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Simple Random Walks in the plane :

An energy based approach

Part II : Identical Initial Fortunes

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August, 2018

Abstract

The basic settings are the same as in Part I, but here each player has an identical initial fortune, and the game stops if one of the players gets ruined.

We consider a ± 1 game, with identical initial fortunes. Using a new, energy-based, approach, we investigate the value of the fortune of each player after n games. We give a complete description of each possible value and its probability; the tools used are Chebychev's polynomial of first and second kind, operator theory and trigonometry. We also investigate the asymptotic behavior when $n \to +\infty$: the profile of each fortune is concave and the total energy tends to 0 exponentially fast.

This behavior is quite different from the case of unbounded initial fortunes, or bounded fortune for one of the players only.

I. Presentation

The basic settings are defined in Part I. Recall that we consider a simple random walk in the plane: a game, with two players. It is defined by a r.v. X taking the values ± 1 with probability $\frac{1}{2}$. The player A wins if $X=1$ and then he receives 1 Euro from the player B, and conversely if $X=-1$. Both players have an initial fortune denoted by F, same for both. The game stops if any of the players sees his fortune equal to zero. The question is: what is the probability distribution of the earnings after n steps ?

II. Notation

We refer to Part I. We set $S_0 = 0$, and for $N \geq 1$, $S_N = \sum_{n=1}^{N} X_n$. Instead of a random walk with multiple possible paths, we consider that we have the propagation of an energy, with the following rules :

- At time $n = 0$, the origin receives an energy equal to 1 ;
- At time $n = 1$, this energy is divided into two: each point $(1,0)$ and $(-1,0)$ receives an energy equals to $\frac{1}{2}$ and so on.

More generally, the energy of a point of coordinates (n,k) in the plane is equal to the probability that the random walk hits this point. It will be denoted by $e(n,k)$.

As we already did in Part I, we restrict ourselves to even values of the time $(2n)$. We defined in Part I :

$$f(n,k) = e(2n,2k)$$

with the general propagation rule :

$$f(n+1,k) = \frac{1}{4} f(n,k-1) + \frac{1}{2} f(n,k) + \frac{1}{4} f(n,k+1)$$

Due to the barrier, this propagation rule will of course be modified: We insert the symmetric barriers $y = \pm(2\xi + 1)$; the reason why we work here with odd values will be apparent later. We consider that if a path hits any of the barriers, its energy is absorbed and disappears. So, our original question may be stated as: Given a time n, what is the distribution of energy on the vertical $x = n$?

We will restrict ourselves to the upper half-plane, since, by symmetry, the results are identical in the lower half plane. We first consider the simple case $\xi = 1$.

III. Case $\xi = 1$

The barrier is at the value ± 3. At the time $n = 2$, the energy distribution is: $e(2,2) = \frac{1}{4}$, $e(2,0) = \frac{1}{2}$, $e(2,-2) = \frac{1}{4}$ (the barrier plays no role). So we get $f(1,1) = \frac{1}{4}$, $f(1,0) = \frac{1}{2}$ and we have the recurrence relations:

$$f(n+1,0) = \frac{1}{2}\big(f(n,0) + f(n,1)\big), \quad f(n+1,1) = \frac{1}{4}\big(f(n,0) + f(n,1)\big)$$

As we already did in Part I, we set $x_n = \frac{1}{2}\big(f(n,0) + f(n,1)\big)$. Then:

$$x_1 = \frac{1}{2}\big(f(1,0) + f(1,1)\big) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4}\right) = \frac{3}{8}.$$

We obtain the equations:

$f(n+1,0) = x_n$, $\quad f(n+1,1) = \frac{1}{2}x_n$, $\quad$ and $\quad x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{2}x_n\right) = \frac{3}{4}x_n$, $\quad$ which gives $x_n = \frac{1}{2}\left(\frac{3}{4}\right)^n$.

So the energy profile at time $2n$ is:

$$e(2n,0) = f(n,0) = x_{n-1} = \frac{1}{2}\left(\frac{3}{4}\right)^{n-1}$$

$$e(2n,1) = e(2n,-1) = f(n,1) = \frac{1}{2}x_{n-1} = \frac{1}{4}\left(\frac{3}{4}\right)^{n-1}$$

The total energy at the instant $2n$ is the sum of all terms, that is $E(2n) = \left(\frac{3}{4}\right)^{n-1}$; it is exponentially decreasing.

IV. General case, $\xi > 1$

A. Notation

Let W_{2n} be the vertical for $x = 2n,$ that is the set of all points $A_{2n,2k}$, $k = 0,...,n.$

B. Basic equations

We have the initial values:

$f(0,0) = 1$, $f(0,k) = 0$ for $k = 1,...,\xi.$

The recurrence equations are:

$$\begin{cases} f(n+1,0) = \frac{1}{2}\left(f(n,0) + f(n,1)\right) \\ f(n+1,k) = \frac{1}{4}\left(f(n,k-1) + 2f(n,k) + f(n,k+1)\right), k = 1,...,\xi - 1 \\ f(n+1,\xi) = \frac{1}{4}\left(f(n,\xi-1) + f(n,\xi)\right) \end{cases} \tag{1}$$

Recall that the barrier is set at $\pm(2\xi+1)$, so the last non-zero value for f on each vertical is $f(n,\xi)$.

We first study the variation of energy, at a given time, on each vertical.

C. Decrease of the energy on each vertical

Lemma 1. - *For a given time n, the energy is decreasing as a function of k :*

$$f(n,k) \geq f(n,k+1),\ k \geq 0.$$

Proof of Lemma 1

This is true for $n = 0$; let us admit the result for n and prove it for $n+1$.

We have:

$$f(n+1,1) = \frac{1}{4}f(n,0) + \frac{1}{2}f(n,1) + \frac{1}{4}f(n,2) \leq \frac{1}{2}f(n,0) + \frac{1}{2}f(n,1) = f(n+1,0)$$

since $f(n,2) \leq f(n,0)$ by the recurrence assumption. For $1 \leq k \leq \xi - 2$:

$$
\begin{aligned}
f(n+1,k+1) &= \frac{1}{4} f(n,k) + \frac{1}{2} f(n,k+1) + \frac{1}{4} f(n,k+2) \\
&\le \frac{1}{4} f(n,k-1) + \frac{1}{2} f(n,k) + \frac{1}{4} f(n,k+1) \\
&= f(n+1,k)
\end{aligned}
$$

Finally, the property $f(n+1,\xi) \le f(n+1,\xi-1)$ comes from:

$$
\frac{1}{4} f(n,\xi-1) + \frac{1}{4} f(n,\xi) \le \frac{1}{4} f(n,\xi-2) + \frac{1}{2} f(n,\xi-1) + \frac{1}{4} f(n,\xi)
$$

which is clear. So Lemma 1 is proved.

Corollary 2. - *Let* $m < n$ *be two instants; let* $A(2m,2k)$ $k = 0,\dots,\xi$ *be points on the* $2m^{th}$ *vertical* W_{2m} *and let* $B = B(2n,0)$ *be the point on the* x *axis at time* $2n$. *Assume we put energy 1 at one of the points* $A(2m,2k)$ $k = 0,\dots,\xi$. *The energy received by* B *will be maximal if this energy is put at* $A(2m,0)$. *In fact, the energy received by* B *is a decreasing function of* k.

Proof of Corollary 2

This is a simple consequence of Lemma 1, because if we put energy 1 at $A(2m,2k)$, the energy received by B is the same as the energy received by $A(2m,2k)$ if we put energy 1 at B.

Corollary 3. - *Assume we have any distribution of energy* E_{2m} *on the vertical* W_{2m}. *Then the energy received by* B *will be larger if all this energy is concentrated at the single point* A_0.

This is a clear consequence of the previous Corollary. There is a more general statement:

Corollary 4. - *Let* $m < n$ *be two instants, and let* $A_1 = A(m,k_1)$ *and* $A_2 = A(m,k_2)$ *be two points on the same vertical, with* $k_1 < k_2$, *both carrying energy 1. Then, on the vertical* W_{2n}, *the energy coming from the first point is larger than the second, which means that the loss of energy is larger in the second case.*

Another equivalent formulation is:

Corollary 4.b - *Let* $m < n$ *be two instants, and let* V, W *be two distribution of energies at time* m*, with same sum. Assume that for any* k, $\sum_{i \geq k} W(i) \geq \sum_{i \geq k} V(i)$ *(the vector* W *is more concentrated than* V *near the barrier). Then the energy sent by* W *to the vertical at time* n *is smaller than the energy sent by* V.

This corollary is quite intuitive. The second distribution is globally closer to the barrier, so the loss of energy is larger. Another way to say this is as follows: take any distribution of energy, and move any quantity closer to the x axis: this is a "protective" move, in the sense that there will be less energy lost in the future.

We now turn to the behavior on the horizontal direction.

D. Decrease of the energy with time

Lemma 5 - *On the* x *axis, the energy is decreasing: for all* n,

$$f(0,n) \geq f(0,n+1).$$

Proof of Lemma 5

We have $f(0,0) = 1$ and $f(1,0) < 1$; let us admit the decrease until step n and prove it at step $n+1$. We have $f(n,0) - f(n+1,0) = \frac{1}{2}\left(f(n,0) - f(n,1)\right) > 0$ by Lemma 1. This proves Lemma 5.

However, it is not true that the energy is decreasing on all horizontal lines $y = j$; indeed, if $j > 1$, it first increases and then decreases: see Part I.

E. Matrix representation

The system (1) may be viewed as the action of a linear operator on the vector $\left(f(n,0), \ldots, f(n,\xi)\right)$; the matrix, of dimension $(\xi+1) \times (\xi+1)$, is:

$$M_F = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \cdots & \cdots & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

and the energy vector at time $2n$ is :

$$E_{2n} = M_F^n \begin{pmatrix} f(0,0) \\ \vdots \\ f(0,\xi) \end{pmatrix}$$

This matrix representation is not quite useful for further investigation, because it turns out that the matrix M_F has a non-zero kernel ; indeed, the vector :

$$Z = \left(1,-1,1,\dots,(-1)^k,\dots\right)$$

satisfies $M_F Z = 0.$ So we will find another representation, for which the matrix will be invertible, and for which the eigenvalues can be explicitly computed. See the book [BB_Op] for all topics related to operator theory, used here.

F. A proper matrix representation

We start with a new change in coordinates.

1. A second change in coordinates

We introduced in Part I, for any n and k :

$$x(n,k) = \frac{1}{2}\left(f(n,k-1) + f(n,k)\right)$$

For $n = 0,$ using the symmetry:

$$x(n,0) = \frac{1}{2}\left(f(n,-1) + f(n,0)\right) = \frac{1}{2}\left(f(n,0) + f(n,1)\right) = x(n,1)$$

We have $x(n,k)=0$ if $k>n+1$. From now on, we will work mostly with the new coordinates.

2. The propagation problem in the new coordinates

Equations (IV.B.1) above become:

$$\begin{cases} f(n+1,0)=x(n,1) \\ f(n+1,k)=\frac{1}{2}\big(x(n,k)+x(n,k+1)\big), k=1,\dots,\xi-1 \\ f(n+1,\xi)=\frac{1}{2}x(n,\xi) \end{cases} \quad (1)$$

From (1), we deduce:

$$x(n+1,1)=\frac{1}{2}\big(f(n+1,0)+f(n+1,1)\big)=\frac{1}{2}\left(x(n,1)+\frac{1}{2}\big(x(n,1)+x(n,2)\big)\right)=\frac{3}{4}x(n,1)+\frac{1}{4}x(n,2)$$

$$\begin{aligned} x(n+1,k) &= \frac{1}{2}\left(\frac{1}{2}\big(x(n,k-1)+x(n,k)\big)+\frac{1}{2}\big(x(n,k)+x(n,k+1)\big)\right) \\ &= \frac{1}{4}x(n,k-1)+\frac{1}{2}x(n,k)+\frac{1}{4}x(n,k+1) \end{aligned}$$

for $k=2,\dots,\xi-1$, and:

$$x(n+1,\xi)=\frac{1}{4}x(n,\xi-1)+\frac{1}{2}x(n,\xi)$$

So we have the system:

$$\begin{cases} x(n+1,1)=\frac{3}{4}x(n,1)+\frac{1}{4}x(n,2) \\ x(n+1,k)=\frac{1}{4}x(n,k-1)+\frac{1}{2}x(n,k)+\frac{1}{4}x(n,k+1), \text{for } k=2,\dots,\xi-1 \\ x(n+1,\xi)=\frac{1}{4}x(n,\xi-1)+\frac{1}{2}x(n,\xi) \end{cases} \quad (2)$$

with the initial values:

$$x(0,1)=\frac{1}{2}\big(f(0,0)+f(0,1)\big)=\frac{1}{2},\ x(0,k)=\frac{1}{2}\big(f(0,k-1)+f(0,k)\big)=0 \text{ for } k\geq 2.$$

These initial values may be written as a vector $X_0=\left(\frac{1}{2},0,\dots,0\right)$.

The system of equations (2) may be written as a matrix, under the form:

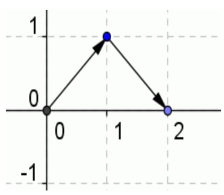
$$\begin{pmatrix} x(n+1,1) \\ x(n+1,2) \\ \dots \\ \dots \\ \dots \\ x(n+1,\xi-1) \\ x(n+1,\xi) \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & \dots & \dots & \dots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \dots & \dots & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \dots & \dots & \dots & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x(n,1) \\ x(n,2) \\ \dots \\ \dots \\ \dots \\ x(n,\xi-1) \\ x(n,\xi) \end{pmatrix} \quad (3)$$

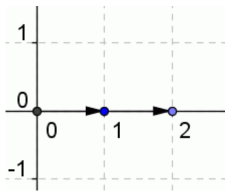
We have a real symmetric matrix, of size ξ, which is denoted by M.

We observe that, in this matrix representation, things are opposite to the physical representation: the first element of the vector X and the first row of the matrix correspond to what happens on the Ox axis; the last element of X and the last row of the matrix correspond to what happens close to the barrier.

We may consider that this is also a propagation problem, with the following properties:

A point may move upwards, horizontally or downwards ; all horizontal arrows have probability $\frac{1}{2}$ except the first one (the one on the x axis) which has probability $\frac{3}{4}$; all oblique arrows (up or down) have probability $\frac{1}{4}$. In this representation, two paths with same origin and same destination do not need to have the same probability. In the picture below, the left path has probability $\frac{1}{4^2}$ and the right path probability $\left(\frac{3}{4}\right)^2$.





Therefore, on the $x(n,k)$ coordinates, a matrix-oriented approach is appropriate, but an approach counting the number of paths is not.

3. Properties of the matrix *M*

The matrix M enjoys many satisfactory properties, which the previous matrix M_F did not have.

Lemma 6. - *The matrix* M *is positive defined.*

Proof of Lemma 6

We have to show that, for all non-zero column-vector X of size ξ, we have $X^t MX > 0$.

Let $X = \begin{pmatrix} x_1 \\ \vdots \\ x_\xi \end{pmatrix}$; we have:

$$MX = \begin{pmatrix} \frac{3}{4}x_1 + \frac{1}{4}x_2 \\ \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3 \\ \vdots \\ \frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1} \\ \vdots \\ \frac{1}{4}x_{\xi-2} + \frac{1}{2}x_{\xi-1} + \frac{1}{4}x_\xi \\ \frac{1}{4}x_{\xi-1} + \frac{1}{2}x_\xi \end{pmatrix}$$

and therefore:

$$\begin{aligned} X^t MX &= \left(\frac{3}{4}x_1 + \frac{1}{4}x_2\right)x_1 + \left(\frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3\right)x_2 + \ldots + \left(\frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1}\right)x_i + \ldots + \\ &\quad + \left(\frac{1}{4}x_{\xi-2} + \frac{1}{2}x_{\xi-1} + \frac{1}{4}x_\xi\right)x_{\xi-1} + \left(\frac{1}{4}x_{\xi-1} + \frac{1}{2}x_\xi\right)x_\xi \\ &= \frac{1}{2}x_1^2 + b + \frac{1}{4}x_\xi^2 \end{aligned}$$

with $b = \frac{1}{4}\left(\left(x_1 + x_2\right)^2 + \left(x_2 + x_3\right)^2 + \ldots + \left(x_{i-1} + x_i\right)^2 + \ldots + \left(x_{\xi-1} + x_\xi\right)^2\right)$

So clearly $X^t MX > 0$ if the x_i are not all equal to 0. This proves Lemma 6.

From Lemma 6 follows that all eigenvalues of M are real and > 0 and that M can be diagonalized in an orthogonal basis made of eigenvectors.

Lemma 7. - *All eigenvalues of* M *are* <1.

Proof of Lemma 7

Let us write the system of equations defining the eigenvalues and eigenvectors, $MX = \lambda X$.

$$\left\{\begin{array}{r} \frac{3}{4}x_1 + \frac{1}{4}x_2 = \lambda x_1 \\ \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3 = \lambda x_2 \\ \vdots \\ \frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1} = \lambda x_i \\ \vdots \\ \frac{1}{4}x_{\xi-2} + \frac{1}{2}x_{\xi-1} + \frac{1}{4}x_\xi = \lambda x_{\xi-1} \\ \frac{1}{4}x_{\xi-1} + \frac{1}{2}x_\xi = \lambda x_\xi \end{array}\right. \qquad (1)$$

It may be written:

$$\left\{\begin{array}{l} x_2 = (4\lambda - 3)x_1 \\ x_3 = (4\lambda - 2)x_2 - x_1 \\ \vdots \\ x_{i+1} = (4\lambda - 2)x_i - x_{i-1} \\ \vdots \\ x_\xi = (4\lambda - 2)x_{\xi-1} - x_{\xi-2} \\ x_{\xi-1} = (4\lambda - 2)x_\xi \end{array}\right. \qquad (2)$$

We know that $x_1 \neq 0$ (if $x_1 = 0$, all $x_i = 0$), so we may assume $x_1 = 1$. Assume $\lambda \geq 1$. From the first equation $x_2 = (4\lambda - 3)x_1$ we deduce $x_2 > x_1 > 0$.

More generally, the equation $x_{i+1} = (4\lambda - 2)x_i - x_{i-1}$ gives:

$$x_{i-1} - x_i = (4\lambda - 3)x_i - x_{i+1} \geq x_i - x_{i+1}$$

that is $x_i - x_{i-1} \le x_{i+1} - x_i$. So the sequence of consecutive differences is increasing. Since $x_2 > x_1$, all differences are positive, the x_i are increasing and are >0. Set $S = \sum_{i=1}^{\xi} x_i$; summing all equations, we get $S - \frac{1}{4} x_\xi = \lambda S$, that is $-\frac{1}{4} x_\xi = (\lambda - 1) S$. But this is a contradiction: $\lambda > 1$, $S > 0$ and $x_\xi > 0$. Lemma 7 is proved.

The results we obtain here for the matrix M do not hold for the previous matrix M_F ; this is why we had to make this change of variables.

4. Precise results on the eigenvalues and eigenvectors

Proposition 8. - *For* $j = 1, \ldots, \xi$, *let* $\vartheta_j = \frac{2j-1}{2\xi+1} \pi$. *The* j^{th} *eigenvalue* λ_j *is:*

$$\lambda_j = \frac{1 + \cos(\vartheta_j)}{2} = \cos^2 \frac{\vartheta_j}{2}$$

and the j^{th} *eigenvector has components :*

$$V_j = \left(\sin(\xi \vartheta_j), \sin((\xi - 1)\vartheta_j), \ldots, \sin(\vartheta_j) \right)$$

Proof of Proposition 8

We have $x_\xi \neq 0$ (otherwise all $x_j' s$ are 0), so we may assume $x_\xi = 1$.

We set $\mu = 2\lambda - 1$, then $\mu < 1$. System (2) above becomes :

$$\begin{cases} x_2 = (2\mu - 1) x_1 \\ x_3 = 2\mu x_2 - x_1 \\ \vdots \\ x_{i+1} = 2\mu x_i - x_{i-1} \\ \vdots \\ x_\xi = 2\mu x_{\xi-1} - x_{\xi-2} \\ x_{\xi-1} = 2\mu x_\xi \end{cases} \qquad (1)$$

We set $y_j = x_{\xi - j}$ for $j = 0, \ldots, \xi - 1$. System (1) becomes:

$$\begin{cases} y_0 = 1 \\ y_1 = 2\mu \\ y_2 = 2\mu y_1 - y_0 \\ \quad\vdots \\ y_j = 2\mu y_{j-1} - y_{j-2} \\ \quad\vdots \\ y_{\xi-1} = 2\mu y_{\xi-2} - y_{\xi-3} \\ y_{\xi-2} = (2\mu - 1)\, y_{\xi-1} \end{cases} \tag{2}$$

Therefore, $y_j = U_j(\mu)$ where U_j is the j^{th} Chebychev's polynomial of second kind, for $j = 0, \ldots, \xi - 1$. The final equation in (2) may be written:

$$U_{\xi-2}(\mu) = (2\mu - 1)U_{\xi-1}(\mu) \tag{3}$$

that is, with $\mu = \cos(\vartheta)$:

$$\frac{\sin((\xi-1)\vartheta)}{\sin(\vartheta)} = (2\cos(\vartheta) - 1)\frac{\sin(\xi\vartheta)}{\sin(\vartheta)}.$$

By Lemma 7, $\sin(\vartheta) \neq 0$, so the above equation is equivalent to:

$$\sin((\xi-1)\vartheta) = (2\cos(\vartheta) - 1)\sin(\xi\vartheta) \tag{4}$$

We have :

$$\sin((\xi-1)\vartheta) - (2\cos(\vartheta) - 1)\sin(\xi\vartheta) = -\sin(\xi\vartheta)\cos(\vartheta) - \cos(\xi\vartheta)\sin(\vartheta) + \sin(\xi\vartheta)$$

Therefore, equation (4) is equivalent to:

$$\sin(\xi\vartheta)(1 - \cos(\vartheta)) = \cos(\xi\vartheta)\sin(\vartheta)$$

or :

$$\tan(\xi\vartheta) = \frac{\sin(\vartheta)}{1 - \cos(\vartheta)} \tag{5}$$

which may be written:

$$\tan(\xi\vartheta) = \frac{1}{\tan\frac{\vartheta}{2}} \tag{6}$$

Therefore:

$$\cos(\xi\vartheta)\cos\frac{\vartheta}{2} - \sin(\xi\vartheta)\sin\frac{\vartheta}{2} = 0$$

which gives:

$$\cos\left(\xi\vartheta + \frac{\vartheta}{2}\right) = 0$$

and this equation has the solutions $\frac{2\xi+1}{2}\vartheta = \frac{\pi}{2} + (j-1)\pi$, $j = 1,...,\xi,$

that is:

$$\vartheta = \frac{(2j-1)\pi}{2\xi+1} \tag{7}$$

as we announced.

Since $U_j(\cos\vartheta) = \frac{\sin((j+1)\vartheta)}{\sin\vartheta}$, after multiplication, we may take, for $j = 1,...,\xi$:

$$V_j = \left(\sin(\xi\vartheta_j), \sin((\xi-1)\vartheta_j), ..., \sin(\vartheta_j)\right) \tag{8}$$

and $\lambda_j = \frac{1+\cos(\vartheta_j)}{2} = \cos^2\frac{\vartheta_j}{2}$. This finishes the proof of Proposition 8.

The first eigenvector, V_1, has all its components real and >0, but all other eigenvectors have some negative component.

Remark. - It follows from the general theory of symmetric matrices, positive defined, that any two eigenvectors V_{j_1}, V_{j_2} are mutually orthogonal, that is:

$$\sum_{l=1}^{\xi} \sin(l\vartheta_{j_1})\sin(l\vartheta_{j_2}) = 0$$

where $\vartheta_{j_1} = \frac{(2j_1-1)\pi}{2\xi+1}, \vartheta_{j_2} = \frac{(2j_2-1)\pi}{2\xi+1}$. This can be checked directly.

We now compute the l_1- norm and the l_2- norm of the eigenvectors.

5. Norms of the eigenvectors

Proposition 9. – *All eigenvectors have the same* $l_1 -$*norm:*

$$\left|V_j\right|_1 = \frac{1}{2}\tan\left(\xi\vartheta_1\right) = \frac{1}{2\tan\frac{\vartheta_1}{2}}$$

All eigenvectors have the same quadratic norm:

$$\left|V_j\right|_2^2 = \frac{\xi}{2} + \frac{1}{4}$$

for $j = 1,\ldots,\xi.$

Proof of Proposition 9

In order to compute the l_1 and l_2 norms of the eigenvectors, let us first observe that all of them are, up to changes of signs, reorderings of the terms of V_1. Indeed, when j changes, the ξ numbers $\sin\left(k\vartheta_j\right)$ ($k = 1,\ldots,\xi$) are reorderings of the ξ numbers $\sin\left(k\vartheta_1\right)$, except for the sign, which may become minus (this does not affect the norms).

So, let us compute $\left|V_1\right|_1$. Apply the matrix M to the eigenvector V_1: by definition, we get $MV_1 = \lambda_1 V_1$ and the loss of energy is $\left(1-\lambda_1\right)\left|V_1\right|_1$. But this loss of energy is also $\frac{1}{4}\sin\left(\vartheta_1\right)$, since the last coordinate of the eigenvector is $\sin\left(\vartheta_1\right)$. So we get:

$$\left|V_1\right|_1 = \frac{1}{2}\frac{\sin\left(\vartheta_1\right)}{1-\cos\left(\vartheta_1\right)} = \frac{1}{2}\tan\left(\xi\vartheta_1\right)$$

$$\left(1-\lambda_1\right)\left|V_1\right|_1 = \frac{1}{4}\sin\left(\vartheta_1\right)$$

But $\lambda_1 = \dfrac{1+\cos\left(\vartheta_1\right)}{2}$, which gives:

$$\left|V_1\right|_1 = \frac{1}{2}\tan\left(\xi\vartheta_1\right) = \frac{1}{2}\frac{\sin\left(\vartheta_1\right)}{1-\cos\left(\vartheta_1\right)} = \frac{1}{2\tan\frac{\vartheta_1}{2}}$$

and this result is valid for all eigenvectors.

Let us now compute $\left|V_1\right|_2^2$. We use the identity:

$$\sum_{k=1}^{\xi}\sin^2(kt)=\frac{\xi}{2}+\frac{1}{4}-\frac{1}{4}\frac{\sin\left((2\xi+1)t\right)}{\sin(t)}$$

which gives, with $\vartheta_1=\dfrac{\pi}{2\xi+1}$:

$$\left|V_1\right|_2^2=\sum_{k=1}^{\xi}\sin^2\left(\frac{k\pi}{2\xi+1}\right)=\frac{\xi}{2}+\frac{1}{4}$$

which proves our claim and finishes the proof of Proposition 9.

6. Decomposition on the basis of eigenvectors

Proposition 10. - *At each step, the energy can be written:*

$$x(n,k)=\frac{2}{2\xi+1}\sum_{j=1}^{\xi}\cos\left(\frac{(2k-1)\vartheta_j}{2}\right)\cos^{2n+1}\frac{\vartheta_j}{2}$$

with $\vartheta_j=\dfrac{2j-1}{2\xi+1}\pi$, $j=1,\dots,\xi$.

Proof of Proposition 10

If we want to compute the energy at the n^{th} step, we start with the initial value $f(0,0)=1$, $f(0,k)=0$, $k=1,\dots,\xi$. So the initial vector is $X_0=\left(\dfrac{1}{2},0,\dots,0\right)$. We decompose this vector on the basis of eigenvectors. We write :

$$X_0=\sum_{j=1}^{\xi}\alpha_j V_j$$

Since the eigenvectors are orthogonal, the coefficients α_j may be computed simply:

$$\alpha_j=\frac{<X_0,V_j>}{\left|V_j\right|_2^2}=\frac{\sin\left(\xi\vartheta_j\right)}{\xi+\dfrac{1}{2}}$$

using Proposition 9. Then, at the n^{th} step (time $2n$), the vector X_n is :

$$X_n = M^n X_0 = M^n \sum_{j=1}^{\xi} \alpha_j V_j = \sum_{j=1}^{\xi} \alpha_j M^n V_j = \sum_{j=1}^{\xi} \alpha_j \lambda_j^n V_j$$

Using the identity $\dfrac{1+\cos(\vartheta_j)}{2} = \cos^2\left(\dfrac{\vartheta_j}{2}\right)$, we obtain the formula:

$$x(n,k) = \frac{2}{2\xi+1} \sum_{j=1}^{\xi} \sin(\xi\vartheta_j) \sin((\xi-k+1)\vartheta_j) \cos^{2n}\frac{\vartheta_j}{2} \tag{1}$$

We observe that:

$$\xi\vartheta_j = (2j-1)\frac{\pi}{2} - \frac{2j-1}{2\xi+1}\frac{\pi}{2} = j\pi - \frac{\pi}{2} - \frac{2j-1}{2\xi+1}\frac{\pi}{2} \tag{2}$$

and therefore:

$$\sin(\xi\vartheta_j) = \sin\left(j\pi - \frac{\pi}{2} - \frac{2j-1}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1}\cos\left(\frac{2j-1}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1}\cos\frac{\vartheta_j}{2} \tag{3}$$

$$\cos(\xi\vartheta_j) = \cos\left(j\pi - \frac{\pi}{2} - \frac{2j-1}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1}\sin\left(\frac{2j-1}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1}\sin\frac{\vartheta_j}{2} \tag{4}$$

More generally:

$$\begin{aligned}\sin((\xi-k+1)\vartheta_j) &= \sin\left(j\pi - \frac{\pi}{2} - \frac{(2k-1)(2j-1)}{2\xi+1}\frac{\pi}{2}\right)\\ &= (-1)^{j-1}\cos\left(\frac{(2k-1)(2j-1)}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1}\cos\frac{(2k-1)\vartheta_j}{2}\end{aligned} \tag{5}$$

$$\begin{aligned}\cos((\xi-k+1)\vartheta_j) &= \cos\left(j\pi - \frac{\pi}{2} - \frac{(2k-1)(2j-1)}{2\xi+1}\frac{\pi}{2}\right)\\ &= (-1)^{j-1}\sin\left(\frac{(2k-1)(2j-1)}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1}\sin\frac{(2k-1)\vartheta_j}{2}\end{aligned} \tag{6}$$

In equation (1), we substitute $\sin(\xi\vartheta_j)$ from (3) and $\sin((\xi-k+1)\vartheta_j)$ from (5). This gives the formula in Proposition 10.

We observe that $x(n,k)$ may be written as a scalar product:

$$x(n,k)=\frac{1}{\xi+\frac{1}{2}}\langle A_k, B_n\rangle$$

with $A_k=\begin{pmatrix}\cos\left((2k-1)\frac{\vartheta_1}{2}\right)\\ \vdots \\ \cos\left((2k-1)\frac{\vartheta_\xi}{2}\right)\end{pmatrix}$, $B_n=\begin{pmatrix}\cos^{2n+1}\frac{\vartheta_1}{2}\\ \vdots \\ \cos^{2n+1}\frac{\vartheta_\xi}{2}\end{pmatrix}$

In the case $\xi=3$, we find numerically:

$$x(n,1)=0.2716\times 0.9505^n+0.1746\times 0.6113^n+0.0538\times 0.1883^n$$
$$x(n,2)=0.2178\times 0.9505^n-0.0969\times 0.6113^n-0.1209\times 0.1883^n$$
$$x(n,3)=0.1209\times 0.9505^n-0.2178\times 0.6113^n+0.0969\times 0.1883^n$$

We can now state the main theorem of this section:

Theorem 11. - *At each step* $2n$, *the energy* $e(2n,2k)=f(n,k)$, $k=0,\dots,\xi$, *is given by:*

$$\begin{cases} f(n,0)=x(n-1,1) \\ f(n,k)=\frac{1}{2}\left(x(n-1,k)+x(n-1,k+1)\right),\ k=1,\dots,\xi-1 \\ f(n,\xi)=\frac{1}{2}x(n-1,\xi) \end{cases}$$

where, for $k=1,\dots,\xi$*:*

$$x(n,k)=\frac{2}{2\xi+1}\sum_{j=1}^{\xi}\cos\left(\frac{(2k-1)\vartheta_j}{2}\right)\cos^{2n+1}\frac{\vartheta_j}{2}$$

and $\vartheta_j=\frac{(2j-1)\pi}{2\xi+1}$, $j=1,\dots,\xi.$

Proof of Theorem 11

It follows immediately from Proposition 10 and the equations relating $f(n,k)$ with $x(n-1,k)$, $x(n-1,k+1)$, namely:

$$\begin{cases} f(n,0) = x(n-1,1) \\ f(n,k) = \frac{1}{2}\left(x(n-1,k) + x(n-1,k+1)\right),\ k = 1,\ldots,\xi-1 \\ f(n,\xi) = \frac{1}{2}x(n-1,\xi) \end{cases}$$

This concludes the proof of Theorem 11.

We now give an expression of $x(n,k)$ which is of combinatorial nature. In what follows, $\binom{n}{k}$ is zero if $k > n$ or if $k < 0$: this will simplify the notation.

Proposition 12. - *At each step, we have :*

$$x(n,k) = \frac{1}{2^{2n+1}}\binom{2n+1}{n-k+1} + \frac{1}{2^{2n+1}}\sum_{u\geq 1}(-1)^u\binom{2n+1}{k+n-u(2\xi+1)}$$

Proof of Proposition 12

We use the following identity:

$$\cos^{2n+1}(x) = \frac{1}{2^{2n}}\sum_{l=0}^{n}\binom{2n+1}{l}\cos\left((2n+1-2l)x\right)$$

which gives :

$$x(n,k) = \frac{2}{2\xi+1}\frac{1}{2^{2n}}\sum_{l=0}^{n}\binom{2n+1}{l}\sum_{j=1}^{\xi}\cos\left(\frac{(2k-1)\vartheta_j}{2}\right)\cos\left((2n+1-2l)\frac{\vartheta_j}{2}\right)$$

that is:

$$x(n,k) = \frac{1}{2\xi+1}\frac{1}{2^{2n}}\sum_{l=0}^{n}\binom{2n+1}{l}\sum_{j=1}^{\xi}\left(\cos\left((k-n-1+l)\vartheta_j\right) + \cos\left((k+n-l)\vartheta_j\right)\right)$$

Let :

$$T_1 = \frac{1}{2\xi+1}\frac{1}{2^{2n}}\sum_{l=0}^{n}\binom{2n+1}{l}\sum_{j=1}^{\xi}\cos\left((k-n-1+l)\vartheta_j\right)$$

For all $k,$ we have the following identities:

$$\sum_{j=1}^{\xi} \cos\left(k\vartheta_j\right) = \frac{1}{2}\left(-1\right)^{k-1} \text{ if } k \text{ is not a multiple of } 2\xi+1$$

$$\sum_{j=1}^{\xi} \cos\left(k\vartheta_j\right) = \xi \text{ if } k \text{ is an even multiple of } 2\xi+1 \text{ (including 0)}$$

$$\sum_{j=1}^{\xi} \cos\left(k\vartheta_j\right) = -\xi \text{ if } k \text{ is an odd multiple of } 2\xi+1.$$

So, we must study the term $k-n-1+l$, $1 \le k \le \xi$, $\quad 0 \le l \le n$.

We have $k-n-1+l=0$ if and only if $l=n-k+1$; we cannot have $k-n-1+l=2\xi+1$, since this is equivalent to $l=n-k+2\xi+2$, but $2\xi-k>0$ and $l \le n$.

Therefore:

$$\begin{aligned}
\left(2\xi+1\right)2^{2n}T_1 &= \sum_{l=0}^{n} \binom{2n+1}{l} \sum_{j=1}^{\xi} \cos\left(\left(k-n-1+l\right)\vartheta_j\right) \\
&= \sum_{\substack{l=0 \\ l\neq n-k+1}}^{n} \binom{2n+1}{l} \sum_{j=1}^{\xi} \cos\left(\left(k-n-1+l\right)\vartheta_j\right) + \binom{2n+1}{n-k+1} \sum_{j=1}^{\xi} \cos\left(\left(k-n-1+n-k+1\right)\vartheta_j\right) \\
&= \frac{1}{2}\left(-1\right)^{k-n} \sum_{\substack{l=0 \\ l\neq n-k+1}}^{n} \binom{2n+1}{l}\left(-1\right)^l + \binom{2n+1}{n-k+1}\xi
\end{aligned}$$

But:

$$\sum_{\substack{l=0 \\ l\neq n-k+1}}^{n} \binom{2n+1}{l}\left(-1\right)^l = \sum_{l=0}^{n} \binom{2n+1}{l}\left(-1\right)^l - \binom{2n+1}{n-k+1}\left(-1\right)^{n-k+1}$$

We have the identity:

$$\sum_{l=0}^{n} \binom{2n+1}{l}\left(-1\right)^l = \left(-1\right)^n \binom{2n}{n}$$

and thus:

$$\sum_{\substack{l=0 \\ l\neq n-k+1}}^{n} \binom{2n+1}{l}\left(-1\right)^l = \left(-1\right)^n \binom{2n}{n} + \left(-1\right)^{n-k} \binom{2n+1}{n-k+1}$$

Therefore:

$$T_1 = \frac{1}{2\xi+1}\frac{1}{2^{2n+1}}\left(\left(-1\right)^k \binom{2n}{n} + \left(2\xi+1\right)\binom{2n+1}{n-k+1}\right)$$

which gives:

$$T_1 = \frac{1}{2\xi+1}\frac{1}{2^{2n+1}}(-1)^k\binom{2n}{n} + \frac{1}{2^{2n+1}}\binom{2n+1}{n-k+1}$$

The same way, let :

$$T_2 = \frac{1}{2\xi+1}\frac{1}{2^{2n}}\sum_{l=0}^{n}\binom{2n+1}{l}\sum_{j=1}^{\xi}\left(\cos(k+n-l)\vartheta_j\right)$$

Since $l \le n$ and $k \ge 1$, the coefficient $k+n-l$ cannot vanish. But we have $k+n-l = 2\xi+1$ for $l = k+n-(2\xi+1)$ and more generally $k+n-l = u(2\xi+1)$ for $l = k+n-u(2\xi+1)$. We need $l \ge 0$, $k+n \ge u(2\xi+1)$, $u \le \dfrac{k+n}{2\xi+1}$, that is $u = 1,\dots,a = \mathrm{int}\left(\dfrac{k+n}{2\xi+1}\right)$.

So:

$$\sum_{j=1}^{\xi}\left(\cos(k+n-l)\vartheta_j\right) = \frac{1}{2}(-1)^{k+n-l-1} \text{ if } l \ne k+n-u(2\xi+1)$$
$$\sum_{j=1}^{\xi}\left(\cos(k+n-l)\vartheta_j\right) = -\xi \text{ if } l = k+n-u(2\xi+1),\ u \text{ odd}$$
$$\sum_{j=1}^{\xi}\left(\cos(k+n-l)\vartheta_j\right) = \xi \text{ if } l = k+n-u(2\xi+1),\ u \text{ even}$$

This gives:

$$(2\xi+1)2^{2n}T_2 = \frac{1}{2}(-1)^{k+n-1}\sum_{\substack{l=0\\ l\ne k+n-u(2\xi+1)}}^{n}(-1)^l\binom{2n+1}{l} + \xi\sum_{u=1}^{a}(-1)^u\binom{2n+1}{k+n-u(2\xi+1)}$$

$$\begin{aligned}(2\xi+1)2^{2n}T_2 = {} & \frac{1}{2}(-1)^{k+n-1}\sum_{l=0}^{n}(-1)^l\binom{2n+1}{l} - \frac{1}{2}(-1)^{k+n-1}\sum_{u=1}^{a}(-1)^{k+n-u(2\xi+1)}\binom{2n+1}{k+n-u(2\xi+1)}\\ & + \xi\sum_{u=1}^{a}(-1)^u\binom{2n+1}{k+n-u(2\xi+1)}\end{aligned}$$

$$(2\xi+1)2^{2n}T_2 = \frac{1}{2}(-1)^{k+n-1}\sum_{l=0}^{n}(-1)^l\binom{2n+1}{l} + \frac{1}{2}\sum_{u=1}^{a}(-1)^u\binom{2n+1}{k+n-u(2\xi+1)} + \xi\sum_{u=1}^{a}(-1)^u\binom{2n+1}{k+n-u(2\xi+1)}$$

$$(2\xi+1)2^{2n+1}T_2 = (-1)^{k-1}\binom{2n}{n} + \sum_{u=1}^{a}(-1)^u\binom{2n+1}{k+n-u(2\xi+1)} + 2\xi\sum_{u=1}^{a}(-1)^u\binom{2n+1}{k+n-u(2\xi+1)}$$

and finally:

$$T_2 = (-1)^{k-1} \frac{1}{(2\xi+1)2^{2n+1}} \binom{2n}{n} + \frac{1}{2^{2n+1}} \sum_{u=1}^{a} (-1)^u \binom{2n+1}{k+n-u(2\xi+1)}$$

So we obtain:

$$x(n,k) = T_1 + T_2 = \frac{1}{2^{2n+1}} \binom{2n+1}{n-k+1} + \frac{1}{2^{2n+1}} \sum_{u=1}^{a} (-1)^u \binom{2n+1}{k+n-u(2\xi+1)}$$

which proves Proposition 12.

The trigonometric expression in Proposition 10 may look less expressive than its combinatorial counterpart, but it is simpler to establish. In our view, the trigonometric presentation is more natural for this subject.

We deduce the probability that, at step $2n$, both players have equal fortune ; this is the value of $e(2n,0)$.

Corollary 13. - *For each* n, *we have:*

$$e(2n,0) = \frac{2}{2\xi+1} \sum_{j=1}^{\xi} \cos^{2n} \frac{\vartheta_j}{2} = \frac{1}{2^{2n-1}} \binom{2n-1}{n} + \frac{1}{2^{2n-1}} \sum_{u\geq 1} (-1)^n \binom{2n+1}{n+1-u(2\xi+1)}$$

Proof of Corollary 13

We saw that $e(2n,0) = f(n,0) = x(n-1,1)$; the result follows from Theorem 11 and Proposition 12.

7. The energy at each step

Using the previous representation, we now give the value of the energy on each vertical.

Proposition 14. – *For each* n, *the quadratic energy satisfies:*

$$|X_n|_2^2 = \frac{1}{2\xi+1} \sum_{j=1}^{\xi} \cos^{4n+2} \left(\frac{2j-1}{2\xi+1} \frac{\pi}{2} \right)$$

Another expression is:

$$|X_n|_2^2 = \frac{1}{2^{4n+3}} \binom{4n+2}{2n+1} + \frac{1}{2^{4n+2}} \sum_{u=u_1}^{u_2} (-1)^u \binom{4n+2}{2n+1-u(2\xi+1)}$$

$$with\ u_1 = 1 + \mathrm{int}\left(\frac{1}{2\xi+1}\right),\ u_2 = \mathrm{int}\left(\frac{2n+1}{2\xi+1}\right).$$

Proof of Proposition 14

We have:

$$X_n = M^n X_0 = \sum_{j=1}^{\xi} \alpha_j M^n V_j = \sum_{j=1}^{\xi} \alpha_j \lambda_j^n V_j$$

and since the V_j's are orthogonal:

$$|X_n|_2^2 = \sum_{j=1}^{\xi} \alpha_j^2 \lambda_j^{2n} |V_j|_2^2$$

that is, using Propositions 8 and 9:

$$|X_n|_2^2 = \frac{1}{2\xi+1} \sum_{j=1}^{\xi} \sin^2\left(\xi\vartheta_j\right) \cos^{4n} \frac{\vartheta_j}{2}.$$

Using formula (3) above (proof of Proposition 10), we obtain:

$$|X_n|_2^2 = \frac{1}{2\xi+1} \sum_{j=1}^{\xi} \cos^{4n+2} \frac{\vartheta_j}{2}$$

as we announced. This proves the first part of Proposition 14.

In order to prove the second part, we write:

$$\cos^{4n+2}\left(\frac{\vartheta}{2}\right) = \frac{1}{2^{4n+2}} \binom{4n+2}{2n+1} + \frac{1}{2^{4n+1}} \sum_{l=0}^{2n} \binom{4n+2}{l} \cos\left(\left(2n+1-l\right)\vartheta\right)$$

$$|X_n|_2^2 = \frac{1}{2^{4n+2}} \frac{\xi}{2\xi+1} \binom{4n+2}{2n+1} + \frac{1}{2^{4n+1}} \frac{1}{2\xi+1} \sum_{l=0}^{2n} \binom{4n+2}{l} \sum_{j=1}^{\xi} \cos\left(\left(2n+1-l\right)\vartheta_j\right)$$

We will make use of the following formulas :

$$\sum_{j=1}^{\xi} \cos\left(k\vartheta_j\right) = \frac{1}{2}\left(-1\right)^{k-1} \text{ if } k \text{ is not a multiple of } 2\xi+1 \ ;$$
$$\sum_{j=1}^{\xi} \cos\left(k\vartheta_j\right) = \left(-1\right)^{u} \xi \text{ if } k = u\left(2\xi+1\right) \text{ is a multiple of } 2\xi+1 \text{ (including 0),}$$

that is:

$$\sum_{j=1}^{\xi} \cos\left(\left(2n+1-l\right)\vartheta_j\right)=\frac{1}{2}\left(-1\right)^l \text{ if } 2n+1-l \text{ is not a multiple of } 2\xi+1;$$

$$\sum_{j=1}^{\xi} \cos\left(\left(2n+1-l\right)\vartheta_j\right)=\left(-1\right)^u \xi \text{ if } 2n+1-l=u\left(2\xi+1\right).$$

The condition $2n+1-l=u\left(2\xi+1\right)$ is equivalent to $l=2n+1-u\left(2\xi+1\right)$. Since $0\le 2n+1-u\left(2\xi+1\right)\le 2n$, we have $u_1=1+\mathrm{int}\left(\frac{1}{2\xi+1}\right)\le u\le u_2=\mathrm{int}\left(\frac{2n+1}{2\xi+1}\right)$.

We have:

$$\sum_{l=0}^{2n}\binom{4n+2}{l}\sum_{j=1}^{\xi}\cos\left(\left(2n+1-l\right)\vartheta_j\right)=\frac{1}{2}\sum_{\substack{l=0\\ l\neq 2n+1-u(2\xi+1)}}^{2n}\left(-1\right)^l\binom{4n+2}{l}+\xi\sum_{u=u_1}^{u_2}\left(-1\right)^u\binom{4n+2}{2n+1-u\left(2\xi+1\right)}$$

and:

$$\sum_{\substack{l=0\\ l\neq 2n+1-u(2\xi+1)}}^{2n}\left(-1\right)^l\binom{4n+2}{l}=\sum_{l=0}^{2n}\left(-1\right)^l\binom{4n+2}{l}+\sum_{u=u_1}^{u_2}\left(-1\right)^u\binom{4n+2}{2n+1-u\left(2\xi+1\right)}$$

But :

$$\sum_{l=0}^{2n}\left(-1\right)^l\binom{4n+2}{l}=\frac{1}{2}\binom{4n+2}{2n+1}$$

which gives:

$$\sum_{\substack{l=0\\ l\neq 2n+1-u(2\xi+1)}}^{2n}\left(-1\right)^l\binom{4n+2}{l}=\frac{1}{2}\binom{4n+2}{2n+1}+\sum_{u=u_1}^{u_2}\left(-1\right)^u\binom{4n+2}{2n+1-u\left(2\xi+1\right)}$$

$$\sum_{l=0}^{2n}\binom{4n+2}{l}\sum_{j=1}^{\xi}\cos\left(\left(2n+1-l\right)\vartheta_j\right)=\frac{1}{4}\binom{4n+2}{2n+1}+\left(\frac{1}{2}+\xi\right)\sum_{u=u_1}^{u_2}\left(-1\right)^u\binom{4n+2}{2n+1-u\left(2\xi+1\right)}$$

$$\left|X_n\right|_2^2=\frac{1}{2^{4n+2}}\frac{\xi}{2\xi+1}\binom{4n+2}{2n+1}+\frac{1}{2^{4n+3}}\frac{1}{2\xi+1}\binom{4n+2}{2n+1}+\frac{1}{2^{4n+2}}\sum_{u=u_1}^{u_2}\left(-1\right)^u\binom{4n+2}{2n+1-u\left(2\xi+1\right)}$$

and finally:

$$\left|X_n\right|_2^2=\frac{1}{2^{4n+3}}\binom{4n+2}{2n+1}+\frac{1}{2^{4n+2}}\sum_{u=u_1}^{u_2}\left(-1\right)^u\binom{4n+2}{2n+1-u\left(2\xi+1\right)}$$

which proves Proposition 14.

We now turn to the l_1- norm (total energy), which requires different arguments.

Proposition 15. – *The total energy on the n^{th} vertical, in coordinates $x(n,k)$, is given by the formula:*

$$|X_n|_1 = \frac{1}{2\xi+1}\sum_{j=1}^{\xi} (-1)^{j-1}\frac{\cos^{2n+2}\frac{\vartheta_j}{2}}{\sin\frac{\vartheta_j}{2}}$$

Proof of Proposition 15

We have $|X_n|_1 = \sum_{k=1}^{\xi} x(n,k) = \frac{1}{\xi+\frac{1}{2}}\sum_{j=1}^{\xi}\sum_{k=1}^{\xi} \cos\left(\frac{(2k-1)\vartheta_j}{2}\right)\cos^{2n+1}\frac{\vartheta_j}{2}$

We use the identity:

$$\sum_{k=1}^{\xi} \cos((2k-1)t) = \frac{1}{2}\frac{\sin(2\xi t)}{\sin t}$$

Therefore:

$$|X_n|_1 = \frac{1}{2\xi+1}\sum_{j=1}^{\xi} \frac{\sin(\xi\vartheta_j)}{\sin\frac{\vartheta_j}{2}}\cos^{2n+1}\frac{\vartheta_j}{2}$$

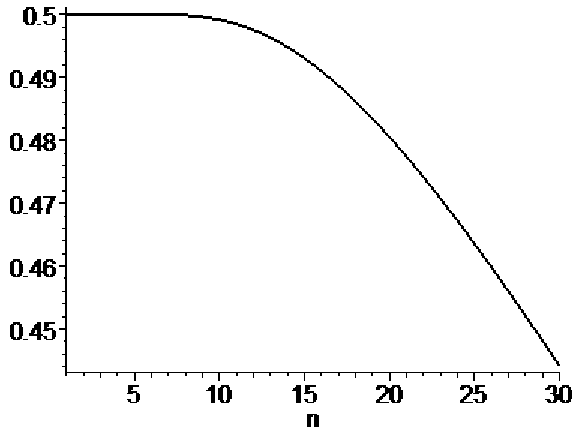
But we saw earlier (Proof of Proposition 10, (3)), that $\sin(\xi\vartheta_j) = (-1)^{j-1}\cos\frac{\vartheta_j}{2}$. So we get:

$$|X_n|_1 = \frac{1}{2\xi+1}\sum_{j=1}^{\xi} (-1)^{j-1}\frac{\cos^{2n+2}\frac{\vartheta_j}{2}}{\sin\frac{\vartheta_j}{2}}$$

which proves Proposition 15. We may also write, using (3) and (4), proof of Proposition 10:

$$|X_n|_1 = \frac{1}{2\xi+1}\sum_{j=1}^{\xi} \frac{\sin^{2n+2}(\xi\vartheta_j)}{\cos(\xi\vartheta_j)}$$

which is the simplest form.



Graph of the energy, for $\xi = 7$ *and* $n = 1, \dots, 30$

Recall that a sequence a_k is said to be concave if, for all k, $a_k \geq \frac{1}{2}\left(a_{k-1} + a_{k+1}\right)$.

Proposition 16. – *At each time, the energy profile is concave.*

Proof of Proposition 16

Let us compute, for fixed n:

$$d_k = x(n,k) - x(n,k+1) = \frac{1}{\xi + \frac{1}{2}} \sum_{j=1}^{\xi} \left(\cos\left(\frac{(2k-1)\vartheta_j}{2}\right) - \cos\left(\frac{(2k+1)\vartheta_j}{2}\right) \right) \cos^{2n+1} \frac{\vartheta_j}{2}$$

Using the identity $\cos p - \cos q = -2 \sin \frac{p+q}{2} \sin \frac{p-q}{2}$, we obtain:

$$d_k = \frac{2}{\xi + \frac{1}{2}} \sum_{j=1}^{\xi} \sin\left(k\vartheta_j\right) \sin\left(\vartheta_j\right) \cos^{2n+1} \frac{\vartheta_j}{2}$$

and:

$$d_k - d_{k+1} = \frac{2}{\xi + \frac{1}{2}} \sum_{j=1}^{\xi} \left(\sin\left(k\vartheta_j\right) - \sin\left((k+1)\vartheta_j\right) \right) \sin\left(\vartheta_j\right) \cos^{2n+1} \frac{\vartheta_j}{2}$$

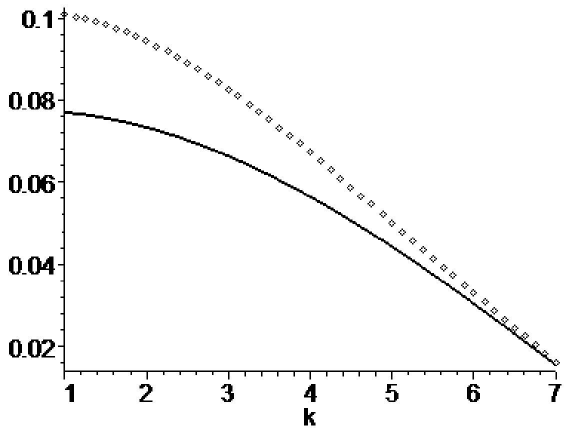
Using the identity $\sin p - \sin q = 2 \sin \frac{p-q}{2} \cos \frac{p+q}{2}$, we obtain:

$$d_k - d_{k+1} = \frac{-4}{\xi + \frac{1}{2}} \sum_{j=1}^{\xi} \sin \frac{\vartheta_j}{2} \cos\left((2k+1) \frac{\vartheta_j}{2} \right) \sin\left(\vartheta_j\right) \cos^{2n+1} \frac{\vartheta_j}{2} < 0 .$$

This shows that $d_k < d_{k+1}$, or $x(n,k) - x(n,k+1) < x(n,k+1) - x(n,k+2)$, that is:

$$x(n,k)+x(n,k+2)<2x(n,k+1)$$

which proves our claim.



The energy profile (in x *), for* $\xi = 7, n = 30$ *(point style) and* $n = 50$ *(line style)*

We observe that the angle $\frac{\vartheta_j}{2}$ satisfies $0<\frac{2j-1}{2\xi+1}\frac{\pi}{2}<\frac{\pi}{2}$ and is increasing as a function of j. Therefore, $\sin\frac{\vartheta_j}{2}$ is positive and increasing, $\cos\frac{\vartheta_j}{2}$ is positive and decreasing, and so is $\cos^{2n+2}\frac{\vartheta_j}{2}$. The series $\sum_{j=1}^{\xi}(-1)^{j-1}\frac{\cos^{2n+2}\frac{\vartheta_j}{2}}{\sin\frac{\vartheta_j}{2}}$ has alternate signs, and its general term is decreasing. So we have the general estimates:

Proposition 17. – *The energy at each step satisfies:*

$$\frac{1}{2\xi+1}\left(\frac{\cos^{2n+2}\left(\frac{1}{2\xi+1}\frac{\pi}{2}\right)}{\sin\left(\frac{1}{2\xi+1}\frac{\pi}{2}\right)}-\frac{\cos^{2n+2}\left(\frac{3}{2\xi+1}\frac{\pi}{2}\right)}{\sin\left(\frac{3}{2\xi+1}\frac{\pi}{2}\right)}\right)<|X_n|_1<\frac{1}{2\xi+1}\frac{\cos^{2n+2}\left(\frac{1}{2\xi+1}\frac{\pi}{2}\right)}{\sin\left(\frac{1}{2\xi+1}\frac{\pi}{2}\right)}$$

If we want to obtain estimates about the energy $e(n,k)$ instead of $x(n,k)$, we come back to system (2.1):

$$\begin{cases} f(n+1,0) = x(n,1) \\ f(n+1,k) = \frac{1}{2}\big(x(n,k) + x(n,k+1)\big), \, k = 1,\dots,\xi-1 \\ f(n+1,\xi) = \frac{1}{2} x(n,\xi) \end{cases}$$

which gives:

$$\sum_{j=1}^{\xi} f(n+1,j) = \frac{3}{2} x(n,1) + \sum_{j=2}^{\xi} x(n,j) \le \frac{3}{2} \sum_{j=1}^{\xi} x(n,j)$$

and, the same way:

$$\begin{aligned} f(n+1,0)^2 + \sum_{j=1}^{\xi-1} f(n+1,j)^2 + f(n,\xi)^2 &= x(n,1)^2 + \frac{1}{4}\sum_{j=1}^{\xi-1}\big(x(n,j) + x(n,j+1)\big)^2 + \frac{1}{4} x(n,\xi)^2 \\ &\le x(n,1)^2 + \frac{1}{2}\sum_{j=1}^{\xi-1} x(n,j)^2 + \frac{1}{2}\sum_{j=1}^{\xi-1} x(n,j+1)^2 + \frac{1}{4} x(n,\xi)^2 \\ &\le \frac{3}{2}\left|X_n\right|_2^2 \end{aligned}$$

Corollary 18. - *Asymptotically when* $n \to +\infty$, *we have the estimate, with* $\vartheta_1 = \frac{\pi}{2\xi+1}$:

$$\left|X_n\right|_1 \sim \frac{1}{2\xi+1} \frac{\cos^{2n+2}\frac{\vartheta_1}{2}}{\sin\frac{\vartheta_1}{2}}$$

For the energy $e(2n,j)$, *we have the estimates:*

$$e(2n+2,0) \sim \frac{2}{2\xi+1}\cos^{2n+2}\frac{\vartheta_1}{2}$$

For $j = 1,\dots,\xi-1$ *:*

$$e(2n+2,2j) = \frac{2}{2\xi+1}\sin\big((\xi-j)\vartheta_1\big)\cos^{2n+2}\frac{\vartheta_1}{2}$$

and for $j = \xi$ *:*

$$e(2n+2,2\xi) \sim \frac{2}{2\xi+1}\sin(\vartheta_1)\cos^{2n+1}\frac{\vartheta_1}{2}$$

and the total energy satisfies:

$$\sum_{j=0}^{\xi} e(2n+2,j) \sim \frac{1}{\xi+\frac{1}{2}} \cos^{2n+1}\frac{\vartheta_1}{2}\left(\cos\frac{\vartheta_1}{2}+\cos\frac{\vartheta_1}{2}\sum_{j=1}^{\xi-1} \sin(j\vartheta_1)+\sin(\vartheta_1)\right)$$

Proof of Corollary 18

The first part follows immediately from Proposition 16. Returning to the energy $f(n,j)=e(2n,2j)$, we find:

$$e(2n+2,0)=x(n,1)\sim \lambda_1^n \frac{\sin^2(\xi\vartheta_1)}{\xi+\frac{1}{2}}=\frac{\sin^2(\xi\vartheta_1)}{\xi+\frac{1}{2}}\cos^{2n}\frac{\vartheta_1}{2}$$

But $\sin(\xi\vartheta_1)=\cos\frac{\vartheta_1}{2}$, which gives the announced formula.

For $j=1,\dots,\xi-1$:

$$e(2n+2,2j)=\frac{1}{2}\big(x(n,j)+x(n,j+1)\big)\sim \frac{\lambda_1^n \sin(\xi\vartheta_1)}{\xi+\frac{1}{2}}\frac{\sin\big((\xi-j+1)\vartheta_1\big)+\sin\big((\xi-j)\vartheta_1\big)}{2}$$

$$=\frac{\lambda_1^n \sin(\xi\vartheta_1)}{\xi+\frac{1}{2}}\sin\big((\xi-j)\vartheta_1\big)\cos\frac{\vartheta_1}{2}=\frac{2}{2\xi+1}\cos^{2n+2}\frac{\vartheta_1}{2}\sin\big((\xi-j)\vartheta_1\big)$$

and finally:

$$e(2n+2,2\xi)=\frac{1}{2}x(n,\xi)\sim \lambda_1^n \frac{\sin(\vartheta_1)\sin(\xi\vartheta_1)}{\xi+\frac{1}{2}}=\frac{2}{2\xi+1}\cos^{2n+1}\frac{\vartheta_1}{2}\sin(\vartheta_1)$$

which proves Corollary 18.

8. Domination principle

Proposition 19. – *Assume that X_0, Y_0 are two initial distributions of energy, on the vertical $x=0$, satisfying $X_0(k)\ge Y_0(k)$ for $k=1,\dots,\xi$. Let X_n, Y_n be the corresponding energies, at time n. Then, for all k, $X_n(k)\ge Y_n(k)$.*

What this proposition says is that, if an energy distribution dominates another one at the initial stage, it will dominate it at any further stage.

Proof of Proposition 19

This is obvious : $X_n - Y_n = M^n\left(X_0 - Y_0\right)$; the vector $X_0 - Y_0$ has all its components which are positive, and the matrix M^n has only positive entries.

Corollary 20. - *The energy at each step satisfies the estimates :*

$$x(n,0) \le \frac{1}{2}\lambda_1^n \quad and \quad x(n,k) \le \frac{\lambda_1^n}{4}\frac{\sin\left((\xi-k)\vartheta_1\right)}{\sin\left(\xi\vartheta_1\right)} \quad for\ k=1,\dots,\xi.$$

with $\lambda_1 = \cos^2\dfrac{\vartheta_1}{2}$.

Proof of Corollary 20

Consider the first eigenvector

$V_1 = \left(\sin\left(\xi\vartheta_1\right), \sin\left((\xi-1)\vartheta_1\right), \dots, \sin\left(\vartheta_1\right)\right)$ with $\vartheta_1 = \dfrac{\pi}{2j-1}$

and let us normalize it differently. We consider:

$$W_1 = \frac{2}{\sin\left(\xi\vartheta_1\right)}V_1 = \left(\frac{1}{2}, \frac{\sin\left((\xi-1)\vartheta_1\right)}{2\sin\left(\xi\vartheta_1\right)}, \dots, \frac{\sin\left(\vartheta_1\right)}{2\sin\left(\xi\vartheta_1\right)}\right)$$

All components are >0, and it dominates the initial vector $X_0 = \left(\dfrac{1}{2}, 0, \dots, 0\right)$. By Proposition 18, the same will hold at each step. But W_1 is an eigenvector, with respect to the eigenvalue λ_1, so we get :

$$x(n,0) \le \frac{1}{2}\lambda_1^n$$

$$x(n,k) \le \frac{\lambda_1^n}{4}\frac{\sin\left((\xi-k)\vartheta_1\right)}{\sin\left(\xi\vartheta_1\right)} \text{ for } k=1,\dots,\xi.$$

as we announced. This proves Corollary 20.

We also deduce from the domination principle the simple estimate:

$$|X_n|_1 \le \frac{\lambda_1^n}{\sin\left(\frac{1}{2\xi+1}\frac{\pi}{2}\right)}$$

Indeed, $|W_1|_1 = \frac{2}{\sin(\xi\vartheta_1)}|V_1|_1 = \frac{1}{\cos(\xi\vartheta_1)} = \frac{1}{\sin\frac{\pi}{2\xi+1}}$. But this estimate is weaker than the one given by Proposition 17.

V. References

[Kalbfleisch] Probability and Statistical Inference, volume 1 : Probability. Springer Texts in Statistics, 1985.

[BB_Op] Bernard Beauzamy : Introduction to Operator Theory and Invariant Subspaces. North Holland, Mathematics Library, 1988.