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# Simple Random Walks in the plane :

# An energy based approach

# Part II : Bounded Initial Fortunes

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**Abstract**

We consider a $\pm 1$ game, with initial bounded fortunes. Using a new, energy-based, approach, we investigate the value of the fortune of each player after $n$ games. We give a complete description of each possible value and its probability ; the tools used are Chebychev's polynomial of first and second kind, operator theory and trigonometry. We also investigate the asymptotic behavior when $n \to +\infty$ : the profile of each fortune is concave and the total energy tends to 0 exponentially fast.

This behaviour is quite different from the case of unbounded initial fortunes, or bounded fortune for one of the players only.

# I. Presentation

The basic settings are defined in [1]. Recall that we consider a simple random walk in the plane: a game, with two players. It is defined by a r.v. $Z$ taking the values $\pm 1$ with probability $\frac{1}{2}$. The player $A$ wins if $Z=1$ and then he receives 1 Euro from the player $B$, and conversely if $Z=-1$. Both players have an initial fortune denoted by $F$, same for both. The game stops if any of the players sees his fortune equal to zero. The question is: what is the probability distribution of the earnings after $n$ steps ?

# II. Notation

We refer to [1]. We set $S_0=0$, and for $n \geq 1$, $S_n = \sum_{i=1}^{n} Z_i$. Instead of a random walk with multiple possible paths, we consider that we have the propagation of an energy, with the following rules :

- At time $n=0$, the origin receives an energy equal to 1 ;
- At time $n=1$, this energy is divided into two: each point $(1,0)$ and $(-1,0)$ receives an energy equals to $\frac{1}{2}$ and so on.

More generally, the energy of a point of coordinates $(i,j)$ in the plane is equal to the probability that the random walk hits this point. It will be denoted by $e(i,j)$.

We insert the two barriers $y=F$ and $y=-F$ and we consider that if a path hits any of the barriers, its energy is absorbed and disappears. So, our original question may be stated as: Given a time $n$, what is the distribution of energy on the vertical $x=n$ ?

As we already did in [1], we restrict ourselves to even values of the time $(2n)$. Conversely, the barrier will be set at an odd value : $F = \pm(2\xi+1)$. We may also restrict ourselves to the upper half-plane, since, by symmetry, the results are identical in the lower half plane.

Let $W_{2n}$ be the vertical for $x=2n$, that is the set of all points $A_{2n,2k}$, $k=-n,\dots,n$. We introduce :

- The total energy on the vertical : $E(W_{2n}) = \sum_{k=-n}^{n} e(2n,2k)$
- The quadratic energy : $E_2(W_{2n}) = \left( \sum_{k=-n}^{n} \left(e(2n,2k)\right)^2 \right)^{1/2}$

For any vector $V=(v_1,\dots,v_N)$, we will use the following classical norms:

$$|V|_1=\sum_{j=1}^{N}|v_j|$$

$$|V|_2=\sqrt{\sum_{j=1}^{N}v_j^2}$$

# III. Statement of Results

We will prove the following:

*A. Case* $\xi=1$

**Theorem 1. -** *For* $\xi=1$ *(barrier at* $\pm 3$*), the energy is :*

$$e(2n,2)=e(2n,-2)=\frac{1}{4}\left(\frac{3}{4}\right)^{n-1},\ e(2n,0)=\frac{1}{2}\left(\frac{3}{4}\right)^{n-1}$$

*and so the total energy at time* $2n$ *is* $E(2n)=\left(\frac{3}{4}\right)^{n-1}$ *and the quadratic energy is* $E_2(2n)=\left(\frac{3}{4}\right)^{n-1}\sqrt{\frac{3}{8}}$.

*B. General case : Stepwise estimates*

**Theorem 2.** - *At each step* $2n$, *the energy* $e(2n,2i)=f(n,i)$, $i=0,\dots,\xi$, *is given by:*

$$\begin{cases} f(n,0)=x(n-1,1) \\ f(n,i)=\frac{1}{2}\big(x(n-1,i)+x(n-1,i+1)\big),\ i=1,\dots,\xi-1 \\ f(n,\xi)=\frac{1}{2}x(n-1,\xi) \end{cases}$$

where, for $i=1,\dots,\xi$:

$$x(n-1,i)=\frac{2}{2\xi+1}\sum_{j=1}^{\xi}\cos\left(\frac{(2i-1)\vartheta_j}{2}\right)\cos^{2n-1}\frac{\vartheta_j}{2}$$

and $\vartheta_j=\dfrac{(2j-1)\pi}{2\xi+1}$, $j=1,\dots,\xi$.

*C. Decrease of the energy as n increases.*

**Theorem 3.** - *The energy* $E\left(W_{2n}\right)$ *is exponentially decreasing when* $n \to +\infty$. *More precisely, it satisfies the estimate:*

$$E\left(W_{2n}\right) \leq \frac{\sqrt{\xi}\sqrt{15}}{2}\left(1-\frac{\pi^2}{16\,\xi^2}\right)^{n-2}$$

*and the quadratic energy satisfies:*

$$E_2\left(W_{2n}\right) \leq \frac{\sqrt{15}}{2}\left(1-\frac{\pi^2}{16\,\xi^2}\right)^{n-2}$$

*D. Asymptotic profile of the energy*

**Theorem 4.** - *For each* $j$, *the limit* $g_j = \lim_{n\to+\infty} \dfrac{e(2n,2j)}{e(2n,0)}$ *exists and satisfies, with* $\vartheta_1 = \dfrac{\pi}{2\xi+1}$ *:*

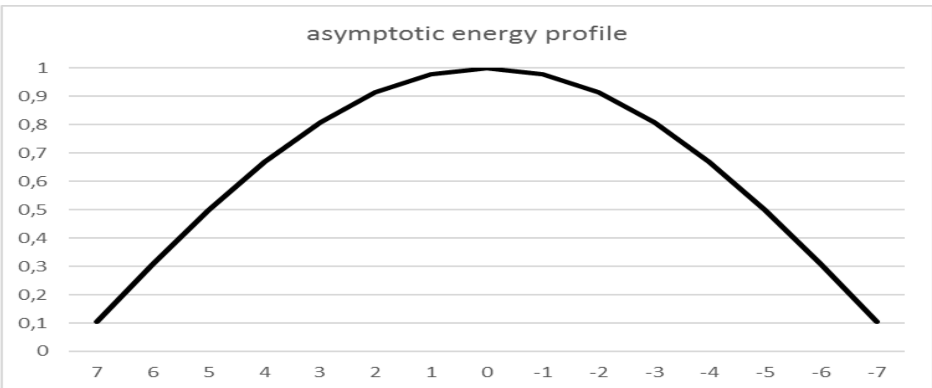
*for* $j = 1,\dots,\xi-1$ *:* $g_j = \sin\left(\left(\xi - j\right)\vartheta_1\right)$

*for* $j = \xi$ *:* $g_\xi = 2\sin\dfrac{\vartheta_1}{2}$

*The sequence* $g_j$ *is concave, which means that :*

$$g_j \geq \frac{1}{2}g_{j-1} + \frac{1}{2}g_{j+1}.$$

Here is an example of an asymptotic energy profile. The barrier is at $\pm 8$. What is drawn is the proportion $g_j$, $-7 \leq j \leq 7$.



We observe that these results are quite different from the case of no barrier (where $e(2n,0) \le \frac{c}{\sqrt{n}}$) or a single barrier (where $e(2n,0) \le \frac{c_\xi}{n^{3/2}}$).

# IV. Proofs of the results

We divide our presentation into several paragraphs.

### 1. Case $\xi = 1$

The barrier is at the value $\pm 3$. At the time $n = 2$, the energy distribution is: $e(2,2) = \frac{1}{4}$, $e(2,0) = \frac{1}{2}$, $e(2,-2) = \frac{1}{4}$ (the barrier plays no role). We set:

$$f(n,i) = e(2n,2i)$$

So we get:

$$f(1,1) = \frac{1}{4},\ f(1,0) = \frac{1}{2}$$

and we have the recurrence relations:

$$f(n+1,0) = \frac{1}{2}\left(f(n,0) + f(n,1)\right),\ f(n+1,1) = \frac{1}{4}\left(f(n,0) + f(n,1)\right)$$

Set:

$$x_n = \frac{1}{2}\left(f(n,0) + f(n,1)\right)$$

Then:

$$x_1 = \frac{1}{2}\left(f(1,0) + f(1,1)\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{4}\right) = \frac{3}{8}$$

We obtain the equations:

$f(n+1,0) = x_n$, $f(n+1,1) = \frac{1}{2}x_n$, and $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{2}x_n\right) = \frac{3}{4}x_n$, which gives $x_n = \frac{1}{2}\left(\frac{3}{4}\right)^n$.

So the energy profile at time $2n$ is:

$$e(2n,0) = f(n,0) = x_{n-1} = \frac{1}{2}\left(\frac{3}{4}\right)^{n-1}$$

$$e(2n,1) = e(2n,-1) = f(n,1) = \frac{1}{2}x_{n-1} = \frac{1}{4}\left(\frac{3}{4}\right)^{n-1}$$

The total energy at the instant $2n$ is the sum of all terms, that is:

$$E(2n) = \left(\frac{3}{4}\right)^{n-1}$$

The quadratic energy is computed the same way, and Theorem 1 is proved. The same definitions of $f(n,i)$ and $x(n,i)$ will be used later.

## 2. A first change of variables

We now investigate the general case: $\xi \geq 2$. We set, as before:

$$f(n,i) = e(2n,2i)$$

with the initial values:

$f(0,0) = 1$, $f(0,i) = 0$ for $i = 1,\ldots,\xi$.

The recurrence equations are:

$$\begin{cases} f(n+1,0) = \frac{1}{2}\left(f(n,0) + f(n,1)\right) \\ f(n+1,i) = \frac{1}{4}\left(f(n,i-1) + 2f(n,i) + f(n,i+1)\right), i = 1,\ldots,\xi - 1 \\ f(n+1,\xi) = \frac{1}{4}\left(f(n,\xi-1) + f(n,\xi)\right) \end{cases} \tag{2.1}$$

(Recall that the barrier is set at $\pm(2\xi+1)$, so the last non-zero value for $f$ on each vertical is $f(n,\xi)$.)

We now study the variation of energy, at a given time, on each vertical.

## 3. Decrease of the energy on each vertical

**Lemma 3.1.** - *For a given time* $n$, *the energy is decreasing as a function of* $i$ *:*

$$f(n,i) \geq f(n,i+1),\ i \geq 0.$$

**Proof of Lemma 3.1**

This is true for $n=0$ ; let us admit the result for $n$ and prove it for $n+1$.

We have:

$$f(n+1,1)=\frac{1}{4}f(n,0)+\frac{1}{2}f(n,1)+\frac{1}{4}f(n,2)\leq\frac{1}{2}f(n,0)+\frac{1}{2}f(n,1)=f(n+1,0)$$

since $f(n,2)\leq f(n,0)$ by the recurrence assumption. For $1\leq i\leq\xi-2$:

$$\begin{aligned}f(n+1,i+1)&=\frac{1}{4}f(n,i)+\frac{1}{2}f(n,i+1)+\frac{1}{4}f(n,i+2)\\&\leq\frac{1}{4}f(n,i-1)+\frac{1}{2}f(n,i)+\frac{1}{4}f(n,i+1)\\&=f(n+1,i)\end{aligned}$$

Finally, the property $f(n+1,\xi)\leq f(n+1,\xi-1)$ comes from:

$$\frac{1}{4}f(n,\xi-1)+\frac{1}{4}f(n,\xi)\leq\frac{1}{4}f(n,\xi-2)+\frac{1}{2}f(n,\xi-1)+\frac{1}{4}f(n,\xi)$$

which is clear. So Lemma 3.1 is proved.

**Corollary 3.2**. - *Let $m<n$ be two instants; let $A(m,i)$ $i=0,\dots,\xi$ be points on the $m^{th}$ vertical and let $B=B(n,0)$ be the point on the $x$ axis at time $n$. Assume we put energy 1 at one of the points $A(m,i)$ $i=0,\dots,\xi$. The energy received by $B$ will be maximal if this energy is put at $A(m,0)$. In fact, the energy received by $B$ is a decreasing function of $i$.*

**Proof of Corollary 3.2**

This is a simple consequence of Lemma 3.1, because if we put energy 1 at $A(m,i)$, the energy received by $B$ is the same as the energy received by $A(m,i)$ if we put energy 1 at $B$.

**Corollary 3.3.** - *Assume we have any distribution of energy $E_m$ on the vertical $W_m$. Then the energy received by $B$ will be larger if all this energy is concentrated at the single point $A_0$.*

This is a clear consequence of the previous Corollary. There is a more general statement:

**Corollary 3.4.** - *Let* $m<n$ *be two instants, and let* $f_1(m,i), f_2(m,i)$ *be two distributions of energy on the vertical* $W_m$. *Assume that the first one is more concentrated near the origin, which means that, for all* $k=0,\dots,\xi$ *:*

$$\sum_{i=0}^{k} f_1(m,i) \geq \sum_{i=0}^{k} f_2(m,i)$$

*Then, for any* $n>m,$ *on the vertical* $W_n$, *the energy coming from the first distribution is larger than the second, which means that the loss of energy is larger in the second case.*

This corollary is quite intuitive. The second distribution is globally closer to the barrier, so the loss of energy is larger. Another way to say this is as follows: take any distribution of energy, and move any quantity closer to the $x$ axis: this is a "protective" move, in the sense that there will be less loss of energy.

We now turn to the behaviour on the horizontal direction

## 4. Decrease of the energy with time

**Lemma 4.1** - *On the* $x$ *axis, the energy is decreasing: for all* $n,$

$$f(0,n) \geq f(0,n+1).$$

**Proof of Lemma 4.1**

We have $f(0,0)=1$ and $f(1,0)<1$; let us admit the decrease until step $n$ and prove it at step $n+1$. We have $f(n,0)-f(n+1,0)=\frac{1}{2}\left(f(n,0)-f(n,1)\right)>0$ by Lemma 3.1. This proves Lemma 4.1.

However, it is not true that the energy is decreasing on all horizontal lines $y=j$ ; indeed, if $j>1,$ it first increases and then decreases.

## 5. A second change in coordinates

We set, for any $n \geq 1$ and $i \geq 2$ :

$$x(n,i)=\frac{1}{2}\left(f(n,i-1)+f(n,i)\right)$$

This definition makes sense even if there is no barrier. We have $x(n,i)=0$ if $i>n+1$.

**Lemma 5.1.** - *If there is no barrier, we have, for any* $n,$ $i \leq n+1$ *:*

$$x(n,i)=\frac{1}{2^{2n+1}}\binom{2n+1}{n+i}$$

**Proof of Lemma 5.1**

Indeed, we have:

$$\begin{aligned}x(n,i)&=\frac{1}{2}\left(f(n,i-1)+f(n,i)\right)=\frac{1}{2}\left(e(2n,2i-2)+e(2n,2i)\right)\\&=\frac{1}{2^{2n+1}}\left(\binom{2n}{n+i-1}+\binom{2n}{n+i}\right)\\&=\frac{1}{2^{2n+1}}\binom{2n+1}{n+i}\end{aligned}$$

using Pascal's formula. This proves Lemma 5.1.

**Lemma 5.2.** - *Let* $X_n=\left(x(n,1),\ldots,x(n,n)\right)$. *We have :*

$$\left|X_n\right|_1=\frac{1}{2} \text{ for all } n$$

$$\left|X_n\right|_2^2=\frac{1}{4^{2n+1}}\sum_{i=1}^{n}\binom{2n+1}{n+i}^2=\frac{1}{2^{4n+3}}\binom{4n+2}{2n+1}$$

Asymptotically, when $n\to+\infty$:

$$\left|X_n\right|_2^2\sim\frac{1}{2\sqrt{2\pi}}\frac{1}{\sqrt{n}}$$

**Proof of Lemma 5.2**

The first statement is just the sum of binomial coefficients. The second is an identity about the sum of squares of binomial coefficients, and the third is an application of Stirling's formula.

From now on, we will work mostly with the new coordinates.

## 6. The propagation problem in the new coordinates

Equations (2.1) become:

$$\begin{cases} f\left(n+1,0\right)=x\left(n,1\right) \\ f\left(n+1,i\right)=\dfrac{1}{2}\left(x\left(n,i\right)+x\left(n,i+1\right)\right),\, i=1,\dots,\xi-1 \\ f\left(n+1,\xi\right)=\dfrac{1}{2}x\left(n,\xi\right) \end{cases} \tag{6.1}$$

We observe that:

$$E_1\left(W_{n+1}\right)=\sum_{i=0}^{\xi}e\left(2n,2i\right)=\sum_{i=0}^{\xi}f\left(n,i\right)\leq\frac{3}{2}\sum_{i=1}^{\xi}x\left(n,i\right)=\frac{3}{2}\left|X_n\right|_1 \tag{6.2}$$

and:

$$E_2^2\left(W_{n+1}\right)=\sum_{i=0}^{\xi}f^2\left(n,i\right)\leq\frac{3}{2}\sum_{i=1}^{\xi}\left(x\left(n,i\right)\right)^2=\frac{3}{2}\left|X_n\right|_2^2 \tag{6.3}$$

From (6.1), we deduce:

$$\begin{aligned} x\left(n+1,1\right)&=\frac{1}{2}\left(f\left(n+1,0\right)+f\left(n+1,1\right)\right)=\frac{1}{2}\left(x\left(n,1\right)+\frac{1}{2}\left(x\left(n,1\right)+x\left(n,2\right)\right)\right) \\ &=\frac{3}{4}x\left(n,1\right)+\frac{1}{4}x\left(n,2\right) \\ x\left(n+1,i\right)&=\frac{1}{2}\left(\frac{1}{2}\left(x\left(n,i-1\right)+x\left(n,i\right)\right)+\frac{1}{2}\left(x\left(n,i\right)+x\left(n,i+1\right)\right)\right) \\ &=\frac{1}{4}x\left(n,i-1\right)+\frac{1}{2}x\left(n,i\right)+\frac{1}{4}x\left(n,i+1\right) \end{aligned}$$

for $i=2,\dots,\xi-1$, and:

$$x\left(n+1,\xi\right)=\frac{1}{4}x\left(n,\xi-1\right)+\frac{1}{2}x\left(n,\xi\right)$$

So we have the system:

$$\begin{cases} x\left(n+1,1\right)=\dfrac{3}{4}x\left(n,1\right)+\dfrac{1}{4}x\left(n,2\right) \\ x\left(n+1,i\right)=\dfrac{1}{4}x\left(n,i-1\right)+\dfrac{1}{2}x\left(n,i\right)+\dfrac{1}{4}x\left(n,i+1\right),\text{for } i=2,\dots,\xi-1 \\ x\left(n+1,\xi\right)=\dfrac{1}{4}x\left(n,\xi-1\right)+\dfrac{1}{2}x\left(n,\xi\right) \end{cases} \tag{6.4}$$

with the initial values:

$x(0,1)=\frac{1}{2}\left(f(0,0)+f(0,1)\right)=\frac{1}{2}$, $x(0,i)=\frac{1}{2}\left(f(0,i-1)+f(0,i)\right)=0$ for $i\geq 2$.

These initial values may be written as a vector:

$$X_0=\left(\frac{1}{2},0,\dots,0\right).$$

The system of equations (6.4) may be written as a matrix, under the form:

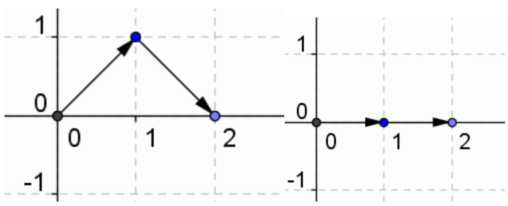
$$\begin{pmatrix} x(n+1,1) \\ x(n+1,2) \\ \dots \\ \dots \\ \dots \\ x(n+1,\xi-1) \\ x(n+1,\xi) \end{pmatrix} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & \dots & \dots & \dots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \dots & \dots & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \dots & \dots & \dots & 0 & \frac{1}{4} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x(n,1) \\ x(n,2) \\ \dots \\ \dots \\ \dots \\ x(n,\xi-1) \\ x(n,\xi) \end{pmatrix} \qquad (6.5)$$

We have a real symmetric matrix, of size $\xi$, which is denoted by $M$.

We observe that, in this matrix representation, things are opposite to the physical representation: the first element of the vector $X$ and the first row of the matrix correspond to what happens on the $Ox$ axis; the last element of $X$ and the last row of the matrix correspond to what happens close to the barrier.

We may consider that this is also a propagation problem, with the following properties:

A point may move upwards, horizontally or downwards ; all horizontal arrows have probability $\frac{1}{2}$ except the first one (the one on the $x$ axis) which has probability $\frac{3}{4}$ ; all oblique arrows (up or down) have probability $\frac{1}{4}$. In this representation, two paths with same origin and same destination do not need to have the same probability. In the picture below, the left path has probability $\frac{1}{4^2}$ and the right path probability $\left(\frac{3}{4}\right)^2$.



Therefore, on the $x(n,i)$ coordinates, a matrix-oriented approach is appropriate, but an approach counting the number of paths is not.

## 7. Properties of the matrix *M*

**Lemma 7.1.** - *The matrix* $M$ *is positive defined.*

**Proof of Lemma 7.1**

We have to show that, for all non-zero column-vector $X$ of size $\xi$, we have:

$$X^t MX > 0$$

Let $X = \begin{pmatrix} x_1 \\ \vdots \\ x_\xi \end{pmatrix}$; we have:

$$MX = \begin{pmatrix} \frac{3}{4}x_1 + \frac{1}{4}x_2 \\ \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3 \\ \vdots \\ \frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1} \\ \vdots \\ \frac{1}{4}x_{\xi-2} + \frac{1}{2}x_{\xi-1} + \frac{1}{4}x_\xi \\ \frac{1}{4}x_{\xi-1} + \frac{1}{2}x_\xi \end{pmatrix}$$

and therefore:

$$\begin{aligned} X^t MX &= \left(\frac{3}{4}x_1 + \frac{1}{4}x_2\right)x_1 + \left(\frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3\right)x_2 + \ldots + \left(\frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1}\right)x_i + \ldots + \\ &\qquad + \left(\frac{1}{4}x_{\xi-2} + \frac{1}{2}x_{\xi-1} + \frac{1}{4}x_\xi\right)x_{\xi-1} + \left(\frac{1}{4}x_{\xi-1} + \frac{1}{2}x_\xi\right)x_\xi \\ &= \frac{1}{4}x_1^2 + b \end{aligned}$$

with $b = \frac{1}{2}\left(x_1^2 + x_1x_2 + x_2^2 + x_2x_3 + x_3^2 + \cdots + x_{i-1}x_i + x_i^2 + \cdots + x_{\xi-2}x_{\xi-1} + x_{\xi-1}^2 + x_{\xi-1}x_\xi + x_\xi^2\right)$

and:

$$
\begin{aligned}
4b &= x_1^2 + x_1^2 + 2x_1x_2 + x_2^2 + x_2^2 + 2x_2x_3 + x_3^2 + \ldots + 2x_{i-1}x_i + x_i^2 + x_i^2 + \ldots + \\
&\qquad + 2x_{\xi-2}x_{\xi-1} + x_{\xi-1}^2 + x_{\xi-1}^2 + 2x_{\xi-1}x_\xi + x_\xi^2 + x_\xi^2 \\
&= x_1^2 + \left(x_1 + x_2\right)^2 + \left(x_2 + x_3\right)^2 + \ldots + \left(x_{i-1} + x_i\right)^2 + \ldots + \left(x_{\xi-1} + x_\xi\right)^2 + x_\xi^2
\end{aligned}
$$

So clearly $X^t MX > 0$ if the $x_i$ are not all equal to 0. This proves Lemma 7.1.

From Lemma 7.1 follows that all eigenvalues of $M$ are real and $> 0$ and that $M$ can be diagonalized in an orthogonal basis made of eigenvectors.

We now study the operator norm of $M$ :

**Proposition 7.2.** - *From $l_1$ into $l_1$, the operator norm of $M$ is 1. From $l_2$ into $l_2$, this norm is* $<1$.

**Proof of Proposition 7.2**

In order to prove the first part, let us take a vector $X$ with $l_1$ norm equal to 1. We want to find the maximum value of $|Y|_1$ with $Y = MX$. This maximum value is obtained when there is no cancellation, that is assuming that all coefficients of $X$ are positive and satisfy $\sum_{i=1}^{\xi} x_i = 1$. But then:

$$
\begin{aligned}
|Y|_1 &= \sum_{i=1}^{\xi} y_i = \frac{3}{4}x_1 + \frac{1}{4}x_2 + \sum_{i=2}^{\xi-1} \frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1} + \frac{1}{4}x_{\xi-1} + \frac{1}{2}x_\xi \\
&= x_1 + x_2 + \cdots + x_{\xi-1} + \frac{3}{4}x_\xi \le 1
\end{aligned}
$$

which proves our claim ; we have $|Y|_1 = |X|_1$ for any vector $X$ for which $x_\xi = 0$.

Let us now turn to the $l_2$ norm. We assume $\sum_{i=1}^{\xi} x_i^2 = 1$ and we want to prove that $\sum_{i=1}^{\xi} y_i^2 < 1$.

But, since $f(t) = t^2$ is a convex function, we have, for all $\alpha_i \ge 0$ with $\sum_i \alpha_i = 1$ :

$$
\left(\sum \alpha_i x_i\right)^2 \le \sum \alpha_i x_i^2
$$

and in particular :

$$
\left(\frac{3}{4}x_1 + \frac{1}{4}x_2\right)^2 \le \frac{3}{4}x_1^2 + \frac{1}{4}x_2^2
$$

$$\left(\frac{1}{4}x_{i-1}+\frac{1}{2}x_i+\frac{1}{4}x_{i+1}\right)^2 \le \frac{1}{4}x_{i-1}^2+\frac{1}{2}x_i^2\,\frac{1}{4}x_{i+1}^2$$

For the last term, $y_\xi = \frac{1}{4}x_{\xi-1}+\frac{1}{2}x_\xi$, the sum of the coefficients is $<1$, so we write:

$$\left(\frac{\frac{1}{4}x_{\xi-1}+\frac{1}{2}x_\xi}{3/4}\right)^2 \le \frac{1}{3}x_{\xi-1}^2+\frac{2}{3}x_\xi^2$$

and therefore:

$$\left(\frac{1}{4}x_{\xi-1}+\frac{1}{2}x_\xi\right)^2 \le \left(\frac{3}{4}\right)^2\left(\frac{1}{3}x_{\xi-1}^2+\frac{2}{3}x_\xi^2\right)=\frac{3}{16}x_{\xi-1}^2+\frac{6}{16}x_\xi^2<\frac{1}{4}x_{\xi-1}^2+\frac{1}{2}x_\xi^2$$

and we add up all terms as we did previously:

$$\sum_{i=1}^{\xi} y_i^2 \le \sum_{i=1}^{\xi-1} x_i^2+\frac{3}{4}x_\xi^2$$

which proves that the operator norm of $M$, from $l_2$ to $l_2$, is $\le 1$. If the operator norm was equal to 1, there would be a case where all inequalities above would be equalities, which implies $x_1=\cdots=x_{\xi-1}$ and $x_\xi=0$. But, for the vector $X=(1,1,\ldots,1,0)$ ; we have $|X|_2^2=\xi-1$ and $|Y|_2^2=\xi-2+\left(\frac{3}{4}\right)^2+\left(\frac{1}{4}\right)^2<\xi-1$.

**Lemma 7.3.** - *All eigenvalues of* $M$ *are* $<1$.

**Proof of Lemma 7.3**

Let us write the system of equations definining the eigenvalues and eigenvectors:

$$MX=\lambda X$$

$$\begin{cases} \frac{3}{4}x_1 + \frac{1}{4}x_2 = \lambda x_1 \\ \frac{1}{4}x_1 + \frac{1}{2}x_2 + \frac{1}{4}x_3 = \lambda x_2 \\ \vdots \\ \frac{1}{4}x_{i-1} + \frac{1}{2}x_i + \frac{1}{4}x_{i+1} = \lambda x_i \\ \vdots \\ \frac{1}{4}x_{\xi-2} + \frac{1}{2}x_{\xi-1} + \frac{1}{4}x_\xi = \lambda x_{\xi-1} \\ \frac{1}{4}x_{\xi-1} + \frac{1}{2}x_\xi = \lambda x_\xi \end{cases} \tag{7.1}$$

It may be written:

$$\begin{cases} x_2 = (4\lambda - 3)x_1 \\ x_3 = (4\lambda - 2)x_2 - x_1 \\ \vdots \\ x_{i+1} = (4\lambda - 2)x_i - x_{i-1} \\ \vdots \\ x_\xi = (4\lambda - 2)x_{\xi-1} - x_{\xi-2} \\ x_{\xi-1} = (4\lambda - 2)x_\xi \end{cases} \tag{7.2}$$

We know that $x_1 \neq 0$ (if $x_1 = 0,$ all $x_i = 0$), so we may assume $x_1 = 1.$ Assume $\lambda \geq 1.$ From the first equation $x_2 = (4\lambda - 3)x_1$ we deduce $x_2 > x_1 > 0$.

More generally, the equation $x_{i+1} = (4\lambda - 2)x_i - x_{i-1}$ gives:

$$x_{i-1} - x_i = (4\lambda - 3)x_i - x_{i+1} \geq x_i - x_{i+1}$$

that is $x_i - x_{i-1} \leq x_{i+1} - x_i.$ So the sequence of consecutive differences is increasing. Since $x_2 > x_1$, all differences are positive, the $x_i$ are increasing and are $> 0.$ Set $S = \sum_{i=1}^{\xi} x_i$ ; summing all equations, we get $S - \frac{1}{4}x_\xi = \lambda S$ , that is $-\frac{1}{4}x_\xi = (\lambda - 1)S.$ But this is a contradiction: $\lambda > 1,$ $S > 0$ and $x_\xi > 0.$ Lemma 7.3 is proved.

## 8. Precise results on the eigenvalues and eigenvectors

**Proposition 8.1.** - *For $j = 1, \ldots, \xi,$ let $\vartheta_j = \frac{2j-1}{2\xi+1}\pi.$ The $j^{th}$ eigenvalue $\lambda_j$ is:*

$$\lambda_j = \frac{1+\cos(\vartheta_j)}{2} = \cos^2\frac{\vartheta_j}{2}$$

*and the* $j^{th}$ *eigenvector has components :*

$$V_j = \left(\sin(\xi\vartheta_j), \sin((\xi-1)\vartheta_j), \ldots, \sin(\vartheta_j)\right)$$

*All vectors have the same* $l_1-$*norm and the same quadratic norm, which are:*

$$|V_j|_1 = \frac{1}{2}\tan(\xi\vartheta_1) = \frac{1}{2\tan\frac{\vartheta_1}{2}}$$

$$|V_j|_2^2 = \frac{\xi}{2} + \frac{1}{4}$$

**Proof of Proposition 8.1**

We have $x_\xi \neq 0$ (otherwise all $x_j{}'s$ are 0), so we may assume $x_\xi = 1$.

We set $\mu = 2\lambda - 1$, then $\mu < 1$. The system (7.2) becomes :

$$\begin{cases} x_2 = (2\mu - 1)x_1 \\ x_3 = 2\mu x_2 - x_1 \\ \quad\vdots \\ x_{i+1} = 2\mu x_i - x_{i-1} \\ \quad\vdots \\ x_\xi = 2\mu x_{\xi-1} - x_{\xi-2} \\ x_{\xi-1} = 2\mu x_\xi \end{cases} \tag{8.1}$$

We set $y_j = x_{\xi-j}$ for $j = 0, \ldots, \xi - 1$. The system (8.1) becomes:

$$\begin{cases} y_0 = 1 \\ y_1 = 2\mu \\ y_2 = 2\mu y_1 - y_0 \\ \quad\vdots \\ y_j = 2\mu y_{j-1} - y_{j-2} \\ \quad\vdots \\ y_{\xi-1} = 2\mu y_{\xi-2} - y_{\xi-3} \\ y_{\xi-2} = (2\mu - 1)\, y_{\xi-1} \end{cases} \tag{8.2}$$

Therefore, $y_j = U_j(\mu)$ where $U_j$ is the $j^{th}$ Chebychev's polynomial of second kind, for $j = 0, \ldots, \xi - 1$. The final equation in (8.2) may be written:

$$U_{\xi-2}(\mu) = (2\mu - 1) U_{\xi-1}(\mu) \tag{8.3}$$

that is, with $\mu = \cos(\vartheta)$:

$$\frac{\sin\left((\xi-1)\vartheta\right)}{\sin(\vartheta)} = \left(2\cos(\vartheta) - 1\right)\frac{\sin(\xi\vartheta)}{\sin(\vartheta)}.$$

By Lemma 7.3, $\sin(\vartheta) \neq 0$, so the above equation is equivalent to:

$$\sin\left((\xi-1)\vartheta\right) = \left(2\cos(\vartheta) - 1\right)\sin(\xi\vartheta) \tag{8.4}$$

We have :

$$\sin\left((\xi-1)\vartheta\right) - \left(2\cos(\vartheta) - 1\right)\sin(\xi\vartheta) = -\sin(\xi\vartheta)\cos(\vartheta) - \cos(\xi\vartheta)\sin(\vartheta) + \sin(\xi\vartheta)$$

Therefore, equation (8.4) is equivalent to:

$$\sin(\xi\vartheta)\left(1 - \cos(\vartheta)\right) = \cos(\xi\vartheta)\sin(\vartheta)$$

or :

$$\tan(\xi\vartheta) = \frac{\sin(\vartheta)}{1 - \cos(\vartheta)} \tag{8.5}$$

which may be written:

$$\tan(\xi\vartheta) = \frac{1}{\tan\frac{\vartheta}{2}} \tag{8.6}$$

Therefore:

$$\cos(\xi\vartheta)\cos\frac{\vartheta}{2} - \sin(\xi\vartheta)\sin\frac{\vartheta}{2} = 0$$

which gives:

$$\cos\left(\xi\vartheta + \frac{\vartheta}{2}\right) = 0$$

and this equation has the solutions $\frac{2\xi+1}{2}\vartheta = \frac{\pi}{2} + (j-1)\pi$, $j = 1, \ldots, \xi$,

that is:

$$\vartheta = \frac{(2j-1)\pi}{2\xi+1} \tag{8.7}$$

as we announced.

Since $U_j(\cos\vartheta) = \frac{\sin((j+1)\vartheta)}{\sin\vartheta}$, after multiplication, we may take, for $j = 1, \ldots, \xi$:

$$V_j = \left(\sin(\xi\vartheta_j), \sin((\xi-1)\vartheta_j), \ldots, \sin(\vartheta_j)\right) \tag{8.6}$$

and $\lambda_j = \frac{1+\cos(\vartheta_j)}{2} = \cos^2\frac{\vartheta_j}{2}$.

We observe that:

$$\xi\vartheta_j = (2j-1)\frac{\pi}{2} - \frac{2j-1}{2\xi+1}\frac{\pi}{2} = j\pi - \frac{\pi}{2} - \frac{2j-1}{2\xi+1}\frac{\pi}{2} \tag{8.7}$$

and therefore:

$$\sin(\xi\vartheta_j) = \sin\left(j\pi - \frac{\pi}{2} - \frac{2j-1}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1}\cos\left(\frac{2j-1}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1}\cos\frac{\vartheta_j}{2} \tag{8.8}$$

$$\cos(\xi\vartheta_j) = \cos\left(j\pi - \frac{\pi}{2} - \frac{2j-1}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1}\sin\left(\frac{2j-1}{2\xi+1}\frac{\pi}{2}\right) = (-1)^{j-1}\sin\frac{\vartheta_j}{2} \tag{8.9}$$

More generally:

$$\begin{aligned}\sin\left(\left(\xi-i+1\right)\vartheta_j\right)&=\sin\left(j\pi-\frac{\pi}{2}-\frac{\left(2i-1\right)\left(2j-1\right)}{2\xi+1}\frac{\pi}{2}\right)\\&=\left(-1\right)^{j-1}\cos\left(\frac{\left(2i-1\right)\left(2j-1\right)}{2\xi+1}\frac{\pi}{2}\right)=\left(-1\right)^{j-1}\cos\frac{\left(2i-1\right)\vartheta_j}{2}\end{aligned}\tag{8.10}$$

$$\begin{aligned}\cos\left(\left(\xi-i+1\right)\vartheta_j\right)&=\cos\left(j\pi-\frac{\pi}{2}-\frac{\left(2i-1\right)\left(2j-1\right)}{2\xi+1}\frac{\pi}{2}\right)\\&=\left(-1\right)^{j-1}\sin\left(\frac{\left(2i-1\right)\left(2j-1\right)}{2\xi+1}\frac{\pi}{2}\right)=\left(-1\right)^{j-1}\sin\frac{\left(2i-1\right)\vartheta_j}{2}\end{aligned}\tag{8.11}$$

In order to compute the $l_1$ and $l_2$ norms of the eigenvectors, let us first observe that all of them are, up to changes of signs, reorderings of the terms of $V_1$. Indeed, when $j$ changes, the $\xi$ numbers $\sin\left(k\vartheta_j\right)$ ($k=1,\dots,\xi$) are reorderings of the $\xi$ numbers $\sin\left(k\vartheta_1\right)$, except for the sign, which may become minus (this does not affect the norms).

So, let us compute $\left|V_1\right|_1$. Apply the matrix $M$ to the eigenvector $V_1$: by definition, we get $MV_1=\lambda_1 V_1$ and the loss of energy is $\left(1-\lambda_1\right)\left|V_1\right|_1$. But this loss of energy is also $\frac{1}{4}\sin\left(\vartheta_1\right)$, since the last coordinate of the eigenvector is $\sin\left(\vartheta_1\right)$. So we get:

$$\left|V_1\right|_1=\frac{1}{2}\frac{\sin\left(\vartheta_1\right)}{1-\cos\left(\vartheta_1\right)}=\frac{1}{2}\tan\left(\xi\vartheta_1\right)$$

$$\left(1-\lambda_1\right)\left|V_1\right|_1=\frac{1}{4}\sin\left(\vartheta_1\right)$$

But $\lambda_1=\frac{1+\cos\left(\vartheta_1\right)}{2}$, which gives:

$$\left|V_1\right|_1=\frac{1}{2}\tan\left(\xi\vartheta_1\right)=\frac{1}{2}\frac{\sin\left(\vartheta_1\right)}{1-\cos\left(\vartheta_1\right)}=\frac{1}{2\tan\frac{\vartheta_1}{2}}$$

and this result is valid for all eigenvectors.

Let us now compute $\left|V_1\right|_2^2$. We have:

$$\left|V_1\right|_2^2=\sum_{k=1}^{\xi}\sin^2\left(\frac{k\pi}{2\xi+1}\right)=\frac{\xi\sin\left(\vartheta_1\right)+\sin\left(\vartheta_1\right)\cos^2\left(\left(\xi+1\right)\vartheta_1\right)-\cos\left(\vartheta_1\right)\sin\left(\left(\xi+1\right)\vartheta_1\right)\cos\left(\left(\xi+1\right)\vartheta_1\right)}{2\sin\left(\vartheta_1\right)}$$
$$=\frac{\xi}{2}+\frac{\cos^2\left(\left(\xi+1\right)\vartheta_1\right)}{2}-\frac{\cos\left(\vartheta_1\right)\sin\left(\left(\xi+1\right)\vartheta_1\right)\cos\left(\left(\xi+1\right)\vartheta_1\right)}{2\sin\left(\vartheta_1\right)}$$
$$=\frac{\xi}{2}-\frac{\cos\left(\left(\xi+1\right)\vartheta_1\right)\sin\left(\xi\vartheta_1\right)}{2\sin\left(\vartheta_1\right)}$$

We need to prove that $\frac{\cos\left(\left(\xi+1\right)\vartheta_1\right)\sin\left(\xi\vartheta_1\right)}{\sin\left(\vartheta_1\right)}=\frac{-1}{2}$. This is equivalent to:

$$2\cos\left(\frac{\xi+1}{2\xi+1}\pi\right)\sin\left(\frac{\xi}{2\xi+1}\pi\right)+\sin\left(\frac{1}{2\xi+1}\pi\right)=0$$

But:

$$2\cos\left(\frac{\xi+1}{2\xi+1}\pi\right)\sin\left(\frac{\xi}{2\xi+1}\pi\right)=\sin\left(\frac{(\xi+1)\pi}{2\xi+1}+\frac{\xi\pi}{2\xi+1}\right)-\sin\left(\frac{\left(\xi+1\right)\pi}{2\xi+1}-\frac{\xi\pi}{2\xi+1}\right)=-\sin\left(\frac{\pi}{2\xi+1}\right)$$

This proves Proposition 8.1.

So the eigenvector satisfies the equations:

$$\frac{3}{4}\sin\left(\xi\vartheta\right)+\frac{1}{4}\sin\left(\left(\xi-1\right)\vartheta\right)=\frac{1+\cos\left(\vartheta\right)}{2}\sin\left(\xi\vartheta\right)$$

$$\frac{1}{4}\sin\left(\left(\xi-j\right)\vartheta\right)+\frac{1}{2}\sin\left(\left(\xi-j+1\right)\vartheta\right)+\frac{1}{4}\sin\left(\left(\xi-j+2\right)\vartheta\right)=\frac{1+\cos\left(\vartheta\right)}{2}\sin\left(\left(\xi-j+1\right)\vartheta\right)$$
for $j=1,\dots,\xi-1$

and finally:

$$\frac{\sin\left(2\vartheta\right)}{4}+\frac{\sin\left(\vartheta\right)}{2}=\frac{1+\cos\left(\vartheta\right)}{2}\sin\left(\vartheta\right)$$

**Remark 1. -** We observe that, for any $n$ and $t$ we have the identity :

$$\frac{\sin\left(\left(n-1\right)t\right)}{4}+\frac{\sin\left(nt\right)}{2}+\frac{\sin\left(\left(n+1\right)t\right)}{4}=\frac{1+\cos\left(t\right)}{2}\sin\left(nt\right)$$

including the case $n=1$ :

$$\frac{\sin(t)}{2}+\frac{\sin(2t)}{4}=\frac{1+\cos(t)}{2}\sin(t)$$

Therefore, the value of $\vartheta$ is determined by the first equation only, that is:

$$\frac{3}{4}\sin(\xi\vartheta)+\frac{1}{4}\sin((\xi-1)\vartheta)=\frac{1+\cos(\vartheta)}{2}\sin(\xi\vartheta)$$

which is equivalent to the equation:

$$\tan(\xi\vartheta)=\frac{\sin(\vartheta)}{1-\cos(\vartheta)}.$$

The first eigenvector, $V_1$, has all its components real and $>0$, but all other eigenvectors have some negative component.

**Remark 2.** - It follows from the general theory of symmetric matrices, positive defined, that any two eingenvectors $V_{j_1}, V_{j_2}$ are mutually orthogonal, that is:

$$\sum_{l=1}^{\xi}\sin\left(l\vartheta_{j_1}\right)\sin\left(l\vartheta_{j_2}\right)=0 \tag{8.7}$$

where $\vartheta_{j_1}=\frac{(2j_1-1)\pi}{2\xi+1}, \vartheta_{j_2}=\frac{(2j_2-1)\pi}{2\xi+1}$. This can be checked directly.

Let us complete the proof of Theorem 2.

## 9. Decomposition on the basis of eigenvectors

**Proposition 9.1.** - *At each step, we have :*

$$x(n,i)=\frac{2}{2\xi+1}\sum_{j=1}^{\xi}\cos\left(\frac{(2i-1)\vartheta_j}{2}\right)\cos^{2n+1}\frac{\vartheta_j}{2}$$

*with* $\vartheta_j=\frac{2j-1}{2\xi+1}\pi$, $j=1,\ldots,\xi$.

**Proof of Proposition 9.1**

If we want to compute the energy at the $n^{th}$ step, we start with the initial value $f(0,0)=1$, $f(0,i)=0$, $i=1,\ldots,\xi$. This gives for the initial vector :

$$X_0 = \left(\frac{1}{2}, 0, \ldots, 0\right)$$

We decompose this vector on the basis of eigenvectors. We write :

$$X_0 = \sum_{j=1}^{\xi} \alpha_j V_j$$

Since the eigenvectors are orthogonal, the coefficients $\alpha_j$ may be computed simply:

$$\alpha_j = \frac{< X_0, V_j >}{\left|V_j\right|_2^2} = \frac{\sin\left(\xi \vartheta_j\right)}{\xi + \frac{1}{2}}$$

Then, at the $n^{th}$ step (time $2n$), the vector $X_n$ is :

$$X_n = M^n X_0 = M^n \sum_{j=1}^{\xi} \alpha_j V_j = \sum_{j=1}^{\xi} \alpha_j M^n V_j = \sum_{j=1}^{\xi} \alpha_j \lambda_j^n V_j$$

Using the identity $\frac{1+\cos\left(\vartheta_j\right)}{2} = \cos^2\left(\frac{\vartheta_j}{2}\right)$, we obtain the formula:

$$x(n,i) = \frac{2}{2\xi+1} \sum_{j=1}^{\xi} \sin\left(\xi \vartheta_j\right) \sin\left(\left(\xi - i + 1\right)\vartheta_j\right) \cos^{2n} \frac{\vartheta_j}{2}$$

using formulas (8.8) and (8.10), this proves Proposition 9.1 and Theorem 2. In this formula, all terms are known, so the energy at each step is explicit.

In the case $\xi = 3,$ we find numerically:

$$x(n,1) = 0.2716 \times 0.9505^n + 0.1746 \times 0.6113^n + 0.0538 \times 0.1883^n$$
$$x(n,2) = 0.2178 \times 0.9505^n - 0.0969 \times 0.6113^n - 0.1209 \times 0.1883^n$$
$$x(n,3) = 0.1209 \times 0.9505^n - 0.2178 \times 0.6113^n + 0.0969 \times 0.1883^n$$

We can study the decrease of the first coordinate :

**Proposition 9.2.** - *For every $n,$ we have :*

$$X_n(1) = \frac{1}{\xi + \frac{1}{2}} \sum_{j=1}^{\xi} \cos^{2n+2}\left(\frac{2j-1}{2\xi+1}\frac{\pi}{2}\right)$$

**Proof of Proposition 9.2**

We start with $X_0 = \left(\frac{1}{2}, 0, \ldots, 0\right)$ and we want to investigate the behaviour of the first coordinate of $X_n = M^n X_0$, denoted by $X_n(1)$. We have:

$$X_0 = \sum_{j=1}^{\xi} \alpha_j V_j \text{ with } \alpha_j = \frac{\sin(\xi \vartheta_j)}{\xi + \frac{1}{2}} = \frac{2}{2\xi+1}(-1)^{j-1} \cos\frac{\vartheta_j}{2}, \ j = 1, \ldots, \xi.$$

We have $X_n = M^n X_0 = \sum_{j=1}^{\xi} \alpha_j \lambda_j^n V_j$

and therefore, since $V_j(1) = \sin(\xi \vartheta_j)$

$$X_n(1) = \frac{1}{\xi + \frac{1}{2}} \sum_{j=1}^{\xi} \sin^2(\xi \vartheta_j) \lambda_j^n$$

and we obtain the explicit formula :

$$X_n(1) = \frac{2}{2\xi+1} \sum_{j=1}^{\xi} \cos^{2n+2}\left(\frac{2j-1}{2\xi+1}\frac{\pi}{2}\right)$$

and Proposition 9.2 is proved.

## 10. The energy at each step

Using the previous representation, we now investigate the behaviour of the energy as a function of $n$.

**Lemma 10.1.** - *The largest eigenvalue of the matrix $M$ satisfies:*

$$\lambda_1 = \frac{1}{2}\left(1 + \cos\left(\frac{\pi}{2\xi+1}\right)\right) \geq 1 - \frac{1}{4}\frac{\pi^2}{(2\xi+1)^2} \sim 1 - \frac{\pi^2}{16\,\xi^2} \text{ when } \xi \to +\infty.$$

**Proof of Lemma 10.1**

This follows from the estimate $\cos(x) \geq 1 - \frac{x^2}{2}$, which proves Lemma 10.1.

Let, for any $\xi$, $V_{\xi,j}$, $j = 1, \ldots, \xi$ be the eigenvectors of the matrix $M$ of size $\xi$.

**Lemma 10.2.** - *We have, when* $\xi \to +\infty$ *:*

$$\left|V_{\xi,j}\right|_1 \sim \frac{2\xi}{\pi}$$

**Proof of Lemma 10.2**

It is enough to prove the Lemma for the first eigenvector, since all eigenvectors have the same $l_1$ norm. We have:

$$\left|V_{\xi,1}\right|_1 = \frac{1}{2}\tan\left(\xi\vartheta_1\right) = \frac{\sin\left(\vartheta_1\right)}{2\left(1-\cos\left(\vartheta_1\right)\right)}$$

But $\vartheta_1 = \dfrac{\pi}{2\xi+1}$, so $\dfrac{\sin\left(\dfrac{\pi}{2\xi+1}\right)}{2\left(1-\cos\left(\dfrac{\pi}{2\xi+1}\right)\right)} \sim \dfrac{2\xi}{\pi}$, as we announced.

From Lemma 7.3 follows that the operator $M$ is a strict contraction in the $l_2$ norm ; indeed we know (see for instance [BB_Op]) that:

$$\left|MX\right|_2 \leq \max\left(\lambda_j\right)\left|X\right|_2$$

**Proposition 10.3** - *The quadratic energy at step* $2n$ *satisfies the estimate:*

$$E_2\left(W_{2n}\right) \leq \frac{1}{2}\sqrt{\frac{3}{2}}\left(1-\frac{\pi^2}{16\xi^2}\right)^{n-1}$$

*and the energy satisfies:*

$$E\left(W_{2n}\right) \leq \sqrt{\frac{3\xi}{8}}\left(1-\frac{\pi^2}{16\xi^2}\right)^{n-1}$$

**Proof of Proposition 10.3**

From the expression $X_0 = \sum_{k=1}^{\xi}\alpha_j V_j$, we deduce:

$$\left|M^n X_0\right|_2^2 = \sum_{j=1}^{\xi}\alpha_j^2\lambda_j^{2n}\left|V_j\right|_2^2$$

with $\alpha_j = \dfrac{\sin(\xi\vartheta_j)}{\xi + \frac{1}{2}}$ and $|V_j|_2^2 = \dfrac{\xi}{2} + \dfrac{1}{4}$

Since $|X_0|_2^2 = \dfrac{1}{4}$, we know that $\sum_{j=1}^{\xi} \alpha_j^2 = \dfrac{\xi}{2} + \dfrac{1}{4}$.

Therefore:

$$\left|M^{n-1}X_0\right|_2^2 = \frac{1}{2\xi+1}\sum_{j=1}^{\xi}\sin^2(\xi\vartheta_j)\lambda_j^{2n-2} \le \lambda_1^{2n-2}\frac{1}{2\xi+1}\sum_{j=1}^{\xi}\sin^2(\xi\vartheta_j) = \frac{\lambda_1^{2n-2}}{4}$$

that is:

$$|X_{n-1}|_2 \le \frac{1}{2}\left(1 - \frac{\pi^2}{16\xi^2}\right)^{n-1} \qquad (10.1)$$

Now, we observe that:

$$E_2(W_{2n}) \le \sqrt{\frac{3}{2}}|X_{n-1}|_2 \qquad (10.2)$$

Indeed,

$$\begin{aligned} f(n,0)^2 + \sum_{i=1}^{\xi-1} f(n,i)^2 + f(n,\xi)^2 &= x(n-1,1)^2 + \frac{1}{4}\sum_{i=1}^{\xi-1}\left(x(n-1,i) + x(n-1,i+1)\right)^2 + \frac{1}{4}x(n-1,\xi)^2 \\ &\le x(n-1,1)^2 + \frac{1}{2}\sum_{i=1}^{\xi-1}x(n-1,i)^2 + \frac{1}{2}\sum_{i=1}^{\xi-1}x(n-1,i+1)^2 + \frac{1}{4}x(n-1,\xi)^2 \\ &\le \frac{3}{2}|X_{n-1}|_2^2 \end{aligned}$$

which proves (10.2).

We deduce from (10.1) and (10.2):

$$E_2(W_{2n}) \le \frac{1}{2}\sqrt{\frac{3}{2}}\left(1 - \frac{\pi^2}{16\xi^2}\right)^{n-1}$$

which proves the first part of Proposition 10.3.

Now, in order to prove the second part, we use Cauchy-Schwarz inequality:

$$\sum_{i=0}^{\xi-1} x(n-1,i) = |X_{n-1}|_1 \le \sqrt{\xi}\,|X_{n-1}|_2$$

using the computation above. This proves Proposition 10.3 and finishes the proof of Theorem 3.

We now turn to the proof of Theorem 4 : asymptotic estimate of the profile on any vertical, when $n \to +\infty$, fixed $\xi$.

## 11. Asymptotic estimates on the energy profile

**Proposition 11.1.** - *Asymptotically when $n \to +\infty$, we have the estimates, with* $\vartheta_1 = \dfrac{\pi}{2\xi+1}$:

$$e(2n+2,0) \sim \frac{2}{2\xi+1}\cos^{2n+2}\frac{\vartheta_1}{2}$$

*For* $j = 1,\ldots,\xi-1$ *:*

$$e(2n+2,2j) = \frac{2}{2\xi+1}\sin\left((\xi-j)\vartheta_1\right)\cos^{2n+2}\frac{\vartheta_1}{2}$$

*and for* $j=\xi$ *:*

$$e(2n+2,2\xi) \sim \frac{2}{2\xi+1}\sin(\vartheta_1)\cos^{2n+1}\frac{\vartheta_1}{2}$$

**Proof of Proposition 11.1**

We work on the vector $X_n = M^n X_0$ ; writing the decomposition $X_0 = \sum_{j=1}^{\xi} \alpha_j V_j$ on the basis of eigenvectors, we obtain:

$$X_n = \sum_{j=1}^{\xi} \alpha_j \lambda_j^n V_j$$

We have $\lambda_1 > \cdots > \lambda_\xi$, so, when $n \to +\infty$, $X_n \sim \alpha_1 \lambda_1^n V_1$.

Asymptotically when $n \to +\infty$, the energy distribution, on the variables $x(n,j)$, is therefore proportional to the first column of the change of basis matrix.

Returning to the energy $f(n,j) = e(2n,2j)$ using formulas (6.1), we find:

$$e(2n+2,0)=x(n,1)\sim\lambda_1^n\frac{\sin^2(\xi\vartheta_1)}{\xi+\frac{1}{2}}=\frac{\sin^2(\xi\vartheta_1)}{\xi+\frac{1}{2}}\cos^{2n}\frac{\vartheta_1}{2}$$

But $\sin(\xi\vartheta_1)=\cos\frac{\vartheta_1}{2}$, which gives the announced formula.

For $j=1,\ldots,\xi-1$ :

$$e(2n+2,2j)=\frac{1}{2}\big(x(n,j)+x(n,j+1)\big)\sim\frac{\lambda_1^n\sin(\xi\vartheta_1)}{\xi+\frac{1}{2}}\frac{\sin\big((\xi-j+1)\vartheta_1\big)+\sin\big((\xi-j)\vartheta_1\big)}{2}$$
$$=\frac{\lambda_1^n\sin(\xi\vartheta_1)}{\xi+\frac{1}{2}}\sin\big((\xi-j)\vartheta_1\big)\cos\frac{\vartheta_1}{2}=\frac{2}{2\xi+1}\cos^{2n+2}\frac{\vartheta_1}{2}\sin\big((\xi-j)\vartheta_1\big)$$

and finally:

$$e(2n+2,2\xi)=\frac{1}{2}x(n,\xi)\sim\lambda_1^n\frac{\sin(\vartheta_1)\sin(\xi\vartheta_1)}{\xi+\frac{1}{2}}=\frac{2}{2\xi+1}\cos^{2n+1}\frac{\vartheta_1}{2}\sin(\vartheta_1)$$

which proves Proposition 11.1.

**Corollary 11.2. -** *For any starting point* $X_0$, *the energy profile at time* $n$, *for the vector* $X_n$, *is asymptotically equivalent to the first eigenvector* $V_1$ *: this profile is proportional to the vector :*

$$V_1=\big(\sin(\xi\vartheta_1),\ldots,\sin\big((\xi-i+1)\vartheta_1\big),\ldots,\sin(\vartheta_1)\big)$$

*This profile is concave, which means that:*

$$g_i\geq\frac{1}{2}g_{i-1}+\frac{1}{2}g_{i+1}$$

**Proof of Corollary 11.2**

The first part follows immediately from Proposition 11.1. In order to prove the concavity, let us compute:

$$P_j=g_i-\frac{1}{2}g_{i-1}-\frac{1}{2}g_{i+1}=\sin\big((\xi-i+1)\vartheta_1\big)-\frac{1}{2}\sin\big((\xi-i+2)\vartheta_1\big)-\frac{1}{2}\sin\big((\xi-i)\vartheta_1\big)$$
$$=\sin\big((\xi-i+1)\vartheta_1\big)-\sin\big((\xi-i+1)\vartheta_1\big)\cos\vartheta_1>0.$$

This proves Corollary 11.2.

## 12. The energy on the boundary

Since we have seen that the energy on each vertical tends to zero, one might wonder where this energy has gone. More precisely, let us assume now that the barrier is not a "black hole", but it keeps the energy it receives, without propagating it further. In other words, it stores the energy it receives. Then we may wonder about the distribution of this energy. The barrier is assumed to be on the $y=\xi+1$ horizontal axis.

**Proposition 12.1.** - *Assume we start with the eigenvector:*

$$V_j=\left(\sin\left(\xi\vartheta_j\right),\dots,\sin\left(\vartheta_j\right)\right)$$

*Then the point of the barrier with coordinates* $\left(n,\xi+1\right)$ *receives the energy:*

$$e_{n,\xi+1}=\frac{1}{4}\lambda_j^{n-1}\sin\left(\vartheta_j\right)$$

**Proof of Proposition 12.1**

By definition, at the first step, the distribution of energy is

$$E_1=\left(\lambda_j\sin\left(\xi\vartheta_j\right),\dots,\lambda_j\sin\left(\vartheta_j\right),\frac{1}{4}\sin\left(\vartheta_j\right)\right)$$

and the loss, kept by the barrier, is $\frac{1}{4}\sin\left(\vartheta_j\right)$. At the second step, the distribution of energy is :

$$E_2=\left(\lambda_j^2\sin\left(\xi\vartheta_j\right),\dots,\lambda_j^2\sin\left(\vartheta_j\right),\frac{1}{4}\lambda_j\sin\left(\vartheta_j\right)\right)$$

and the loss, kept by the barrier, is $\frac{1}{4}\lambda_j\sin\left(\vartheta_j\right)$ ; and so on at further steps and this proves the Proposition.

For the first eigenvector, we check immediately that the total loss of energy is equal to the total initial energy. Indeed, the total initial energy was, by Proposition 8.1:

$$TIE=\sum_{k=1}^{\xi}\sin\left(k\vartheta_1\right)=\frac{1}{2}\tan\left(\xi\vartheta_1\right)$$

and the total loss is, by Proposition 12.1:

$$TLE=\frac{\sin(\vartheta_1)}{4}\sum_{n=1}^{+\infty}\lambda_1^{n-1}=\frac{\sin(\vartheta_1)}{4}\frac{1}{1-\lambda_1}=\frac{\sin(\vartheta_1)}{2(1-\cos(\vartheta_1))}$$

and both quantities coincide.

If we start with another vector, we have to decompose it on the basis of eigenvectors:

$$V=\sum_{j=1}^{\xi}\alpha_j V_j$$

and the loss at step $n$ will be $e_{n,\xi+1}=\frac{1}{4}\sum_{j=1}^{\xi}\alpha_j\sin(\vartheta_j)\lambda_j^{n-1}$.

For instance, if we start with:

$$X_1=(1,0,\ldots,0)$$

we have (see above):

$$\alpha_{1,j}=\frac{\sin(\xi\vartheta_j)}{\frac{\xi}{2}+\frac{1}{4}}$$

and the loss at step $n$ will be:

$$e_{n,\xi+1}=\frac{1}{2\xi+1}\sum_{j=1}^{\xi}\sin(\xi\vartheta_j)\sin(\vartheta_j)\lambda_j^{n-1}=\frac{1}{2\xi+1}\sum_{j=1}^{\xi}(-1)^{j-1}\sin(\vartheta_j)\cos^{2n-1}\frac{\vartheta_j}{2}$$

The sum $\sum_{n=1}^{+\infty}e_{n,\xi+1}$ should be equal to 1 ; this is the case (numerical verification, done with Maple). We also observe that $e_{n,\xi+1}=0$ if $n<\xi$.

If we start with:

$X_i=(0,\ldots 0,1,0,\ldots,0)$ (1 at the $i^{th}$ place), we have:

$$\alpha_{i,j}=\frac{\sin((\xi-i+1)\vartheta_j)}{\frac{\xi}{2}+\frac{1}{4}}$$

and the loss at step $n$ will be:

$$e_{n,\xi+1} = \frac{1}{2\xi+1}\sum_{j=1}^{\xi} \sin\left(\left(\xi - i + 1\right)\vartheta_j\right)\sin\left(\vartheta_j\right)\lambda_j^{n-1}$$

$$= \frac{1}{2\xi+1}\sum_{j=1}^{\xi} \left(-1\right)^{j-1}\cos\frac{\left(2i-1\right)\vartheta_j}{2}\sin\left(\vartheta_j\right)\cos^{2(n-1)}\frac{\vartheta_j}{2}$$

# V. References


[Kalbfleisch] Probability and Statistical Inference, volume 1 : Probability. Springer Texts in Statistics, 1985.

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