\pi^2}{512}}$$

Proof of Theorem 8.1

The factor $3/2$ comes (see the estimates (6.2) and (6.3) in Part II) from the conversion of the variable X to the energy.

Similar results hold for any barrier of the form $f\left(x\right)=\pm c\sqrt{x}$. The statement is:

Theorem 8.2. - *Let* $b(x)=\pm c\sqrt{x}$ *be the barrier, with* $c\ge 1$. *At the end of the period* $t_n=\left[\left(\frac{n}{c}\right)^2\right]$, *the total energy satisfies:*

$$E_{t_n}\le\frac{3}{2}n^{-\frac{\pi^2}{8c^2}}$$

In other words, at any instant N, the energy satisfies:

$$E_N \leq \frac{3}{2} c^{-\frac{\pi^2}{8c^2}} N^{-\frac{\pi^2}{16c^2}}$$

III. Study of the barrier $b(x)=\sqrt{xLog(x)}$

1. Transition times

In this case, t_n is defined by $b(t_n)=n$; $t_n Log(t_n)=n^2$; $t_n=b^{-1}(n)$. We denote by β the inverse function of b (which is well-defined, since b is increasing). The duration of the n^{th} period is, with the notation of the previous paragraphs:

$$l_n = t_{n+1} - t_n = \beta(n+1) - \beta(n) \approx \beta'(n)$$

2. Decrease of energy

For this barrier, we prove:

Theorem 2.1. - *The energy left after the n^{th} period tends to 0 when $n \to +\infty$. More precisely, at the end of the n^{th} period, the total energy satisfies:*

$$E_{t_n} \leq \frac{3}{2}\left(Log(N)\right)^{\frac{-\pi^2}{16}}$$

Proof of Theorem 2.1

Let us first see the qualitative version. We know that (see II.7 above):

$$Log\left(\lambda_n^{l_n}\right) \leq -l_n \frac{\pi^2}{16n^2}$$

So we have to show that:

$$\sum \frac{l_n}{n^2} = +\infty .$$

We have:

$$b'(x) = \frac{1}{2} \frac{1+Log(x)}{\sqrt{xLog(x)}}$$

$$\beta'(n) = \frac{1}{b'(t_n)} = \frac{2n}{1+Log(t_n)}$$

Therefore:

$$\frac{l_n}{n^2} \approx \frac{2}{n\left(1+Log\left(t_n\right)\right)}$$

We have $t_n < n^2$, which implies $Log\left(t_n\right) < 2Log\left(n\right)$, and finally:

$$\frac{2}{n\left(1+Log\left(t_n\right)\right)} > \frac{2}{n\left(1+2Log\left(n\right)\right)} \approx \frac{1}{nLog\left(n\right)}$$, general term of a divergent series.

More quantitatively, the energy left at the end of the N^{th} period (in the variable X) satisfies:

$$E_N = \exp\left(\sum_{n=2}^{N} Log\left(\lambda_n^{l_n}\right)\right) \leq \exp\left(-\sum_{n=2}^{N} \frac{\pi^2 l_n}{16n^2}\right) \leq \exp\left(-\frac{\pi^2}{16}\sum_{n=2}^{N} \frac{1}{nLog\left(n\right)}\right)$$
$$\approx \exp\left(-\frac{\pi^2}{16} Log\left(Log\left(N\right)\right)\right) = \left(Log\left(N\right)\right)^{\frac{-\pi^2}{16}}$$

which proves Theorem 2.1.

IV. Energy profile

In all these cases, the energy profile during the n^{th} period is approximately proportional to the first eigenvector, and this approximation is more and more accurate when $n \to +\infty$. This means that, during the n^{th} period, the energy profile (in the X variable) is proportional to :

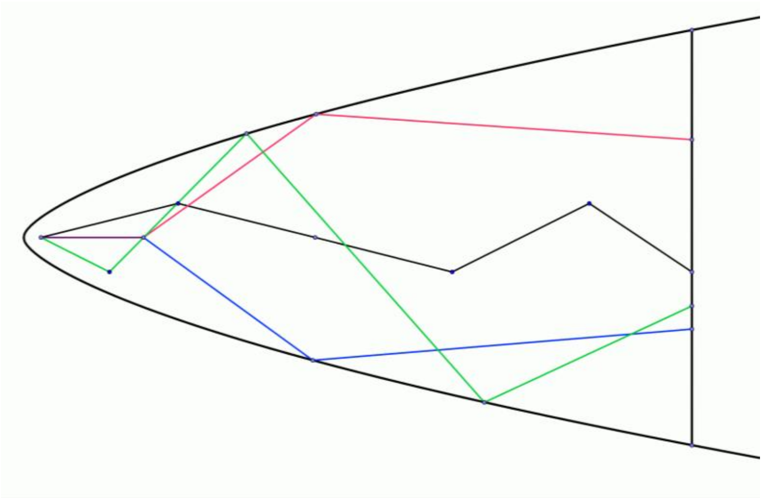
$$V_n = \left(\sin\left(n\vartheta\right),\ldots,\sin\left(\vartheta\right)\right)$$

with $\vartheta = \frac{\pi}{2n+1}$.

V. Comparison with Khinchine's curves

Let $b\left(x\right)$ be the barrier ; here $b\left(x\right) = \sqrt{xLog\left(x\right)}$. Let N be any instant. The fact that the energy left after the instant N tends to 0 when $N \to +\infty$ may at first sight look contradictory with Khintchine's result, according to which $k\left(x\right) = \sqrt{2xLog\left(Log\left(x\right)\right)}$ is a security curve, since our barrier b is above Khintchine's barrier k. But in fact, there is no contradiction. Let us explain the situation more in detail.

We distinguish between 4 types of paths (see picture):



A : all paths which never touch b nor $-b$ before the instant N (such a path is drawn in black).

B^+ : all paths which touch b but do not touch $-b$ before the instant N (such a path is drawn in red).

B^- : all paths which touch $-b$ but do not touch b before the instant N (such a path is drawn in blue).

C : all paths which touch both b and $-b$ before the instant N (such a path is drawn in green).

Of course, these four sets are disjoint, and their union represents all possible paths.

What we saw, for $b(x)=\sqrt{xLog(x)}$, is that:

$$P\left(B^+ \cup B^- \cup C\right) \to 1 \text{ when } N \to +\infty.$$

In other words, it becomes more and more unlikely that a path never touches the barrier or its opposite.

This barrier is above Khinchine's security curve, which is:

$$k(x)=\sqrt{2xLog\left(Log(x)\right)}$$

which means that the probability to touch b after the time N tends to 0 when $N \to +\infty$. In other words, almost every path returns near Khinchine's curve k infinitely many times, but this is not so for the barrier b.

This is not contradictory with our result. It means simply that, for instance :

$P(B^+) \to 0.1$ when $N \to +\infty$

$P(B^-) \to 0.1$ when $N \to +\infty$

$P(C) \to 0.8$ when $N \to +\infty$

So the total probability of hitting $\pm b$ tends to 1 when $N \to +\infty$ (our result), but the probability to hit either b or $-b$ after time N tends to 0 when $N \to +\infty$.

VI. Proof of the main Theorem, first statement

We have given all ingredients for the proof, except that we have to convert a statement given in terms of integral into a statement given in terms of a series. This is done by means fo the following Proposition.

Proposition 1. - *Let $b(x)$ be a barrier, that is a positive, differentiable, strictly increasing function, tending to $+\infty$ when $x \to +\infty$. Let, for each n, $[t_n, t_{n+1}[$ be the interval on which the discretization of b takes the value n (this is the n^{th} period), and let $l_n = t_{n+1} - t_n$ be the duration of this period. Then the integral $\int_A^{+\infty} \frac{dx}{b^2(x)}$ diverges at infinity if and only if the series $\sum_{n=1}^{+\infty} \frac{l_n}{n^2}$ diverges.*

Proof of Proposition 1

Let $\beta = b^{-1}$ be the inverse function of the function b (this inverse exists since b is strictly increasing). It is also positive, differentiable and strictly increasing. We have, by definition $t_n = \beta(n)$ and therefore:

$$l_n = t_{n+1} - t_n = \beta(n+1) - \beta(n) \approx \beta'(n) = \frac{1}{b'(t_n)}.$$

So we may write:

$$\sum_{n=1}^{+\infty} \frac{l_n}{n^2} \approx \sum_{n=1}^{+\infty} \frac{1}{n^2 b'(t_n)} \approx \int_1^{+\infty} \frac{1}{y^2 b'(\beta(y))} dy$$

Set $y = b(x)$, $dy = b'(x)dx$. The above integral becomes:

$$\int_1^{+\infty} \frac{1}{y^2 b'(\beta(y))} dy = \int_A^{+\infty} \frac{b'(x)}{b^2(x) b'(x)} dx = \int_A^{+\infty} \frac{dx}{b^2(x)},$$

which proves Proposition 1.

Quantitative statement

We saw that the energy left at the end of the N^{th} period satisfies:

$$E_N \le \exp\left(-\frac{\pi^2}{16}\sum_{n=2}^{N}\frac{l_n}{n^2}\right)$$

But :

$$\sum_{n=1}^{N}\frac{l_n}{n^2} \approx \int_1^{N+1} \frac{dy}{y^2 b'(\beta(y))} = \int_{\beta(1)}^{\beta(N+1)} \frac{dx}{b^2(x)} = \int_1^{t_{N+1}} \frac{dx}{b^2(x)}$$

and finally:

$$E_N \le \exp\left(-\frac{\pi^2}{16}\int_1^{t_{N+1}} \frac{dx}{b^2(x)}\right)$$

which proves our claim.

VII. Proof of the Main Theorem, converse statement

We want to show that if the integral diverges at infinity, the game may continue indefinitely (the remaining energy does not tend to zero). This part is much simpler than the previous one.

First of all, we have seen (§ above) that the convergence of the integral is equivalent to the convergence of the series $\sum_{n=1}^{+\infty}\frac{l_n}{n^2} < +\infty$, which corresponds to the energy, assuming it is carried by the first eigenvector only at each period.

The energy left after N periods will be $E_{t_N} \ge \exp\left(-\sum_{n=1}^{N}\frac{l_n}{n^2}\right)$ if we can prove that this energy is larger than the energy carried, during each period, by the first eigenvector of the corresponding matrix.

Proposition 1. - *We penalize ourselves if, at the end of each period, we replace the first eigenvector of this period by the first eigenvector of the next period, with same normalization.*

Proof of Proposition 1

Let us explain the statement more in detail. Let simply V be the first eigenvector during the n^{th} period. Recall that, when normalized in l_1 norm:

$$V = 2\tan\frac{\vartheta_1}{2}\left(\sin\left(n\vartheta_1\right),\dots,\sin\left(\vartheta_1\right)\right)$$

with $\vartheta_1 = \dfrac{\pi}{2n+1}$; when we start the $(n+1)^{st}$ period, it becomes:

$$V = 2\tan\frac{\vartheta_1}{2}\left(\sin\left(n\vartheta_1\right),\dots,\sin\left(\vartheta_1\right),0\right)$$

The first eigenvector of the $(n+1)^{st}$ period is:

$$W = 2\tan\frac{\vartheta_2}{2}\left(\sin\left((n+1)\vartheta_2\right),\dots,\sin\left(\vartheta_2\right)\right)$$

with $\vartheta_2 = \dfrac{\pi}{2n+3}$.

When we say that we "penalize ourselves", it means that the energy will be likely to disappear more easily with W than with V. So, if we perform this replacement at each step and, at the end, get a non-zero energy, it means that the whole game produces a non-zero energy.

In practice, using Part II, Corollary 3.4, it means that the "tail" of W contains more energy than the tail of V ; in simpler terms, W is globally closer to the barrier. More precisely, for any $k \le n+1$, let us define the tails made of the last k terms:

$$V_k = 2\tan\frac{\vartheta_1}{2}\left(0,\sin\left(\vartheta_1\right),\dots,\sin\left((k-1)\vartheta_1\right)\right)$$

$$W_k = 2\tan\frac{\vartheta_2}{2}\left(\sin\left(\vartheta_2\right),\dots,\sin\left(k\vartheta_2\right)\right)$$

in order to prove the Theorem, all we need to show is :

Lemma 2. - *For any* k, $k = 1,\dots,n+1$, $|W_k|_1 \ge |V_k|_1$.

Proof of Lemma 2

We use the identity:

$$\sum_{j=1}^{k}\sin\left(j\vartheta\right) = \frac{\sin\left(k\vartheta\right) - \sin\left((k+1)\vartheta\right) + \sin\left(\vartheta\right)}{2\left(1-\cos\left(\vartheta\right)\right)} \qquad (1)$$

We have to show that:

$$\tan\left(\frac{\vartheta_1}{2}\right)\frac{\sin\left((k-1)\vartheta_1\right)-\sin\left(k\vartheta_1\right)+\sin\left(\vartheta_1\right)}{1-\cos\left(\vartheta_1\right)} \leq \tan\left(\frac{\vartheta_2}{2}\right)\frac{\sin\left(k\vartheta_2\right)-\sin\left((k+1)\vartheta_2\right)+\sin\left(\vartheta_2\right)}{1-\cos\left(\vartheta_2\right)} \quad (2)$$

Using the identity $\dfrac{\tan\frac{t}{2}}{1-\cos(t)} = \dfrac{1}{\sin(t)}$, (2) is equivalent to:

$$\frac{\sin\left((k-1)\vartheta_1\right)-\sin\left(k\vartheta_1\right)+\sin\left(\vartheta_1\right)}{\sin\left(\vartheta_1\right)} \leq \frac{\sin\left(k\vartheta_2\right)-\sin\left((k+1)\vartheta_2\right)+\sin\left(\vartheta_2\right)}{\sin\left(\vartheta_2\right)} \quad (3)$$

Or:

$$\frac{\sin\left((k-1)\vartheta_1\right)-\sin\left(k\vartheta_1\right)}{\sin\left(\vartheta_1\right)} \leq \frac{\sin\left(k\vartheta_2\right)-\sin\left((k+1)\vartheta_2\right)}{\sin\left(\vartheta_2\right)} \quad (4)$$

Using the identity $\sin(p)-\sin(q) = 2\cos\dfrac{p+q}{2}\sin\left(\dfrac{p-q}{2}\right)$, (4) becomes:

$$\cos\left((2k-1)\vartheta_1\right) \geq \cos\left((2k+1)\vartheta_2\right) \quad (5)$$

But the angles in (5) are smaller than π, so the cosine is decreasing. Therefore, (5) is equivalent to:

$$(2k-1)\vartheta_1 \leq (2k+1)\vartheta_2 \qquad (6)$$

That is:

$$\frac{(2k-1)\pi}{2n+1} \leq \frac{(2k+1)\pi}{2n+3} \qquad (7)$$

which itself is equivalent to:

$$4k \leq 4n+2$$

which is satisfied for $k \leq n.$ For $k = n+1,$ the l_1 norms are equal by definition. This proves Lemma 2, Proposition 1, and finishes the proof of the Main Theorem.