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Simple Random Walks in the plane:

An energy-based approach

Part II: Identical Initial Fortunes

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Abstract

The basic settings are the same as in Part I, but here each player has an identical initial fortune, and the game stops if one of the players gets ruined.

We consider a ± 1 game, with identical initial fortunes. Using a new, energy-based, approach, we investigate the value of the fortune of each player after n games. We give a complete description of each possible value and its probability; the tools used are Chebychev's polynomial of first and second kind, operator theory and trigonometry. We also investigate the asymptotic behavior when $n \to +\infty$: the profile of each fortune is concave, and the total energy tends to 0 exponentially fast.

This behavior is quite different from the case of unbounded initial fortunes, or bounded fortune for one of the players only.

I. Presentation

The basic settings are defined in Part I. Recall that we consider a simple random walk in the plane: a game, with two players. It is defined by a r.v. X taking the values ± 1 with probability $\frac{1}{2}$. The player A wins if $X=1$ and then he receives 1 Euro from the player B, and conversely if $X=-1$. Both players have an initial fortune denoted by F, same for both. The game stops if any of the players sees his fortune equal to zero. The question is: what is the probability distribution of the earnings after n steps?

II. Notation

We refer to Part I. We set $S_0=0$, and for $N\geq 1$, $S_N=\sum_{n=1}^{N} X_n$. Instead of a random walk with multiple possible paths, we consider that we have the propagation of an energy, with the following rules:

- At time $n=0$, the origin receives an energy equal to 1;
- At time $n=1$, this energy is divided into two: each point $(1,0)$ and $(-1,0)$ receives an energy equals to $\frac{1}{2}$ and so on.

More generally, the energy of a point of coordinates (n,k) in the plane is equal to the probability that the random walk hits this point. It will be denoted by $e(n,k)$.

As we already did in Part I, we restrict ourselves to even values of the time $(2n)$. We defined in Part I:

$$f(n,k)=e(2n,2k)$$

with the general propagation rule, for $k\geq 1$:

$$f(n+1,k)=\frac{1}{4}f(n,k-1)+\frac{1}{2}f(n,k)+\frac{1}{4}f(n,k+1).$$

Due to the barrier, this propagation rule will of course be modified: We insert the symmetric barriers $y=\pm(2\xi+1)$; the reason why we work here with odd values will be apparent later. We consider that if a path hits any of the barriers, its energy is absorbed and disappears. So, our original question may be stated as: Given a time n, what is the distribution of energy on the vertical $x=n$?

We will restrict ourselves to the upper half-plane, since, by symmetry, the results are identical in the lower half plane. We first consider the simple case $\xi = 1$.

III. Case $\xi = 1$

The barrier is at the value ± 3. At the time $n = 2$, the energy distribution is: $e(2,2) = \frac{1}{4}$, $e(2,0) = \frac{1}{2}$, $e(2,-2) = \frac{1}{4}$ (the barrier plays no role). So, we get $f(1,1) = \frac{1}{4}$, $f(1,0) = \frac{1}{2}$ and we have the recurrence relations:

$$f(n+1,0) = \frac{1}{2}\left(f(n,0) + f(n,1)\right),\ f(n+1,1) = \frac{1}{4}\left(f(n,0) + f(n,1)\right).$$

As we already did in Part I, we set $x_n = \frac{1}{2}\left(f(n,0) + f(n,1)\right)$. Then:

$$x_1 = \frac{1}{2}\left(f(1,0) + f(1,1)\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4}\right) = \frac{3}{8}.$$

We obtain the equations:

$f(n+1,0) = x_n$, $f(n+1,1) = \frac{1}{2}x_n$, $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{2}x_n\right) = \frac{3}{4}x_n$, which gives $x_n = \frac{1}{2}\left(\frac{3}{4}\right)^n$.

So, the energy profile at time $2n$ is:

$$e(2n,0) = f(n,0) = x_{n-1} = \frac{1}{2}\left(\frac{3}{4}\right)^{n-1}$$

$$e(2n,1) = e(2n,-1) = f(n,1) = \frac{1}{2}x_{n-1} = \frac{1}{4}\left(\frac{3}{4}\right)^{n-1}$$

The total energy at the instant $2n$ is the sum of all terms, that is $E(2n) = \left(\frac{3}{4}\right)^{n-1}$; it is exponentially decreasing.

IV. General case, $\xi > 1$

A. Notation

Let W_{2n} be the vertical for $x = 2n,$ that is the set of all points $A_{2n,2k}$, $k = 0,...,n.$

B. Basic equations

We have the initial values:

$f(0,0) = 1$, $f(0,k) = 0$ for $k = 1,...,\xi.$

The recurrence equations are:

$$\begin{cases} f(n+1,0) = \frac{1}{2}\left(f(n,0) + f(n,1)\right) \\ f(n+1,k) = \frac{1}{4}\left(f(n,k-1) + 2f(n,k) + f(n,k+1)\right), k = 1,...,\xi - 1 \\ f(n+1,\xi) = \frac{1}{4}\left(f(n,\xi-1) + f(n,\xi)\right) \end{cases} \quad (1)$$

Recall that the barrier is set at $\pm(2\xi+1)$, so the last non-zero value for f on each vertical is $f(n,\xi)$.

We first study the variation of energy, at a given time, on each vertical.

C. Decrease of the energy on each vertical

Lemma 1. - *For a given time* $n,$ *the energy is decreasing as a function of* k *:*

$$f(n,k) \geq f(n,k+1),\ k \geq 0.$$

Proof of Lemma 1

This is true for $n = 0$; let us admit the result for n and prove it for $n+1$.

We have:

$$f(n+1,1) = \frac{1}{4}f(n,0) + \frac{1}{2}f(n,1) + \frac{1}{4}f(n,2) \leq \frac{1}{2}f(n,0) + \frac{1}{2}f(n,1) = f(n+1,0)$$

since $f(n,2) \leq f(n,0)$ by the recurrence assumption. For $1 \leq k \leq \xi - 2$:

$$
\begin{aligned}
f(n+1,k+1) &= \frac{1}{4} f(n,k) + \frac{1}{2} f(n,k+1) + \frac{1}{4} f(n,k+2) \\
&\leq \frac{1}{4} f(n,k-1) + \frac{1}{2} f(n,k) + \frac{1}{4} f(n,k+1) \\
&= f(n+1,k)
\end{aligned}
$$

Finally, the property $f(n+1,\xi) \leq f(n+1,\xi-1)$ comes from:

$$\frac{1}{4} f(n,\xi-1) + \frac{1}{4} f(n,\xi) \leq \frac{1}{4} f(n,\xi-2) + \frac{1}{2} f(n,\xi-1) + \frac{1}{4} f(n,\xi)$$

which is clear. So, Lemma 1 is proved.

Corollary 2. - *Let $m < n$ be two instants; let $A(2m,2k)$, $k = 0,\ldots,\xi$, be points on the $2m^{th}$ vertical W_{2m} and let $B = B(2n,0)$ be the point on the x axis at time $2n$. Assume we put energy 1 at one of the points $A(2m,2k)$, $k = 0,\ldots,\xi$. The energy received by B will be maximal if this energy is put at $A(2m,0)$. In fact, the energy received by B is a decreasing function of k.*

Proof of Corollary 2

This is a simple consequence of Lemma 1, because if we put energy 1 at $A(2m,2k)$, the energy received by B is the same as the energy received by $A(2m,2k)$ if we put energy 1 at B.

Corollary 3. - *Assume we have any distribution of energy E_{2m} on the vertical W_{2m}. Then the energy received by B will be larger if all this energy is concentrated at the single point A_0.*

This is a clear consequence of the previous Corollary. There is a more general statement:

Corollary 4. - *Let $m < n$ be two instants, and let $A_1 = A(m,k_1)$ and $A_2 = A(m,k_2)$ be two points on the same vertical, with $k_1 < k_2$, both carrying energy 1. Then, on the vertical W_{2n}, the energy coming from the first point is larger than the second, which means that the loss of energy is larger in the second case.*

Another equivalent formulation is:

Corollary 4.b - *Let* $m<n$ *be two instants, and let* V,W *be two distribution of energies at time* m*, with same sum. Assume that for any* k*,* $\sum_{i\geq k} W(i) \geq \sum_{i\geq k} V(i)$ *(the vector* W *is more concentrated than* V *near the barrier). Then the energy sent by* W *to the vertical at time* n *is smaller than the energy sent by* V.

This corollary is quite intuitive. The second distribution is globally closer to the barrier, so the loss of energy is larger. Another way to say this is as follows: take any distribution of energy and move any quantity closer to the x axis: this is a "protective" move, in the sense that there will be less energy lost in the future.

We now turn to the behavior on the horizontal direction.

D. Decrease of the energy with time

Lemma 5 - *On the* x *axis, the energy is decreasing: for all* n*,*

$$f(0,n) \geq f(0,n+1).$$

Proof of Lemma 5

We have $f(0,0)=1$ and $f(1,0)<1$; let us admit the decrease until step n and prove it at step $n+1$. We have $f(n,0)-f(n+1,0)=\frac{1}{2}\left(f(n,0)-f(n,1)\right)>0$ by Lemma 1. This proves Lemma 5.

However, it is not true that the energy is decreasing on all horizontal lines $y=j$; indeed, if $j>1$, it first increases and then decreases: see Part I.

E. Matrix representation

The system (1) may be viewed as the action of a linear operator on the vector $\left(f(n,0),\dots,f(n,\xi)\right)$; the matrix, of dimension $(\xi+1)\times(\xi+1)$, is:

$$M_F = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \cdots & \cdots & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

and the energy vector at time $2n$ is:

$$E_{2n} = M_F^n \begin{pmatrix} f(0,0) \\ \vdots \\ f(0,\xi) \end{pmatrix}$$

This matrix representation is not quite useful for further investigation, because it turns out that the matrix M_F has a non-zero kernel; indeed, the vector:

$$Z = (1,-1,1,-1,\ldots)$$

satisfies $M_F Z = 0.$ So we will find another representation, for which the matrix will be invertible, and for which the eigenvalues can be explicitly computed. See the book [BB_Op] for all topics related to operator theory, used here.

F. A proper matrix representation

We start with a new change in coordinates.

1. A second change in coordinates

We introduced in Part I, for any n and k:

$$x(n,k) = \frac{1}{2}\big(f(n,k-1) + f(n,k)\big)$$

For $n = 0,$ using the symmetry:

$$x(n,0) = \frac{1}{2}\big(f(n,-1) + f(n,0)\big) = \frac{1}{2}\big(f(n,0) + f(n,1)\big) = x(n,1)$$

We have $x(n,k)=0$ if $k>n+1$. From now on, we will work mostly with the new coordinates.

2. The propagation problem in the new coordinates

- Case $\xi=2$

Equations (IV.B.1) above become:

$$\begin{cases} f(n+1,0)=x(n,1) \\ f(n+1,1)=\frac{1}{2}x(n,1)+\frac{1}{2}x(n,2) \\ f(n+1,2)=\frac{1}{2}x(n,2) \end{cases} \qquad (1)$$

From (1), we deduce:

$$x(n+1,1)=\frac{1}{2}\left(f(n+1,0)+f(n+1,1)\right)=\frac{1}{2}\left(x(n,1)+\frac{1}{2}\left(x(n,1)+x(n,2)\right)\right)=\frac{3}{4}x(n,1)+\frac{1}{4}x(n,2)$$

$$\begin{aligned} x(n+1,2)&=\frac{1}{2}f(n+1,1)+\frac{1}{2}f(n+1,2)=\frac{1}{2}\left(\frac{1}{2}x(n,1)+\frac{1}{2}x(n,2)\right)+\frac{1}{2}\left(\frac{1}{2}x(n,2)\right) \\ &=\frac{1}{4}x(n,1)+\frac{1}{2}x(n,2) \end{aligned}$$

So, we have the system:

$$\begin{cases} x(n+1,1)=\frac{3}{4}x(n,1)+\frac{1}{4}x(n,2) \\ x(n+1,2)=\frac{1}{4}x(n,1)+\frac{1}{2}x(n,2) \end{cases} \qquad (2)$$

with the initial values:

$$x(0,1)=\frac{1}{2}\left(f(0,0)+f(0,1)\right)=\frac{1}{2},\ x(0,2)=\frac{1}{2}\left(f(0,1)+f(0,2)\right)=0$$

These initial values may be written as a vector $X_0=\left(\frac{1}{2},0\right)$.

The system of equations (2) may be written as a matrix, under the form:

$$\begin{pmatrix} x(n+1,1) \\ x(n+1,2) \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x(n,1) \\ x(n,2) \end{pmatrix} \qquad (3)$$

- General case: $\xi > 2$

Equations (IV.B.1) above become:

$$\begin{cases} f(n+1,0) = x(n,1) \\ f(n+1,k) = \frac{1}{2}\big(x(n,k) + x(n,k+1)\big), k = 1,\ldots,\xi-1 \\ f(n+1,\xi) = \frac{1}{2} x(n,\xi) \end{cases} \qquad (1)$$

From (1), we deduce:

$$x(n+1,1) = \frac{1}{2}\big(f(n+1,0) + f(n+1,1)\big) = \frac{1}{2}\left(x(n,1) + \frac{1}{2}\big(x(n,1) + x(n,2)\big)\right) = \frac{3}{4}x(n,1) + \frac{1}{4}x(n,2)$$

$$\begin{aligned} x(n+1,k) &= \frac{1}{2}\left(\frac{1}{2}\big(x(n,k-1) + x(n,k)\big) + \frac{1}{2}\big(x(n,k) + x(n,k+1)\big)\right) \\ &= \frac{1}{4}x(n,k-1) + \frac{1}{2}x(n,k) + \frac{1}{4}x(n,k+1) \end{aligned}$$

for $k = 2,\ldots,\xi-1$, and:

$$x(n+1,\xi) = \frac{1}{4}x(n,\xi-1) + \frac{1}{2}x(n,\xi)$$

So, we have the system:

$$\begin{cases} x(n+1,1) = \frac{3}{4}x(n,1) + \frac{1}{4}x(n,2) \\ x(n+1,k) = \frac{1}{4}x(n,k-1) + \frac{1}{2}x(n,k) + \frac{1}{4}x(n,k+1), \text{for } k = 2,\ldots,\xi-1 \\ x(n+1,\xi) = \frac{1}{4}x(n,\xi-1) + \frac{1}{2}x(n,\xi) \end{cases} \qquad (2)$$

with the initial values:

$$x(0,1) = \frac{1}{2}\big(f(0,0) + f(0,1)\big) = \frac{1}{2},\ x(0,k) = \frac{1}{2}\big(f(0,k-1) + f(0,k)\big) = 0,\ \text{for } k \geq 2.$$

These initial values may be written as a vector $X_0 = \left(\frac{1}{2}, 0, ..., 0 \right)$.

The system of equations (2) may be written as a matrix, under the form:

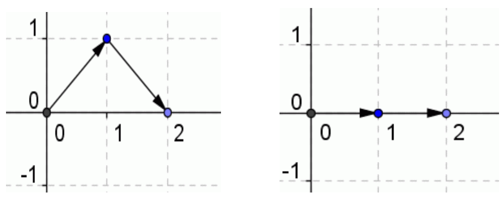
$$\begin{pmatrix} x(n+1,1) \\ x(n+1,2) \\ ... \\ ... \\ ... \\ x(n+1,\xi-1) \\ x(n+1,\xi) \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & ... & ... & ... & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & ... & ... & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & ... & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & ... & ... & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & ... & ... & ... & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x(n,1) \\ x(n,2) \\ ... \\ ... \\ ... \\ x(n,\xi-1) \\ x(n,\xi) \end{pmatrix} \qquad (3)$$

We have a real symmetric matrix, of size ξ, which is denoted by M.

We observe that, in this matrix representation, things are opposite to the physical representation: the first element of the vector X and the first row of the matrix correspond to what happens on the Ox axis; the last element of X and the last row of the matrix correspond to what happens close to the barrier.

We may also consider this situation as a propagation problem, with the following properties:

A point may move upwards, horizontally or downwards; all horizontal arrows have probability $\frac{1}{2}$, except the first one (the one on the x axis) which has probability $\frac{3}{4}$; all oblique arrows (up or down) have probability $\frac{1}{4}$. In this representation, two paths with same origin and same destination do not need to have the same probability. In the picture below, the left path has probability $\frac{1}{4^2}$ and the right path probability $\left(\frac{3}{4} \right)^2$.



Therefore, on the $x(n,k)$ coordinates, a matrix-oriented approach is appropriate, but an approach counting the number of paths is not.

We immediately deduce from the system (2) above that, if a vector X is made of positive, decreasing, coordinates (that is $x_1 \geq x_2 \geq \cdots \geq x_\xi$), the same will hold for its image $Y = MX$. In particular, the largest component of Y is the first. The first component of Y is smaller than the first component of X, since:

$$y_1 - x_1 = \frac{3}{4}x_1 + \frac{1}{4}x_2 - x_1 = -\frac{1}{4}x_1 + \frac{1}{4}x_2 \leq 0.$$

More generally, let us define a "bottom segment" (in the matrix notation) as a segment of the form $[1,k]$ for some $k, 1 \leq k \leq \xi,$ and, conversely, a "top segment" as a segment of the form $[k,\xi]$. A bottom segment includes the special cases of the x axis only and the whole strip $[1,\xi]$. Then, if a vector is made of positive, decreasing, coordinates, any bottom segment will see its energy decrease from one step to the next. Indeed, the energy it sends (one quarter of the top value of the segment) is smaller than the energy it receives (one quarter of the value of the point above the segment).

We observe that this is not true for a top segment, which may very well see its energy increase from one step to the next. Indeed, a top segment loses its energy in two ways: it sends some energy to the bottom, and some is absorbed by the barrier. But the latter may be 0, or be very small, and the bottom segment may receive more energy than it loses.

3. Properties of the matrix *M*

The matrix M enjoys many satisfactory properties, which the previous matrix M_F did not have.

Lemma 6. - *The matrix M is positive defined.*

Proof of Lemma 6

We have to show that, for all non-zero column-vector X of size ξ, we have $X^t MX > 0$.

Let $X = \begin{pmatrix} x_1 \\ \vdots \\ x_\xi \end{pmatrix}$; we have:

$$MX = \begin{pmatrix} \frac{3}{4}x_1 + \frac{1}{4}x_2 \\ \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3 \\ \vdots \\ \frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1} \\ \vdots \\ \frac{1}{4}x_{\xi-2} + \frac{1}{2}x_{\xi-1} + \frac{1}{4}x_\xi \\ \frac{1}{4}x_{\xi-1} + \frac{1}{2}x_\xi \end{pmatrix}$$

and therefore:

$$\begin{aligned} X^t MX &= \left(\frac{3}{4}x_1 + \frac{1}{4}x_2\right)x_1 + \left(\frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3\right)x_2 + \ldots + \left(\frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1}\right)x_i + \ldots + \\ &\quad + \left(\frac{1}{4}x_{\xi-2} + \frac{1}{2}x_{\xi-1} + \frac{1}{4}x_\xi\right)x_{\xi-1} + \left(\frac{1}{4}x_{\xi-1} + \frac{1}{2}x_\xi\right)x_\xi \\ &= \frac{1}{2}x_1^2 + b + \frac{1}{4}x_\xi^2 \end{aligned}$$

with $b = \frac{1}{4}\left(\left(x_1 + x_2\right)^2 + \left(x_2 + x_3\right)^2 + \ldots + \left(x_{i-1} + x_i\right)^2 + \ldots + \left(x_{\xi-1} + x_\xi\right)^2\right).$

So, clearly $X^t MX > 0$ if the x_i's are not all equal to 0. This proves Lemma 6.

From Lemma 6 follows that all eigenvalues of M are real and >0 and that M can be diagonalized in an orthogonal basis made of eigenvectors.

Lemma 7. - *All eigenvalues of* M *are* <1.

Proof of Lemma 7

Let us write the system of equations defining the eigenvalues and eigenvectors, $MX = \lambda X$.

$$\begin{cases} \frac{3}{4}x_1 + \frac{1}{4}x_2 = \lambda x_1 \\ \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3 = \lambda x_2 \\ \vdots \\ \frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1} = \lambda x_i \\ \vdots \\ \frac{1}{4}x_{\xi-2} + \frac{1}{2}x_{\xi-1} + \frac{1}{4}x_\xi = \lambda x_{\xi-1} \\ \frac{1}{4}x_{\xi-1} + \frac{1}{2}x_\xi = \lambda x_\xi \end{cases} \qquad (1)$$

It may be written:

$$\begin{cases} x_2 = (4\lambda - 3)x_1 \\ x_3 = (4\lambda - 2)x_2 - x_1 \\ \vdots \\ x_{i+1} = (4\lambda - 2)x_i - x_{i-1} \\ \vdots \\ x_\xi = (4\lambda - 2)x_{\xi-1} - x_{\xi-2} \\ x_{\xi-1} = (4\lambda - 2)x_\xi \end{cases} \qquad (2)$$

We know that $x_1 \neq 0$ (if $x_1 = 0$, all $x_i = 0$), so we may assume $x_1 = 1$. Assume $\lambda \geq 1$. From the first equation $x_2 = (4\lambda - 3)x_1$ we deduce $x_2 > x_1 > 0$.

More generally, the equation $x_{i+1} = (4\lambda - 2)x_i - x_{i-1}$ gives:

$$x_{i-1} - x_i = (4\lambda - 3)x_i - x_{i+1} \geq x_i - x_{i+1}$$

that is $x_i - x_{i-1} \leq x_{i+1} - x_i$. So the sequence of consecutive differences is increasing. Since $x_2 > x_1$, all differences are positive, the x_i are increasing and are > 0. Set $S = \sum_{i=1}^{\xi} x_i$; summing all equations, we get $S - \frac{1}{4}x_\xi = \lambda S$, that is $-\frac{1}{4}x_\xi = (\lambda - 1)S$. But this is a contradiction: $\lambda > 1$, $S > 0$ and $x_\xi > 0$. Lemma 7 is proved.

The results we obtain here for the matrix M do not hold for the previous matrix M_F ; this is why we had to make this change of variables.

4. Precise results on the eigenvalues and eigenvectors

Proposition 8. - *For* $j=1,\dots,\xi,$ *let* $\vartheta_j = \dfrac{2j-1}{2\xi+1}\pi$. *The* j^{th} *eigenvalue* λ_j *is:*

$$\lambda_j = \frac{1+\cos\left(\vartheta_j\right)}{2} = \cos^2\frac{\vartheta_j}{2}$$

and the j^{th} *eigenvector has components:*

$$V_j = \left(\sin\left(\xi\vartheta_j\right), \sin\left(\left(\xi-1\right)\vartheta_j\right), \dots, \sin\left(\vartheta_j\right)\right).$$

Proof of Proposition 8

We have $x_\xi \neq 0$ (otherwise all $x_j's$ are 0), so we may assume $x_\xi = 1$.

We set $\mu = 2\lambda - 1$, then $\mu < 1$. System (2) above becomes:

$$\begin{cases} x_2 = \left(2\mu-1\right)x_1 \\ x_3 = 2\mu x_2 - x_1 \\ \quad\vdots \\ x_{i+1} = 2\mu x_i - x_{i-1} \\ \quad\vdots \\ x_\xi = 2\mu x_{\xi-1} - x_{\xi-2} \\ x_{\xi-1} = 2\mu x_\xi \end{cases} \tag{1}$$

We set $y_j = x_{\xi-j}$ for $j = 0,\dots,\xi-1$. System (1) becomes:

$$\begin{cases} y_0 = 1 \\ y_1 = 2\mu \\ y_2 = 2\mu y_1 - y_0 \\ \quad\vdots \\ y_j = 2\mu y_{j-1} - y_{j-2} \\ \quad\vdots \\ y_{\xi-1} = 2\mu y_{\xi-2} - y_{\xi-3} \\ y_{\xi-2} = \left(2\mu-1\right)y_{\xi-1} \end{cases} \tag{2}$$

Therefore, $y_j = U_j\left(\mu\right)$ where U_j is the j^{th} Chebychev's polynomial of second kind, for $j = 0,\dots,\xi-1$. The final equation in (2) may be written:

$$U_{\xi-2}(\mu)=(2\mu-1)U_{\xi-1}(\mu) \tag{3}$$

that is, with $\mu=\cos(\vartheta)$:

$$\frac{\sin((\xi-1)\vartheta)}{\sin(\vartheta)}=(2\cos(\vartheta)-1)\frac{\sin(\xi\vartheta)}{\sin(\vartheta)}.$$

By Lemma 7, $\sin(\vartheta)\neq 0,$ so the above equation is equivalent to:

$$\sin((\xi-1)\vartheta)=(2\cos(\vartheta)-1)\sin(\xi\vartheta) \tag{4}$$

We have:

$$\sin((\xi-1)\vartheta)-(2\cos(\vartheta)-1)\sin(\xi\vartheta)=-\sin(\xi\vartheta)\cos(\vartheta)-\cos(\xi\vartheta)\sin(\vartheta)+\sin(\xi\vartheta).$$

Therefore, equation (4) is equivalent to:

$$\sin(\xi\vartheta)(1-\cos(\vartheta))=\cos(\xi\vartheta)\sin(\vartheta)$$

or:

$$\tan(\xi\vartheta)=\frac{\sin(\vartheta)}{1-\cos(\vartheta)} \tag{5}$$

which may be written:

$$\tan(\xi\vartheta)=\frac{1}{\tan\frac{\vartheta}{2}} \tag{6}$$

Therefore:

$$\cos(\xi\vartheta)\cos\frac{\vartheta}{2}-\sin(\xi\vartheta)\sin\frac{\vartheta}{2}=0$$

which gives:

$$\cos\left(\xi\vartheta+\frac{\vartheta}{2}\right)=0,$$

and this equation has the solutions $\frac{2\xi+1}{2}\vartheta=\frac{\pi}{2}+(j-1)\pi$, $j=1,\dots,\xi,$

that is:

$$\vartheta = \frac{(2j-1)\pi}{2\xi+1} \tag{7}$$

as we announced.

Since $U_j(\cos\vartheta) = \frac{\sin((j+1)\vartheta)}{\sin\vartheta}$, after multiplication, we may take, for $j = 1,\ldots,\xi$:

$$V_j = \left(\sin(\xi\vartheta_j), \sin((\xi-1)\vartheta_j), \ldots, \sin(\vartheta_j)\right) \tag{8}$$

and $\lambda_j = \frac{1+\cos(\vartheta_j)}{2} = \cos^2\frac{\vartheta_j}{2}$. This finishes the proof of Proposition 8.

Another formulation for the components of the eigenvectors is:

Proposition 8b. – *The components of the j^{th} eigenvector, V_j, may be written:*

$$V_j = (-1)^{j-1}\left(\cos\left(\frac{\vartheta_j}{2}\right), \ldots, \cos\left(\frac{(2k-1)\vartheta_j}{2}\right), \ldots, \cos\left(\frac{(2\xi-1)\vartheta_j}{2}\right)\right)$$

Proof of Proposition 8b

Indeed, we observe that:

$$\xi\vartheta_j = (2j-1)\frac{\pi}{2} - \frac{2j-1}{2\xi+1}\frac{\pi}{2} = j\pi - \frac{\pi}{2} - \frac{2j-1}{2\xi+1}\frac{\pi}{2}$$

and therefore:

$$\sin(\xi\vartheta_j) = \sin\left(j\pi - \frac{\pi}{2} - \frac{2j-1}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1}\cos\left(\frac{2j-1}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1}\cos\frac{\vartheta_j}{2}$$

More generally:

$$\begin{aligned}\sin((\xi-k+1)\vartheta_j) &= \sin\left(j\pi - \frac{\pi}{2} - \frac{(2k-1)(2j-1)}{2\xi+1}\frac{\pi}{2}\right) \\ &= (-1)^{j-1}\cos\left(\frac{(2k-1)(2j-1)}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1}\cos\frac{(2k-1)\vartheta_j}{2}\end{aligned}$$

This proves Proposition 8b.

The first eigenvector, V_1, has all its components real and >0, but all other eigenvectors have some negative component.

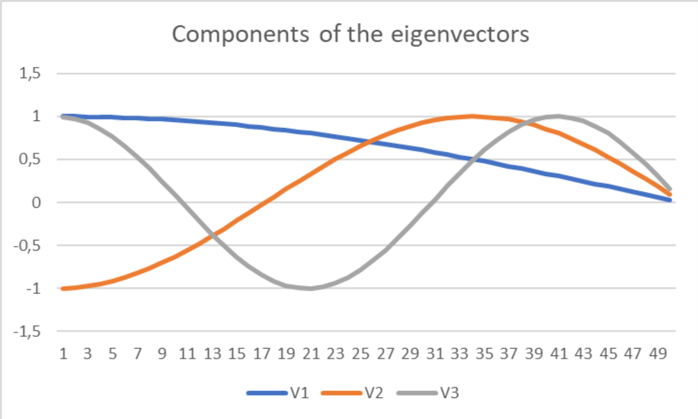


Fig. : the components of V_1, V_2, V_3, *for* $\xi = 50$

Remark. - It follows from the general theory of symmetric matrices, positive defined, that any two eigenvectors V_{j_1}, V_{j_2} are mutually orthogonal, that is:

$$\sum_{l=1}^{\xi} \sin\left(l\vartheta_{j_1}\right)\sin\left(l\vartheta_{j_2}\right) = 0$$

where $\vartheta_{j_1} = \dfrac{(2j_1 - 1)\pi}{2\xi + 1}, \vartheta_{j_2} = \dfrac{(2j_2 - 1)\pi}{2\xi + 1}$. This can be checked directly.

We now compute the $l_2 -$ norm of the eigenvectors.

5. Norms of the eigenvectors

Proposition 9. – *All eigenvectors have the same quadratic norm:*

$$\left|V_j\right|_2^2 = \frac{\xi}{2} + \frac{1}{4},$$

for $j = 1, \ldots, \xi.$

Proof of Proposition 9

Let us compute $\left|V_j\right|_2^2$. We use the identity:

$$\sum_{k=1}^{\xi} \sin^2(kt) = \frac{\xi}{2} + \frac{1}{4} - \frac{1}{4}\frac{\sin\left((2\xi + 1)t\right)}{\sin(t)}$$

which gives, with $t=\dfrac{(2j-1)\pi}{2\xi+1}$:

$$\left|V_j\right|_2^2=\sum_{k=1}^{\xi}\sin^2\left(\frac{k(2j-1)\pi}{2\xi+1}\right)=\frac{\xi}{2}+\frac{1}{4}-\frac{1}{4}\frac{\sin\left((2j-1)\pi\right)}{\sin\left(\frac{(2j-1)\pi}{2\xi+1}\right)}=\frac{\xi}{2}+\frac{1}{4}$$, as we announced.

For any vector $V=\left(v_1,\dots,v_\xi\right)$, we denote by $s(V)=\sum_{j=1}^{\xi}v_j$ the sum of its components. It can be considered as the "energy" carried by the vector, but one should remember that some components may be negative.

Indeed, for any vector X, we have a Hilbert space decomposition :

$$X=\sum_{j=1}^{\xi}\alpha_j V_j$$

Since the eigenvectors are orthogonal, the coefficients α_j may be computed simply:

$$\alpha_j=\frac{<X,V_j>}{\left|V_j\right|_2^2}$$

Assume X is a vector with positive coefficients ; then $s(X)$ is the energy carried by this vector. But, from the above decomposition, we get:

$$s(X)=\sum_{j=1}^{\xi}\alpha_j s\left(V_j\right)$$

and $s\left(V_j\right)$ may be regarded as the energy carried by the vector V_j, though this vector does not have all its coefficients positive.

Proposition 10. – *All eigenvectors satisfy:*

$$s\left(V_j\right)=\frac{1}{2}\frac{1}{\tan\left(\frac{\vartheta_j}{2}\right)}=\frac{1}{2}\tan\left(\xi\vartheta_j\right)$$

Proof of Proposition 10

The eigenvector V_j carries the energy $s(V_j)$. Apply the matrix M to this eigenvector. The loss of energy is $\frac{1}{4}\sin(\vartheta_j)$, since the last coordinate of the eigenvector is $\sin(\vartheta_j)$. But, by definition, we have $MV_j = \lambda_j V_j$ and the loss of energy is $(1-\lambda_j)s(V_j)$. So, we get:

$$s(V_j) = \frac{1}{4}\frac{\sin(\vartheta_j)}{1-\cos^2\left(\frac{\vartheta_j}{2}\right)} = \frac{1}{4}\frac{\sin(\vartheta_j)}{\sin^2\left(\frac{\vartheta_j}{2}\right)} = \frac{1}{2}\frac{\sin\left(\frac{\vartheta_j}{2}\right)\cos\left(\frac{\vartheta_j}{2}\right)}{\sin^2\left(\frac{\vartheta_j}{2}\right)} = \frac{1}{2}\frac{1}{\tan\left(\frac{\vartheta_j}{2}\right)} = \frac{1}{2}\tan(\xi\vartheta_j)$$

which proves our claim.

Let us give an example, for $\xi = 10$:

	W 1	W 2	W 3	W 4	W 5	W 6	W 7	W 8	W 9	W 10
theta (rd)	0,150	0,449	0,748	1,047	1,346	1,646	1,945	2,244	2,543	2,842
components	0,149	0,434	0,680	0,866	0,975	0,997	0,931	0,782	0,563	0,295
	0,295	0,782	0,997	0,866	0,434	-0,149	-0,680	-0,975	-0,931	-0,563
	0,434	0,975	0,782	0,000	-0,782	-0,975	-0,434	0,434	0,975	0,782
	0,563	0,975	0,149	-0,866	-0,782	0,295	0,997	0,434	-0,680	-0,931
	0,680	0,782	-0,563	-0,866	0,434	0,931	-0,295	-0,975	0,149	0,997
	0,782	0,434	-0,975	0,000	0,975	-0,434	-0,782	0,782	0,434	-0,975
	0,866	0,000	-0,866	0,866	0,000	-0,866	0,866	0,000	-0,866	0,866
	0,931	-0,434	-0,295	0,866	-0,975	0,563	0,149	-0,782	0,997	-0,680
	0,975	-0,782	0,434	0,000	-0,434	0,782	-0,975	0,975	-0,782	0,434
	0,997	-0,975	0,931	-0,866	0,782	-0,680	0,563	-0,434	0,295	-0,149
s(V)	6,672	2,191	1,274	0,866	0,627	0,464	0,341	0,241	0,154	0,075

Only the first vector has all its coefficients positive and distinct. For the others, some components may be repeated, for instance $W_4(2) = W_4(3)$. We also observe that the energy is decreasing from V_1 to V_ξ : this is clear, since $0 < \frac{\vartheta_j}{2} < \frac{\pi}{2}$ and ϑ_j increases.

We observe that the l_1 norms differ from one vector to the other. Here are these norms in the previous case:

norme 1	6,672	6,572	6,672	6,062	6,572	6,672	6,672	6,572	6,672	6,672

6. Decomposition on the basis of eigenvectors

Proposition 11. – Let $\vartheta_j = \frac{2j-1}{2\xi+1}\pi$, $j = 1,\dots,\xi$. *At the initial step, the proportion of energy carried by the* j^{th} *eigenvector is given by the formula:*

$$s_j\left(X_0\right)=\frac{1}{2\xi+1}(-1)^{j-1}\frac{\cos^2\left(\frac{\vartheta_j}{2}\right)}{\sin\left(\frac{\vartheta_j}{2}\right)}$$

More generally, at the n^{th} *step, the proportion of energy carried by the* j^{th} *eigenvector is given by the formula:*

$$s_j\left(M^nX_0\right)=\frac{(-1)^{j-1}}{2\xi+1}\frac{\cos^{2n+2}\left(\frac{\vartheta_j}{2}\right)}{\sin\left(\frac{\vartheta_j}{2}\right)}$$

At the n^{th} *step, the energy at each point can be written:*

$$x(n,k)=\frac{2}{2\xi+1}\sum_{j=1}^{\xi}\cos\left(\frac{(2k-1)\vartheta_j}{2}\right)\cos^{2n+1}\frac{\vartheta_j}{2}$$

The total energy on the n^{th} *vertical is:*

$$\left|X_n\right|_1=s\left(X_n\right)=\sum_{j=1}^{\xi}s_j\left(M^nX_0\right)=\frac{1}{2\xi+1}\sum_{j=1}^{\xi}(-1)^{j-1}\frac{\cos^{2n+2}\frac{\vartheta_j}{2}}{\sin\frac{\vartheta_j}{2}}.$$

The quadratic energy on the n^{th} *vertical is:*

$$\left|X_n\right|_2^2=\frac{1}{2\xi+1}\sum_{j=1}^{\xi}\cos^{4n+2}\left(\frac{2j-1}{2\xi+1}\frac{\pi}{2}\right)$$

Proof of Proposition 11

We start with the initial value $f(0,0)=1,\ f(0,k)=0,\ k=1,...,\xi.$ So, the initial vector is $X_0=\left(\frac{1}{2},0,...,0\right)$. We decompose this vector on the basis of eigenvectors. We write:

$$X_0=\sum_{j=1}^{\xi}\alpha_jV_j$$

and we obtain:

$$\alpha_j = \frac{< X_0, V_j >}{\left|V_j\right|_2^2} = \frac{\sin\left(\xi \vartheta_j\right)}{\xi + \frac{1}{2}} = \frac{(-1)^{j-1} \cos\left(\frac{\vartheta_j}{2}\right)}{\xi + \frac{1}{2}}$$

using Proposition 8b. We see that the coefficients α_j have alternate signs and are decreasing in absolute value, for $j = 1, \ldots, \xi$.

The energy can be written:

$$s\left(X_0\right) = \sum_{j=1}^{\xi} \alpha_j s\left(V_j\right)$$

that is, using Proposition 10:

$$s\left(X_0\right) = \frac{1}{2\xi + 1} \sum_{j=1}^{\xi} \sin\left(\xi \vartheta_j\right) \tan\left(\xi \vartheta_j\right) = \frac{1}{2\xi + 1} \sum_{j=1}^{\xi} (-1)^{j-1} \frac{\cos^2\left(\frac{\vartheta_j}{2}\right)}{\sin\left(\frac{\vartheta_j}{2}\right)}$$

This proves our first claim.

We see that the proportion of energy carried by the j^{th} eigenvector has alternate sign and is decreasing in absolute value.

At the n^{th} step (time $2n$), the vector X_n is:

$$X_n = M^n X_0 = M^n \sum_{j=1}^{\xi} \alpha_j V_j = \sum_{j=1}^{\xi} \alpha_j M^n V_j = \sum_{j=1}^{\xi} \alpha_j \lambda_j^n V_j \qquad (6)$$

This gives:

$$s\left(X_n\right) = \sum_{j=1}^{\xi} \alpha_j \lambda_j^n s\left(V_j\right) = \frac{1}{2\xi + 1} \sum_{j=1}^{\xi} (-1)^{j-1} \frac{\cos^{2n+2}\left(\frac{\vartheta_j}{2}\right)}{\sin\left(\frac{\vartheta_j}{2}\right)}$$

which proves our second claim.

Let $E\left(n, V_j\right)$ be the energy carried by the eigenvector V_j at step n. The previous result gives:

$$E(n,V_j)=\frac{(-1)^{j-1}}{2\xi+1}\frac{\cos^{2n+1}\left(\frac{\vartheta_j}{2}\right)}{\sin\left(\frac{\vartheta_j}{2}\right)}$$

If we want to find the energy at each place $x(n,k)$, we simply have to identify the coordinates in (6). Recall that the k^{th} component of V_j $(k=1,\ldots,\xi)$ is:

$$\sin\left((\xi-k+1)\vartheta_j\right)=(-1)^{j-1}\cos\frac{(2k-1)\vartheta_j}{2}$$

So we get:

$$x(n,k)=\sum_{j=1}^{\xi}\alpha_j\lambda_j^n V_j(k)=\sum_{j=1}^{\xi}\frac{(-1)^{j-1}\cos\frac{\vartheta_j}{2}}{\xi+\frac{1}{2}}\cos^{2n}\left(\frac{\vartheta_j}{2}\right)(-1)^{j-1}\cos\frac{(2k-1)\vartheta_j}{2}$$

$$=\frac{1}{\xi+\frac{1}{2}}\sum_{j=1}^{\xi}\cos^{2n+1}\left(\frac{\vartheta_j}{2}\right)\cos\frac{(2k-1)\vartheta_j}{2}$$

which proves our third claim. The fourth claim is an obvious consequence of the second, summing upon j. The fifth claim follows from the formula:

$$|X_n|_2^2=\frac{1}{2\xi+1}\sum_{j=1}^{\xi}\sin^2\left(\xi\vartheta_j\right)\cos^{4n}\frac{\vartheta_j}{2}.$$

We observe that $x(n,k)$ may be written as a scalar product:

$$x(n,k)=\frac{1}{\xi+\frac{1}{2}}\langle A_k,B_n\rangle\text{, with }A_k=\begin{pmatrix}\cos\left((2k-1)\frac{\vartheta_1}{2}\right)\\ \vdots\\ \cos\left((2k-1)\frac{\vartheta_\xi}{2}\right)\end{pmatrix},\ B_n=\begin{pmatrix}\cos^{2n+1}\frac{\vartheta_1}{2}\\ \vdots\\ \cos^{2n+1}\frac{\vartheta_\xi}{2}\end{pmatrix}.$$

In the case $\xi=3$, we find numerically:

$$x(n,1)=0.2716\times0.9505^n+0.1746\times0.6113^n+0.0538\times0.1883^n$$
$$x(n,2)=0.2178\times0.9505^n-0.0969\times0.6113^n-0.1209\times0.1883^n$$
$$x(n,3)=0.1209\times0.9505^n-0.2178\times0.6113^n+0.0969\times0.1883^n$$

We can now state the main theorem of this section:

Theorem 12. - *At each step* $2n$, *the energy* $e(2n,2k)=f(n,k)$, $k=0,\dots,\xi$, *is given by:*

$$\begin{cases} f(n,0)=x(n-1,1) \\ f(n,k)=\frac{1}{2}\big(x(n-1,k)+x(n-1,k+1)\big),\ k=1,\dots,\xi-1 \\ f(n,\xi)=\frac{1}{2}x(n-1,\xi) \end{cases}$$

where, for $k=1,\dots,\xi$*:*

$$x(n,k)=\frac{2}{2\xi+1}\sum_{j=1}^{\xi}\cos\left(\frac{(2k-1)\vartheta_j}{2}\right)\cos^{2n+1}\frac{\vartheta_j}{2}$$

and $\vartheta_j=\frac{(2j-1)\pi}{2\xi+1}$, $j=1,\dots,\xi.$

Proof of Theorem 12

It follows immediately from Proposition 11 and the equations relating $f(n,k)$ with $x(n-1,k)$, $x(n-1,k+1)$, namely:

$$\begin{cases} f(n,0)=x(n-1,1) \\ f(n,k)=\frac{1}{2}\big(x(n-1,k)+x(n-1,k+1)\big),\ k=1,\dots,\xi-1 \\ f(n,\xi)=\frac{1}{2}x(n-1,\xi) \end{cases}$$

This concludes the proof of Theorem 12.

7. Combinatorial expressions

So far, the results have been given in trigonometric terms. We now show that they may be expressed in terms of combinatorics.

In what follows, $\binom{n}{k}$ is zero if $k>n$ or if $k<0$: this will simplify the notation.

Proposition 13. - *At each step, we have:*

$$x(n,k)=\frac{1}{2^{2n+1}}\binom{2n+1}{n-k+1}+\frac{1}{2^{2n+1}}\sum_{u\geq 1}(-1)^u\binom{2n+1}{k+n-u(2\xi+1)}$$

Proof of Proposition 13

We use the identity:

$$\cos^{2n+1}(x) = \frac{1}{2^{2n}} \sum_{l=0}^{n} \binom{2n+1}{l} \cos\left((2n+1-2l)x\right)$$

which gives:

$$x(n,k) = \frac{2}{2\xi+1} \frac{1}{2^{2n}} \sum_{l=0}^{n} \binom{2n+1}{l} \sum_{j=1}^{\xi} \cos\left(\frac{(2k-1)\vartheta_j}{2}\right) \cos\left((2n+1-2l)\frac{\vartheta_j}{2}\right)$$

that is:

$$x(n,k) = \frac{1}{2\xi+1} \frac{1}{2^{2n}} \sum_{l=0}^{n} \binom{2n+1}{l} \sum_{j=1}^{\xi} \left(\cos\left((k-n-1+l)\vartheta_j\right) + \cos\left((k+n-l)\vartheta_j\right)\right)$$

Let:

$$T_1 = \frac{1}{2\xi+1} \frac{1}{2^{2n}} \sum_{l=0}^{n} \binom{2n+1}{l} \sum_{j=1}^{\xi} \cos\left((k-n-1+l)\vartheta_j\right)$$

For all k, we have the following identities:

$\sum_{j=1}^{\xi} \cos(k\vartheta_j) = \frac{1}{2}(-1)^{k-1}$ if k is not a multiple of $2\xi+1$

$\sum_{j=1}^{\xi} \cos(k\vartheta_j) = \xi$ if k is an even multiple of $2\xi+1$ (including 0)

$\sum_{j=1}^{\xi} \cos(k\vartheta_j) = -\xi$ if k is an odd multiple of $2\xi+1$.

So, we must study the term $k-n-1+l$, $1 \le k \le \xi$, $0 \le l \le n$.

We have $k-n-1+l=0$ if and only if $l=n-k+1$; we cannot have $k-n-1+l=2\xi+1$, since this is equivalent to $l=n-k+2\xi+2$, but $2\xi-k>0$ and $l \le n$.

Therefore:

$$\left(2\xi+1\right)2^{2n}T_1=\sum_{l=0}^{n}\binom{2n+1}{l}\sum_{j=1}^{\xi}\cos\left(\left(k-n-1+l\right)\vartheta_j\right)$$

$$=\sum_{\substack{l=0\\ l\neq n-k+1}}^{n}\binom{2n+1}{l}\sum_{j=1}^{\xi}\cos\left(\left(k-n-1+l\right)\vartheta_j\right)+\binom{2n+1}{n-k+1}\sum_{j=1}^{\xi}\cos\left(\left(k-n-1+n-k+1\right)\vartheta_j\right)$$

$$=\frac{1}{2}\left(-1\right)^{k-n}\sum_{\substack{l=0\\ l\neq n-k+1}}^{n}\binom{2n+1}{l}\left(-1\right)^{l}+\binom{2n+1}{n-k+1}\xi$$

But:

$$\sum_{\substack{l=0\\ l\neq n-k+1}}^{n}\binom{2n+1}{l}\left(-1\right)^{l}=\sum_{l=0}^{n}\binom{2n+1}{l}\left(-1\right)^{l}-\binom{2n+1}{n-k+1}\left(-1\right)^{n-k+1}$$

We have the identity:

$$\sum_{l=0}^{n}\binom{2n+1}{l}\left(-1\right)^{l}=\left(-1\right)^{n}\binom{2n}{n}$$

and thus:

$$\sum_{\substack{l=0\\ l\neq n-k+1}}^{n}\binom{2n+1}{l}\left(-1\right)^{l}=\left(-1\right)^{n}\binom{2n}{n}+\left(-1\right)^{n-k}\binom{2n+1}{n-k+1}$$

Therefore:

$$T_1=\frac{1}{2\xi+1}\frac{1}{2^{2n+1}}\left(\left(-1\right)^{k}\binom{2n}{n}+\left(2\xi+1\right)\binom{2n+1}{n-k+1}\right)$$

which gives:

$$T_1=\frac{1}{2\xi+1}\frac{1}{2^{2n+1}}\left(-1\right)^{k}\binom{2n}{n}+\frac{1}{2^{2n+1}}\binom{2n+1}{n-k+1}$$

The same way, let:

$$T_2=\frac{1}{2\xi+1}\frac{1}{2^{2n}}\sum_{l=0}^{n}\binom{2n+1}{l}\sum_{j=1}^{\xi}\left(\cos\left(k+n-l\right)\vartheta_j\right)$$

Since $l \le n$ and $k \ge 1$, the coefficient $k+n-l$ cannot vanish. But we have $k+n-l=2\xi+1$ for $l=k+n-(2\xi+1)$ and more generally $k+n-l=u(2\xi+1)$ for $l=k+n-u(2\xi+1)$. We need $l \ge 0$, $k+n \ge u(2\xi+1)$, $u \le \dfrac{k+n}{2\xi+1}$, that is $u=1,\ldots,a=\text{int}\left(\dfrac{k+n}{2\xi+1}\right)$.

So:

$$\sum_{j=1}^{\xi}\left(\cos(k+n-l)\vartheta_j\right)=\frac{1}{2}(-1)^{k+n-l-1} \text{ if } l \ne k+n-u(2\xi+1)$$
$$\sum_{j=1}^{\xi}\left(\cos(k+n-l)\vartheta_j\right)=-\xi \text{ if } l=k+n-u(2\xi+1),\ u \text{ odd}$$
$$\sum_{j=1}^{\xi}\left(\cos(k+n-l)\vartheta_j\right)=\xi \text{ if } l=k+n-u(2\xi+1),\ u \text{ even}$$

This gives:

$$(2\xi+1)2^{2n}T_2=\frac{1}{2}(-1)^{k+n-1}\sum_{\substack{l=0\\ l\ne k+n-u(2\xi+1)}}^{n}(-1)^l\binom{2n+1}{l}+\xi\sum_{u=1}^{a}(-1)^u\binom{2n+1}{k+n-u(2\xi+1)}$$

$$(2\xi+1)2^{2n}T_2=\frac{1}{2}(-1)^{k+n-1}\sum_{l=0}^{n}(-1)^l\binom{2n+1}{l}-\frac{1}{2}(-1)^{k+n-1}\sum_{u=1}^{a}(-1)^{k+n-u(2\xi+1)}\binom{2n+1}{k+n-u(2\xi+1)}$$
$$+\xi\sum_{u=1}^{a}(-1)^u\binom{2n+1}{k+n-u(2\xi+1)}$$

$$(2\xi+1)2^{2n}T_2=\frac{1}{2}(-1)^{k+n-1}\sum_{l=0}^{n}(-1)^l\binom{2n+1}{l}+\frac{1}{2}\sum_{u=1}^{a}(-1)^u\binom{2n+1}{k+n-u(2\xi+1)}+\xi\sum_{u=1}^{a}(-1)^u\binom{2n+1}{k+n-u(2\xi+1)}$$

$$(2\xi+1)2^{2n+1}T_2=(-1)^{k-1}\binom{2n}{n}+\sum_{u=1}^{a}(-1)^u\binom{2n+1}{k+n-u(2\xi+1)}+2\xi\sum_{u=1}^{a}(-1)^u\binom{2n+1}{k+n-u(2\xi+1)}$$

and finally:

$$T_2=(-1)^{k-1}\frac{1}{(2\xi+1)2^{2n+1}}\binom{2n}{n}+\frac{1}{2^{2n+1}}\sum_{u=1}^{a}(-1)^u\binom{2n+1}{k+n-u(2\xi+1)}$$

So, we obtain:

$$x(n,k)=T_1+T_2=\frac{1}{2^{2n+1}}\binom{2n+1}{n-k+1}+\frac{1}{2^{2n+1}}\sum_{u=1}^{a}(-1)^u\binom{2n+1}{k+n-u(2\xi+1)}$$

which proves Proposition 13.

The trigonometric expression in Proposition 11 may look less expressive than its combinatorial counterpart, but it is simpler to establish. In our view, the trigonometric presentation is more natural for the present subject.

Proposition 14. – *For each* n, *the quadratic energy satisfies:*

$$|X_n|_2^2 = \frac{1}{2^{4n+3}}\binom{4n+2}{2n+1} + \frac{1}{2^{4n+2}}\sum_{u=u_1}^{u_2} (-1)^u \binom{4n+2}{2n+1-u(2\xi+1)}$$

with $u_1 = 1+\operatorname{int}\left(\frac{1}{2\xi+1}\right)$, $u_2 = \operatorname{int}\left(\frac{2n+1}{2\xi+1}\right)$.

Proof of Proposition 14

We write:

$$\cos^{4n+2}\left(\frac{\vartheta}{2}\right) = \frac{1}{2^{4n+2}}\binom{4n+2}{2n+1} + \frac{1}{2^{4n+1}}\sum_{l=0}^{2n} \binom{4n+2}{l}\cos\left((2n+1-l)\vartheta\right)$$

$$|X_n|_2^2 = \frac{1}{2^{4n+2}}\frac{\xi}{2\xi+1}\binom{4n+2}{2n+1} + \frac{1}{2^{4n+1}}\frac{1}{2\xi+1}\sum_{l=0}^{2n} \binom{4n+2}{l}\sum_{j=1}^{\xi} \cos\left((2n+1-l)\vartheta_j\right)$$

We will make use of the following formulas:

$\sum_{j=1}^{\xi} \cos(k\vartheta_j) = \frac{1}{2}(-1)^{k-1}$ if k is not a multiple of $2\xi+1$;

$\sum_{j=1}^{\xi} \cos(k\vartheta_j) = (-1)^u\,\xi$ if $k = u(2\xi+1)$ is a multiple of $2\xi+1$ (including 0),

that is:

$\sum_{j=1}^{\xi} \cos\left((2n+1-l)\vartheta_j\right) = \frac{1}{2}(-1)^l$ if $2n+1-l$ is not a multiple of $2\xi+1$;

$\sum_{j=1}^{\xi} \cos\left((2n+1-l)\vartheta_j\right) = (-1)^u\,\xi$ if $2n+1-l = u(2\xi+1)$.

The condition $2n+1-l = u(2\xi+1)$ is equivalent to $l = 2n+1-u(2\xi+1)$. Since $0 \le 2n+1-u(2\xi+1) \le 2n$, we have $u_1 = 1+\operatorname{int}\left(\frac{1}{2\xi+1}\right) \le u \le u_2 = \operatorname{int}\left(\frac{2n+1}{2\xi+1}\right)$.

We have:

$$\sum_{l=0}^{2n} \binom{4n+2}{l} \sum_{j=1}^{\xi} \cos\left(\left(2n+1-l\right)\vartheta_j\right) = \frac{1}{2} \sum_{\substack{l=0 \\ l\neq 2n+1-u(2\xi+1)}}^{2n} (-1)^l \binom{4n+2}{l} + \xi \sum_{u=u_1}^{u_2} (-1)^u \binom{4n+2}{2n+1-u(2\xi+1)}$$

and:

$$\sum_{\substack{l=0 \\ l\neq 2n+1-u(2\xi+1)}}^{2n} (-1)^l \binom{4n+2}{l} = \sum_{l=0}^{2n} (-1)^l \binom{4n+2}{l} + \sum_{u=u_1}^{u_2} (-1)^u \binom{4n+2}{2n+1-u(2\xi+1)}$$

But:

$$\sum_{l=0}^{2n} (-1)^l \binom{4n+2}{l} = \frac{1}{2}\binom{4n+2}{2n+1}$$

which gives:

$$\sum_{\substack{l=0 \\ l\neq 2n+1-u(2\xi+1)}}^{2n} (-1)^l \binom{4n+2}{l} = \frac{1}{2}\binom{4n+2}{2n+1} + \sum_{u=u_1}^{u_2} (-1)^u \binom{4n+2}{2n+1-u(2\xi+1)}$$

$$\sum_{l=0}^{2n} \binom{4n+2}{l} \sum_{j=1}^{\xi} \cos\left(\left(2n+1-l\right)\vartheta_j\right) = \frac{1}{4}\binom{4n+2}{2n+1} + \left(\frac{1}{2}+\xi\right) \sum_{u=u_1}^{u_2} (-1)^u \binom{4n+2}{2n+1-u(2\xi+1)}$$

$$\left|X_n\right|_2^2 = \frac{1}{2^{4n+2}} \frac{\xi}{2\xi+1}\binom{4n+2}{2n+1} + \frac{1}{2^{4n+3}} \frac{1}{2\xi+1}\binom{4n+2}{2n+1} + \frac{1}{2^{4n+2}} \sum_{u=u_1}^{u_2} (-1)^u \binom{4n+2}{2n+1-u(2\xi+1)}$$

and finally:

$$\left|X_n\right|_2^2 = \frac{1}{2^{4n+3}}\binom{4n+2}{2n+1} + \frac{1}{2^{4n+2}} \sum_{u=u_1}^{u_2} (-1)^u \binom{4n+2}{2n+1-u(2\xi+1)}$$

which proves Proposition 14.

8. Consequences

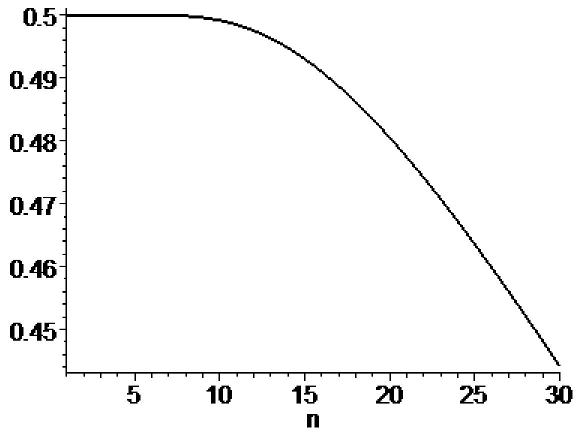
We deduce the probability that, at step $2n$, both players have equal fortune; this is the value of $e\left(2n,0\right)$.

Corollary 15. - *For each* n, *we have:*

$$e(2n,0)=\frac{2}{2\xi+1}\sum_{j=1}^{\xi}\cos^{2n}\frac{\vartheta_j}{2}=\frac{1}{2^{2n-1}}\binom{2n-1}{n}+\frac{1}{2^{2n-1}}\sum_{u\geq 1}(-1)^n\binom{2n+1}{n+1-u(2\xi+1)}$$

Proof of Corollary 15

We saw that $e(2n,0)=f(n,0)=x(n-1,1)$; the result follows from Theorem 12 and Proposition 13.



The total energy (in x), for $\xi=7$ *and* $n=1,...,30$

Recall that a sequence a_k is said to be concave if, for all k, $a_k \geq \frac{1}{2}(a_{k-1}+a_{k+1})$.

Proposition 16. – *At each time, the energy profile is concave.*

Proof of Proposition 16

Let us compute, for fixed n:

$$d_k = x(n,k)-x(n,k+1)=\frac{1}{\xi+\frac{1}{2}}\sum_{j=1}^{\xi}\left(\cos\left(\frac{(2k-1)\vartheta_j}{2}\right)-\cos\left(\frac{(2k+1)\vartheta_j}{2}\right)\right)\cos^{2n+1}\frac{\vartheta_j}{2}$$

Using the identity $\cos p-\cos q=-2\sin\frac{p+q}{2}\sin\frac{p-q}{2}$, we obtain:

$$d_k=\frac{2}{\xi+\frac{1}{2}}\sum_{j=1}^{\xi}\sin(k\vartheta_j)\sin(\vartheta_j)\cos^{2n+1}\frac{\vartheta_j}{2}$$

and:

$$d_k - d_{k+1} = \frac{2}{\xi + \frac{1}{2}} \sum_{j=1}^{\xi} \left(\sin\left(k\vartheta_j\right) - \sin\left(\left(k+1\right)\vartheta_j\right) \right) \sin\left(\vartheta_j\right) \cos^{2n+1} \frac{\vartheta_j}{2}$$

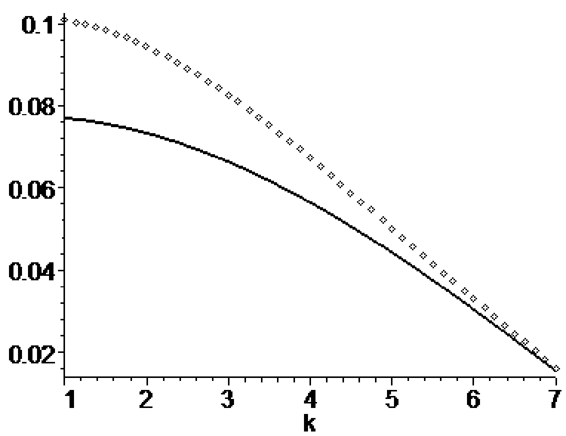
Using the identity $\sin p - \sin q = 2\sin\frac{p-q}{2}\cos\frac{p+q}{2}$, we obtain:

$$d_k - d_{k+1} = \frac{-4}{\xi + \frac{1}{2}} \sum_{j=1}^{\xi} \sin\frac{\vartheta_j}{2} \cos\left(\left(2k+1\right)\frac{\vartheta_j}{2} \right) \sin\left(\vartheta_j\right) \cos^{2n+1} \frac{\vartheta_j}{2} < 0 .$$

This shows that $d_k < d_{k+1}$, or $x\left(n,k\right) - x\left(n,k+1\right) < x\left(n,k+1\right) - x\left(n,k+2\right)$, that is:

$$x\left(n,k\right) + x\left(n,k+2\right) < 2x\left(n,k+1\right)$$

which proves our claim.



The energy profile (in x), for $\xi = 7, n = 30$ (point style) and $n = 50$ (line style)

As we already said, the angle $\frac{\vartheta_j}{2}$ satisfies $0 < \frac{2j-1}{2\xi+1}\frac{\pi}{2} < \frac{\pi}{2}$ and is increasing as a function of j. Therefore, $\sin\frac{\vartheta_j}{2}$ is positive and increasing, $\cos\frac{\vartheta_j}{2}$ is positive and decreasing, and so is $\cos^{2n+2}\frac{\vartheta_j}{2}$. The series $\sum_{j=1}^{\xi} \left(-1\right)^{j-1} \frac{\cos^{2n+2}\frac{\vartheta_j}{2}}{\sin\frac{\vartheta_j}{2}}$ has alternate signs, and its general term is decreasing. So, we have the general estimates:

Corollary 17. – *The energy at each step satisfies:*

$$\frac{1}{2\xi+1}\left(\frac{\cos^{2n+2}\left(\frac{1}{2\xi+1}\frac{\pi}{2}\right)}{\sin\left(\frac{1}{2\xi+1}\frac{\pi}{2}\right)}-\frac{\cos^{2n+2}\left(\frac{3}{2\xi+1}\frac{\pi}{2}\right)}{\sin\left(\frac{3}{2\xi+1}\frac{\pi}{2}\right)}\right)<\left|X_n\right|_1<\frac{1}{2\xi+1}\frac{\cos^{2n+2}\left(\frac{1}{2\xi+1}\frac{\pi}{2}\right)}{\sin\left(\frac{1}{2\xi+1}\frac{\pi}{2}\right)}$$

We state the upper estimate in a form which will be helpful to us in Part IV. In the results presented above, we started with an energy equal to 1, put at the origin, in the standard coordinates; it became an energy equal to $\frac{1}{2}$ in the x coordinates. If we want to investigate the total loss, we should start with an energy equal to 1, at the origin, in the x coordinates. So we get:

Corollary 17b. - *Assume that the barrier is set at $2\xi+1$; let $\lambda_1=\cos^2\left(\frac{1}{2\xi+1}\frac{\pi}{2}\right)$ be the first eigenvalue of the matrix M in this dimension. Assume we start with an energy equal to 1 at the origin. Then, at any step n, the total remaining energy satisfies the estimate:*

$$E_n\le\frac{4}{\pi}\lambda_1^n$$

Proof of Corollary 17b

If we start with an energy equal to 1, we have, by Corollary 17:

$$E_n<\frac{2}{2\xi+1}\frac{\cos^2\left(\frac{1}{2\xi+1}\frac{\pi}{2}\right)}{\sin\left(\frac{1}{2\xi+1}\frac{\pi}{2}\right)}\lambda_1^n$$

The function $f(x)=x\frac{\cos^2\left(\frac{\pi}{2}x\right)}{\sin\left(\frac{\pi}{2}x\right)}$ is decreasing on the interval $0<x<\frac{1}{3}$, so the maximum value is obtained when $x\to 0$, that is $n\to+\infty$, and this limit is $\frac{2}{\pi}$. This proves Corollary 17b.

If we want to obtain estimates about the energy $e(n,k)$ instead of $x(n,k)$, we come back to system (2.1):

$$\begin{cases} f(n+1,0) = x(n,1) \\ f(n+1,k) = \frac{1}{2}\left(x(n,k) + x(n,k+1)\right), k = 1,\ldots,\xi - 1 \\ f(n+1,\xi) = \frac{1}{2}x(n,\xi) \end{cases}$$

which gives:

$$\sum_{j=1}^{\xi} f(n+1,j) = \frac{3}{2}x(n,1) + \sum_{j=2}^{\xi} x(n,j) \le \frac{3}{2}\sum_{j=1}^{\xi} x(n,j)$$

and, the same way:

$$\begin{aligned} f(n+1,0)^2 + \sum_{j=1}^{\xi-1} f(n+1,j)^2 + f(n,\xi)^2 &= x(n,1)^2 + \frac{1}{4}\sum_{j=1}^{\xi-1}\left(x(n,j) + x(n,j+1)\right)^2 + \frac{1}{4}x(n,\xi)^2 \\ &\le x(n,1)^2 + \frac{1}{2}\sum_{j=1}^{\xi-1} x(n,j)^2 + \frac{1}{2}\sum_{j=1}^{\xi-1} x(n,j+1)^2 + \frac{1}{4}x(n,\xi)^2 \\ &\le \frac{3}{2}|X_n|_2^2 \end{aligned}$$

Corollary 18. - *Asymptotically when* $n \to +\infty$, *we have the estimate, with* $\vartheta_1 = \frac{\pi}{2\xi+1}$:

$$|X_n|_1 \sim \frac{1}{2\xi+1} \frac{\cos^{2n+2}\frac{\vartheta_1}{2}}{\sin\frac{\vartheta_1}{2}}$$

For the energy $e(2n,j)$, *we have the estimates:*

$$e(2n+2,0) \sim \frac{2}{2\xi+1}\cos^{2n+2}\frac{\vartheta_1}{2}$$

and, for $j = 1,\ldots,\xi-1$*:*

$$e(2n+2,2j) = \frac{2}{2\xi+1}\sin\left((\xi-j)\vartheta_1\right)\cos^{2n+2}\frac{\vartheta_1}{2}$$

and for $j = \xi$ *:*

$$e(2n+2,2\xi) \sim \frac{2}{2\xi+1}\sin(\vartheta_1)\cos^{2n+1}\frac{\vartheta_1}{2}$$

and the total energy satisfies:

$$\sum_{j=0}^{\xi} e(2n+2,j) \sim \frac{1}{\xi+\frac{1}{2}} \cos^{2n+1}\frac{\vartheta_1}{2}\left(\cos\frac{\vartheta_1}{2}+\cos\frac{\vartheta_1}{2}\sum_{j=1}^{\xi-1} \sin(j\vartheta_1)+\sin(\vartheta_1)\right).$$

Proof of Corollary 18

The first part follows immediately from Proposition 16.

Returning to the energy $f(n,j)=e(2n,2j)$, we find:

$$e(2n+2,0)=x(n,1)\sim \lambda_1^n \frac{\sin^2(\xi\vartheta_1)}{\xi+\frac{1}{2}} = \frac{\sin^2(\xi\vartheta_1)}{\xi+\frac{1}{2}}\cos^{2n}\frac{\vartheta_1}{2}$$

But $\sin(\xi\vartheta_1)=\cos\frac{\vartheta_1}{2}$, which gives the announced formula. For $j=1,\dots,\xi-1$:

$$e(2n+2,2j)=\frac{1}{2}\big(x(n,j)+x(n,j+1)\big)\sim \frac{\lambda_1^n \sin(\xi\vartheta_1)}{\xi+\frac{1}{2}}\frac{\sin\big((\xi-j+1)\vartheta_1\big)+\sin\big((\xi-j)\vartheta_1\big)}{2}$$

$$=\frac{\lambda_1^n \sin(\xi\vartheta_1)}{\xi+\frac{1}{2}}\sin\big((\xi-j)\vartheta_1\big)\cos\frac{\vartheta_1}{2}=\frac{2}{2\xi+1}\cos^{2n+2}\frac{\vartheta_1}{2}\sin\big((\xi-j)\vartheta_1\big)$$

and finally:

$$e(2n+2,2\xi)=\frac{1}{2}x(n,\xi)\sim \lambda_1^n \frac{\sin(\vartheta_1)\sin(\xi\vartheta_1)}{\xi+\frac{1}{2}}=\frac{2}{2\xi+1}\cos^{2n+1}\frac{\vartheta_1}{2}\sin(\vartheta_1)$$

which proves Corollary 18. We see that, asymptotically when $n\to+\infty$, the energy is carried by the first eigenvector.

9. Domination principle

Proposition 19. – *Assume that* X_0, Y_0 *are two initial distributions of energy, on the vertical* $x=0$, *satisfying* $X_0(k)\geq Y_0(k)$, *for* $k=1,\dots,\xi$. *Let* X_n, Y_n *be the corresponding energies, at time* n. *Then, for all* k, $X_n(k)\geq Y_n(k)$.

What this proposition says is that, if an energy distribution dominates another one at the initial stage, it will dominate it at any further stage.

Proof of Proposition 19

This is obvious: $X_n - Y_n = M^n (X_0 - Y_0)$; the vector $X_0 - Y_0$ has all its components which are positive, and the matrix M^n has only positive entries.

Corollary 20. - *The energy at each step satisfies the estimates:*

$$x(n,0) \le \frac{1}{2}\lambda_1^n \quad \textit{and} \quad x(n,k) \le \frac{\lambda_1^n}{4}\frac{\sin((\xi-k)\vartheta_1)}{\sin(\xi\vartheta_1)} \quad \textit{for } k = 1,\dots,\xi.$$

with $\lambda_1 = \cos^2 \frac{\vartheta_1}{2}$.

Proof of Corollary 20

Consider the first eigenvector

$V_1 = (\sin(\xi\vartheta_1), \sin((\xi-1)\vartheta_1), \dots, \sin(\vartheta_1))$ with $\vartheta_1 = \frac{\pi}{2\xi+1}$

and let us normalize it differently. We consider:

$$W_1 = \left(\frac{1}{2}, \frac{\sin((\xi-1)\vartheta_1)}{2\sin(\xi\vartheta_1)}, \dots, \frac{\sin(\vartheta_1)}{2\sin(\xi\vartheta_1)}\right)$$

All components are >0, and it dominates the initial vector $X_0 = \left(\frac{1}{2}, 0, \dots, 0\right)$. By Proposition 19, the same will hold at each step. But W_1 is an eigenvector, with respect to the eigenvalue λ_1, so we get:

$$x(n,0) \le \frac{1}{2}\lambda_1^n$$

$$x(n,k) \le \frac{\lambda_1^n}{4}\frac{\sin((\xi-k)\vartheta_1)}{\sin(\xi\vartheta_1)} \quad \text{for } k = 1,\dots,\xi.$$

as we announced. This proves Corollary 20.

We also deduce from the domination principle the simple estimate:

$$|X_n|_1 \le \frac{\lambda_1^n}{\sin\left(\frac{1}{2\xi+1}\frac{\pi}{2}\right)}$$

Indeed, $\left|W_1\right|_1 = \dfrac{2}{\sin\left(\xi\vartheta_1\right)}\left|V_1\right|_1 = \dfrac{1}{\cos\left(\xi\vartheta_1\right)} = \dfrac{1}{\sin\dfrac{\pi}{2\xi+1}}$. But this estimate is weaker than the one given by Corollary 19.

V. Counting the paths

Let us see what our result implies, in terms of number of paths at a given time. We recall that the barrier has been set at $\pm(2\xi+1)$.

We have to refer to the original settings of a ± 1 RW, because in our "all-even approach", where a path may go up 2 points, stay at the same level, or go down 2 points, all paths do not have the same probability (to stay at the same level is twice more likely than going up or down). So, we distinguish between:

- The Random Walk (RW), taking values ± 1 with probability $\frac{1}{2}$;
- The 3-values Random Process (3VRP), taking values $+2$ with probability $\frac{1}{4}$, 0 with probability $\frac{1}{2}$, -2 with probability $\frac{1}{4}$.

A "path" is, by definition, a set of values taken by the RW.

The maximal value a path may take (without being annihilated by the barrier) is 2ξ and the minimal value is -2ξ; in other words, we may say that the remaining paths are "confined" in the strip $[-2\xi, 2\xi]$.

If M is the matrix, of size $\xi \times \xi$:

$$M = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & \cdots & \cdots & \cdots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \cdots & \cdots & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \cdots & \cdots & \cdots & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

and $X_0 = \left(\frac{1}{2}, 0, \ldots, 0\right)$, let $X_{n-1} = M^{n-1} X_0$. Then, the system:

$$\begin{cases} e(2n,0) = f(n,0) = x(n-1,1) \\ e(2n,\pm 2k) = f(n,k) = \frac{1}{2}x(n-1,k) + \frac{1}{2}x(n-1,k+1),\ k = 1,\ldots,\xi - 1 \text{ (if } \xi \geq 2\text{)} \\ e(2n,\pm 2\xi) = f(n,\xi) = \frac{1}{2}x(n-1,\xi) \end{cases}$$

may be represented by a matrix P_0 with $\xi + 1$ rows and ξ columns:

$$P_0 = \begin{pmatrix} 1 & 0 & \cdots & & \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \vdots \\ & \cdots & 0 & \frac{1}{2} & \frac{1}{2} \\ & & \cdots & 0 & \frac{1}{2} \end{pmatrix}$$

Let $F_n = P_0 M_{n-1} X_0$; this is the column-vector of the $f(n,k)$.

The total number of paths, confined in the strip $[-2\xi, 2\xi]$, reaching the vertical W_{2n}, is therefore:

$$Nb(paths) = f(n,0) + 2\sum_{k=1}^{\xi-1} f(n,k).$$

Example: Assume $\xi = 2$ and $n = 8$; the number of paths reaching the vertical W_{16}, confined in the strip $[-4, 4]$, is 38 000.

VI. References

[Kalbfleisch] Probability and Statistical Inference, volume 1: Probability. Springer Texts in Statistics, 1985.

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