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Simple Random Walks in the Plane:

An Energy-Based Approach

Part I : Basic Facts

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I. Introduction

We consider a simple random walk in the plane : a sequence of random variables X_n with values $\pm 1,$ probability $1/2$ in each case. Let $S_N = \sum_{n=1}^{N} X_n$ be the sum of the first N variables. This random walk can be viewed as a game between two players A and B; at the n^{th} step, the first player receives 1 Euro from the second player if $X_n = +1$ and conversely if $X_n = 1.$ So the sum S_N represents the increase of fortune of A compared to B at the end of N games ; this increase may of course be positive or negative. At the initial moment, we set $S_0 = 0.$ Besides that, each player has an initial fortune, which is finite, or infinite in a theoretical setting. The game may stop when one of the players is ruined (his fortune becomes equal to 0). The general question is to study the behavior of S_N (possible values, with their probabilities), the duration of the game, depending upon the initial fortunes, and the asymptotic behavior, when $N \to +\infty$.

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The behavior of S_N is determined by laws of Nature: one may repeat the experiment and check the results. But, at the same time, these laws are axiomatically defined, as we just did. Such random walks are probably the only example of laws of Nature which may be axiomatically defined: all laws in Physics are otherwise empirical. This remark, in itself, justifies a careful study of the situation: are the usual methods appropriate ? are there better ones ?

Among the many existing results on this topic, let us mention in particular two which are well-known:

- Feller's "Gambler's ruin" ; see [Feller]. The problem may be stated as follows : given an initial fortune and a barrier, what is the probability to reach the barrier without having first reached the barrier $y=0$ (which means ruin) ? The gambler's ruin does not care about a specific time, whereas we compute the probability for each specific time. We thank Doron Zeilberger for useful discussions about this comparison.

- Asymptotic results : Khintchin's law of the iterated logarithm (1924); see [Khintchin]: almost surely, when $n \to +\infty$:

$$\limsup \frac{S_n}{\sqrt{2nLog\left(Log\left(n\right)\right)}} = +1 \text{ and } \liminf \frac{S_n}{\sqrt{2nLog\left(Log\left(n\right)\right)}} = -1$$

Such results are probabilistic in nature and not quantitative at all. Here, on the contrary, we will present a new approach to such problems, which is "energy based" and not probabilistic. This will allow us to develop a unified framework, and to obtain quantitative estimates which were not known previously.

Indeed, the probabilistic appearance of Khinchin's laws is misleading. Looking at such a statement, everyone has the impression that, for a given player, there are some unknown forces which will, sooner or later, bring his fortune close to Khintchin's curves (a Khintchin curve is of the form $y = \pm\sqrt{2x\,Log\left(Log\left(x\right)\right)}$, of course). This is completely wrong ; at any time, the game is only governed by the ± 1 rule, with equal probability.

What Khintchin's laws say, and, more generally, what any result about random walks says, is that there are more paths with some properties than paths with other properties. They are not individual results about each path; they are results about the number of paths with a given characteristic. Such results are in fact of combinatorial nature. For instance, at time n, the proportion of paths which never touched the curve $y=\sqrt{x}$ tends to 0 when $n \to +\infty$.

Our approach relies upon a concept derived from "energy absorption". In this framework, we also introduce tools derived from Analysis, special functions, and Operator Theory.

II. Basic settings

A. Preliminary tools

At any time $n,$ we have of course $|S_n| \leq n.$ The values of S_n are even if n is even, and are odd if n is odd.

The following Lemma simply reflects the combinatorics (see for instance [Feller]):

Lemma 1. - *Let* $A_{n,k}$ *be the point of coordinates* $(n,k),$ *with* $k=-n,\dots,n.$ *The number of paths from 0 to* $A_{n,k}$ *is:*

$$N\left(0 \to A_{n,k}\right) = \binom{n}{\frac{n+k}{2}}$$

Proof of Lemma 1

If we want to reach this point in n steps, we need x times the value 1 and y times the value -1, with $x+y=n$ and $x-y=k$, which gives $x=\frac{n+k}{2}, \; y=\frac{n-k}{2}$. So there are $\binom{n}{x}$ possible paths, which proves the result.

When no confusion is possible, we will write $N(n,k)$ instead of $N\left(0 \to A_{n,k}\right).$

At a given time $n,$ we have 2^n paths starting at 0. Given a property, for instance "to be above the x axis at time n", we may count the number of paths which satisfy this property. Dividing by the total number $2^n,$ we have the proportion of paths satisfying the property. This proportion, in its turn, may be viewed as a probability: in this case, the probability that the player $A,$ at the instant $n,$ has positive gains ($S_n > 0$). So, the probabilities may always be viewed as proportion of paths, and conversely.

There is always a difficulty in such statements, and one should be very careful about that: do we mean "at time n precisely", or do we mean "at all times $k \leq n$" ? Both, as we will see later, are completely different. The first type of statement is usually easy to obtain; the second type is much harder.

As an example of statement of the first type, we have:

Lemma 2. – *For all* $n \geq 1$, $P(S_n \geq 0) > \frac{1}{2}$.

Proof of Lemma 2

This is clear, since $P(S_n > 0) = P(S_n < 0)$, $P(S_n < 0) + P(S_n = 0) + P(S_n > 0) = 1$ and $P(S_n \geq 0) = P(S_n > 0) + P(S_n = 0)$.

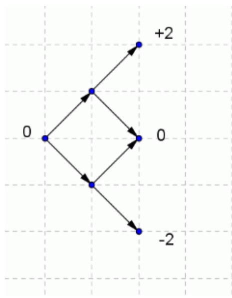
We already have two equivalent points of view: probability and proportions; we will introduce a third one, based upon the energy.

B. Introducing the energy

We consider that, at time 0, a unit of energy is put at the origin. This unit will then divide itself in two halves, at time $n=1$, one at the point $(1,1)$ and one at the point $(1,-1)$. More generally, every time a division point is met, the available energy divides equally into the two possible paths. So, for instance, at the time $n=2$, 3 points will receive some energy, namely $(2,2)$ receives 1/4, $(2,0)$ receives 1/2, $(2,2)$ receives 1/4. At any step, in this configuration, the sum is always 1.

In what follows, we will almost always restrict ourselves to the case where n is even. This means that the elementary game consists in two repetitions, $X_1 + X_2$, with :

$$P(X_1 + X_2 = -2) = \frac{1}{4},\ P(X_1 + X_2 = 0) = \frac{1}{2},\ P(X_1 + X_2 = 2) = \frac{1}{4} \qquad (1)$$

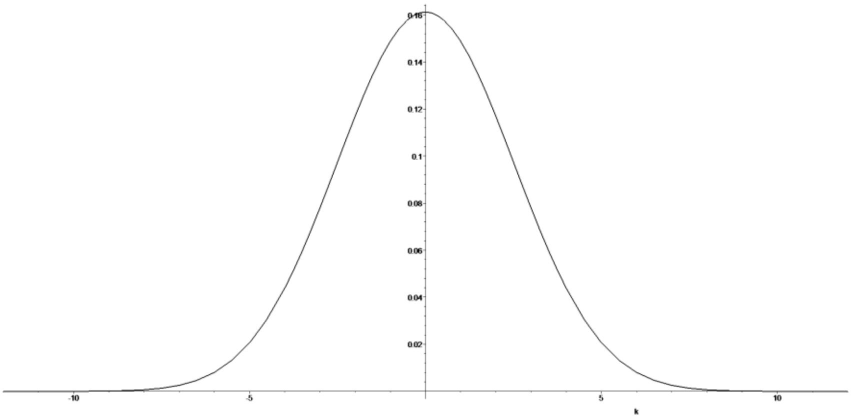


So an energy put at any point will divide into four : one fourth 2 steps above, one half at the same level, one fourth 2 steps below: see picture.

In this basic setting, since the energy 1 is put at O and since there is a total of 2^{2n} possible paths $N(2n,2k) = N(0 \to A_{2n,2k})$ at time $2n$, each point $A_{2n,2k}$ receives an amount of energy, denoted by $e(A_{2n,2k})$, or simply by $e(2n,2k)$, equal to:

$$e(2n,2k) = P(S_{2n} = 2k) = \frac{1}{2^{2n}}\binom{2n}{n+k} \qquad (2)$$

We see that the repartition of energy is given by a binomial law: there is more energy at the central points and very little energy at the extreme points $A_{2n,2n}$ and $A_{2n,-2n}$.



Example of energy distribution for $n=12$

Since we restrict ourselves to the even values of $n,k,$ let $f(n,k)=e(2n,2k)$ be the energy put at the point of coordinates $2n,2k.$ It satisfies for any $k=-n,...,n$:

$$f(n,k)=\frac{1}{4}f(n-1,k-1)+\frac{1}{2}f(n-1,k)+\frac{1}{4}f(n-1,k+1) \tag{3}$$

Now, we observe, using the symmetry of the process, that:

$$f(n,0)=\frac{1}{2}f(n-1,0)+\frac{1}{2}f(n-1,1) \tag{4}$$

Therefore, we will consider equations (3) and (4) for $k=0,...,n$ only.

In the next paragraphs, we investigate the repartition of energy, on horizontal lines and on diagonals.

C. Horizontal lines

We first study the decrease of probability on each horizontal line. The probability to reach $(2n,2k)$ is $f(n,k)=\frac{1}{2^{2n}}\binom{2n}{n+k}$ and the probability to reach $(2n+2,2k)$ is $f(n+1,k)=\frac{1}{2^{2n+2}}\binom{2n+2}{n+k+1}$. The condition $f(n+1,k)\leq f(n,k)$ is equivalent to:

$$\frac{1}{4}\binom{2n+2}{n+k+1}\leq\binom{2n}{n+k}$$

which, after simplification, reduces to:

$$n+1\geq 2k^2$$

So, for fixed k, the probability first increases and then decreases. For a given k, $f(n,k) \to 0$ when $n \to +\infty$.

D. Diagonals

We investigate the probability to reach a point $(2n+2k, 2k)$, that is the $2k^{th}$ point on the $2n^{th}$ diagonal. We use only even values, as before. The 0^{th} diagonal, denoted by D_0, contains 1 at the origin an then $\frac{1}{4^k}$ at the $2k^{th}$ place. So the values are decreasing. The probability to reach $A_{2n+2k,2k}$ is $f(n+k,k) = \frac{1}{2^{2n+2k}}\binom{2n+2k}{n+2k} = \frac{1}{2^{2n+2k}}\binom{2n+2k}{n}$, which is decreasing in k, for fixed n.

E. Further changes of variables

We introduce a new notation, which will be useful in Part II.

We set, for any $n \geq 1$ and $k \leq n$:

$$x(n,k) = \frac{1}{2}\left(f(n,k-1) + f(n,k) \right)$$

Lemma 3. - *We have, for any n, $k \leq n$:*

$$x(n,k) = \frac{1}{2^{2n+1}}\binom{2n+1}{n+k}$$

Proof of Lemma 3

Indeed, we have:

$$x(n,k) = \frac{1}{2}\left(f(n,k-1) + f(n,k) \right) = \frac{1}{2^{2n+1}}\left(\binom{2n}{n+k-1} + \binom{2n}{n+k} \right) = \frac{1}{2^{2n+1}}\binom{2n+1}{n+k}$$

using Pascal's formula. This proves Lemma 3.

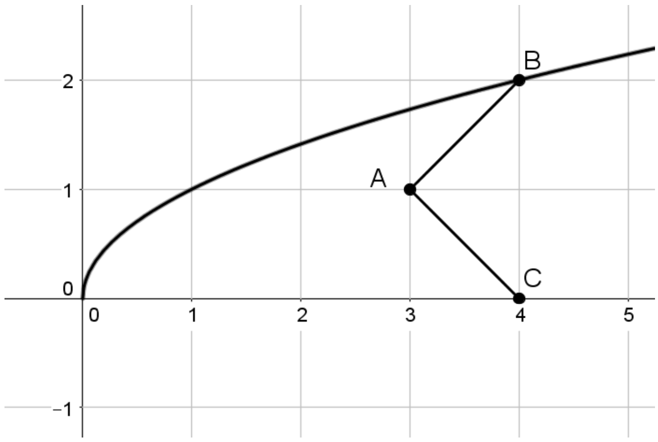
III. Introducing a barrier

People usually think that, most of the time, the gain will go to $+\infty$ or to $-\infty$: either you are in a good day, or in a bad day. But this is not true at all, and the reality is much more complex, as we now see. In order to study this question, we will see how often the RW goes above any horizontal line.

A. General definition

In the preliminary approach, the total amount of energy remains the same at each time step. Now, we introduce a curve, $y=\varphi(x)$ located in the upper half-plane (the same holds for the lower half-plane, of course), and we want to investigate the probability that the random walk, up to time n, remains constantly below this curve, which means that $S(j)<\varphi(j)$ for all $j=1,\ldots,n$. Later, we will investigate the probability to remain between the curve and its symmetric, which means $|S(j)|<\varphi(j)$, or, more generally, to remain between two curves : $-\varphi_1(j)<S(j)<\varphi_2(j)$.

Our representation, in order to investigate this phenomenon, will be the fact that the curve φ absorbs the energy. This means that, for any path which touches the curve, the corresponding energy disappears.



Example of energy absorption

In this example, the point A sends its energy to both B and C, but B is on the curve we have introduced, so this part of the energy disappears, and we are left with $e(C)=\frac{1}{2}e(A)$.

The curve we introduce will be called the critical curve. It may be considered as a "black frontier" (in the sense of a black hole), meaning that it absorbs all energy it receives, and sends back nothing.

We have:

Proposition 4. - *Let* $y=\varphi(x)$ *be any critical curve, in the upper half-plane. The total energy left, at time* n*, is equal to the total probability to reach any of the points* $A_{n,k}$ *below the curve, that is* $k<\varphi(n)$*, without ever touching the curve at any time before (* $j \le n$ *).*

Proof of Proposition 4

This is a mere rephrasing of the disappearance of energy. Any time a path touches the curve, it is annihilated, so what remains is the set of paths which never touched the curve.

If a time n is fixed, and a curve φ is fixed, we will call <u>admissible</u> a path with never touches it (at any time $j \le n$). For any point A in the plane, let $N_{ad}(A)$ be the number of admissible paths, starting at 0, which reach A, and $p_{ad}(A)=\frac{N_{ad}(A)}{2^n}$ the probability to reach A by an admissible path. Proposition 4 states that:

$$\sum_{k=-n}^{n} e\left(A_{n,k}\right)=\sum_{k<\varphi(n)} p_{ad}\left(A_{n,k}\right)$$

B. The case of an horizontal line

We now compute the number of admissible paths when the critical curve is a simple horizontal line segment. As we already said, we restrict ourselves to the even case.

To say that the critical curve is set at $y=2\xi$ means that this is the original fortune of player B, and that he will be ruined if $S_{2n}=2\xi$ (the fortune of B is now equal to 0).

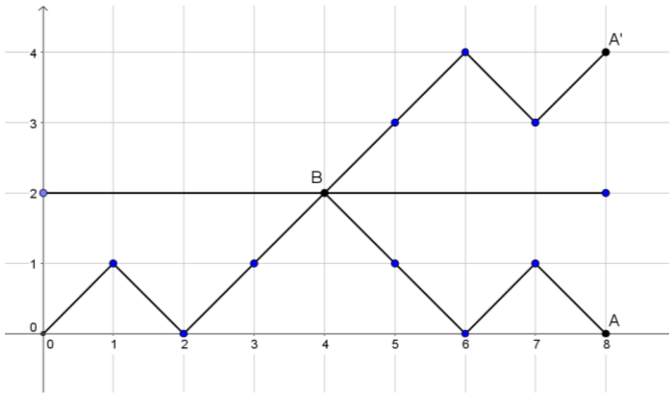
Let $y=2\xi$ be an horizontal line and $W_{2n}=\left\{A_{2n,2y};-n \le y \le n\right\}$ be the vertical segment for $x=2n$. We denote by E_{2n} the total energy on this vertical : $E_{2n}=\sum_{k=-n}^{n} e\left(A_{2n,2k}\right)$. In this setting, E_{2n} is the probability that the game reaches time $2n$, or, in other words, did not stop earlier.

The following Proposition is known as the "reflection principle":

Proposition 5. - *Let* $y=2\xi$ *(* $\xi \ge 0$ *) be an horizontal line segment. Let* $A_{2n,2k}$*, with coordinates* $(2n,2k)$*, be any point that the random walk may reach, with* $k<\xi$*. The number of paths, starting at 0, finishing at* $A_{2n,2k}$*, which touch the horizontal segment at a time before* $2n$ *is* $N(2n,4\xi-2k)$*, where* $A_{2n,4\xi-2k}$ *is the symmetric of* $A_{2n,2k}$ *with respect to the line segment.*

Proof of Proposition 5

Let B be the first time a path touches the segment (there may be several). There are as many paths from B to A than from B to A', symmetric of A with respect to the barrier.



The reflection principle

The symmetric of $A_{2n,2k}$ is $A_{2n,4\xi-2k}$. So the number of paths which touch the segment $y=2\xi$ at any time before n is, by Lemma 1 :

$$N(2n,4\xi-2k)=\binom{2n}{n+2\xi-k}$$

This proves Proposition 5.

Corollary 6. - Assume $0<k<\xi$. **T**he number of paths, starting at 0, which reach $A_{2n,2k}$ without touching the segment $y=2\xi$ at any time $m\le n$ is:

$$N(2n,2k\,;S_m<2\xi,m=1,\ldots,n)=\binom{2n}{n+k}-\binom{2n}{n+2\xi-k}$$

Proposition 7. - *Assume that our critical curve is the line segment* $y=2\xi$, $\xi\ge1$. *The energy left at time* $2n$ *is:*

$$E_{2n}=\frac{1}{2^{2n}}\sum_{k=-\xi}^{\xi-1}\binom{2n}{n+k}$$

Proof of Proposition 7

The critical line segment $y=2\xi$ has two effects :

- No point $A_{2n,2k}$ above this segment, that is $k\ge\xi$, receives any energy at all ; there is a drop of total energy equal to the probability to reach this point;

– For every point strictly below this segment, that is $k < \xi$, there is a drop of energy equal to the probability to reach its symmetric.

This gives:

$$E_{2n} = 1 - \frac{1}{2^{2n}} \sum_{k \geq \xi} \binom{2n}{n+k} - \frac{1}{2^{2n}} \sum_{k \leq \xi - 1} \binom{2n}{n+2\xi - k} = 1 - \frac{1}{2^{2n}} \sum_{k \geq \xi} \binom{2n}{n+k} - \frac{1}{2^{2n}} \sum_{j \geq \xi + 1} \binom{2n}{n+j}$$
$$= 1 - \frac{1}{2^{2n}} \sum_{k \geq \xi} \binom{2n}{n+k} - \frac{1}{2^{2n}} \sum_{j \geq \xi + 1} \binom{2n}{n-j} = \frac{1}{2^{2n}} \sum_{k=-\xi}^{\xi-1} \binom{2n}{n+k}$$

This proves Proposition 7.

In this setting, E_{2n} is the probability that the game has not stopped at time $2n$, which means that the barrier was not touched, which means, in familiar words, that player B, who had an initial fortune of 2ξ Euros, has not been ruined so far. The quantity $1 - E_{2n}$ is the probability that the player B gets ruined before time $2n$.

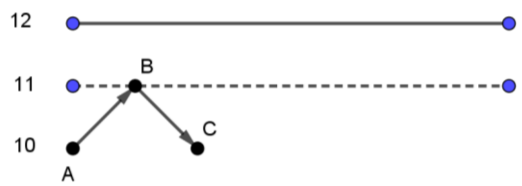
Let us assume for example that $\xi = 100$ Euros, so the initial fortune of B is 200 Euros. Using Proposition 7, we find that if $n = 10\,000$, $E_{2n} = 0.84$ and if $n = 100\,000$, $E_{2n} = 0.35$. In other words, even with a small initial fortune, B is not going to get ruined quickly.

If B's initial fortune is 1000 Euros, he has probability $\frac{1}{2}$ to stay in the game for $n = 5\,092\,958$ time steps and probability 0.95 to last at least $n = 1\,410\,791$ time steps.

It is clear that $E_{2n} \to 0$ when $n \to +\infty$. Indeed, in the expression $E_{2n} = \frac{1}{2^{2n}} \sum_{j=-\xi}^{\xi} \binom{2n}{n+j}$, there is a fixed number of terms and each term tends to 0 when $n \to +\infty$. We will make this statement quantitative later (see the paragraph "Gaussian Interpretation").

C. Different positions of the barrier

Let us see what difference it makes when the barrier is set at 2ξ or $2\xi + 1$.



Let us look at the figure above ; the point A satisfies $y=10$. If we put the barrier at $y=11$ (dotted line), then the vector $\overrightarrow{AB}$ does not exist, and the energy in C is $\frac{1}{4}$ of the energy of A. If we put the barrier at $y=12$ (upper line), the vector $\overrightarrow{AB}$ exists, and the energy in C is $\frac{1}{2}$ of the energy of A. This will modify the entries (first and last row) of the matrix describing the process, and will be studied in detail in Parts II and III.

D. Present and past times

The following Corollary relates the behavior at time $<n$ with the behavior at time n. It will be useful later.

Corollary 9. - *For any integer ξ and any $n \geq 1$, we have:*

$$P\left(\exists m \leq n, S_{2m} \geq 2\xi\right) \leq 2P\left(S_{2n} \geq 2\xi\right)$$

Proof of Corollary 9

The left hand side is the probability to touch the horizontal line before time $2n$; its value is, by Proposition 7, is $1-E_{2n}=\frac{1}{2^{2n}}\left(\sum_{k\geq\xi}\binom{2n}{n+k}+\sum_{k<\xi}\binom{2n}{n+2\xi-k}\right)$. For the right hand side, we have:

$P\left(S_{2n} \geq 2\xi\right)=\frac{1}{2^{2n}}\sum_{k\geq\xi}\binom{2n}{n+k}$; but $\sum_{k<\xi}\binom{2n}{n+2\xi-k}\leq\sum_{k\geq\xi}\binom{2n}{n+k}$. This proves Corollary 9.

We give another proof, which is not of combinatorial type, but purely probabilistic. It comes from [Velenik], §2.3. The comparison between both types of proof is interesting.

For any fixed x real, we set:

$$\sigma_x=\inf\left\{k \geq 0, S_k > x\right\}$$

This is the first time when the sequence S_k is above the value x. The events $\left\{\sigma_x=k\right\}$ are mutually disjoint, and we have:

$$P\left\{\exists k \leq n, S_k > x\right\}=\sum_{k=1}^{n}P\left\{\sigma_x=k\right\}$$

For $k=1,\ldots,n$, we introduce the event:

$$U_k=\left\{S_k \leq S_n\right\}.$$

The events $\{\sigma_x = k\} \cap U_k$, $k = 1,\dots,n$, are a partition of the event $\{S_n > x\}$; indeed, they are disjoint and their union is the set $\{S_n > x\}$: if $S_n > x$, there is a k, $1 \le k \le n$ such that $S_k > x$.

Therefore:

$$P\{S_n > x\} = \sum_{k=1}^{n} P(\{\sigma_x = k\} \cap U_k) = \sum_{k=1}^{n} P(\sigma_x = k) P(U_k)$$

Indeed, the event $\sigma_x = k$ depends upon $X_1,\dots,X_k$, and the event U_k can be writtten $S_n - S_k > 0$, that is $X_{k+1} + \cdots + X_n > 0$; so, it is independent from $X_1,\dots,X_k$.

We have:

$$\sum_{k=1}^{n} P(\sigma_x = k) P(U_k) \ge \min_{k=1,\dots,n} P(U_k) \sum_{k=1}^{n} P(\sigma_x = k)$$

But $P(U_k) = P(X_{k+1} + \cdots + X_n > 0) > \frac{1}{2}$ by Lemma 2 above: all partial sums have the same law. Therefore:

$$\sum_{k=1}^{n} P(\sigma_x = k) P(U_k) \ge \frac{1}{2} \sum_{k=1}^{n} P(\sigma_x = k) = \frac{1}{2} P(\cup(\sigma_x = k))$$

But the set $\cup(\sigma_x = k)$ can be described by the fact that there is a k, $1 \le k \le n$, such that $S_k > x$. This proves Corollary 10.

E. The x axis as a special case

We are interested by the situation where the barrier is the x axis; in terms of fortunes, it corresponds to the case where B has no initial fortune at all, and there is no restriction on A. Our question is: what is the probability that the game lasts at least $2n$ moves ? The player B is ruined if the random walk touches the x axis. Of course, in order that the game initially starts, the player B must win the first two games. So the starting point is $A_{2,-2}$ which is reached with probability $1/4$. Then the game should not touch the x axis, and the probability is the same as for a starting point $A(4,0)$ and a barrier at $2\xi = 2$.

Proposition 10. - *Assume that B has no initial fortune. The probability that the game lasts at least $2n$ moves is:*

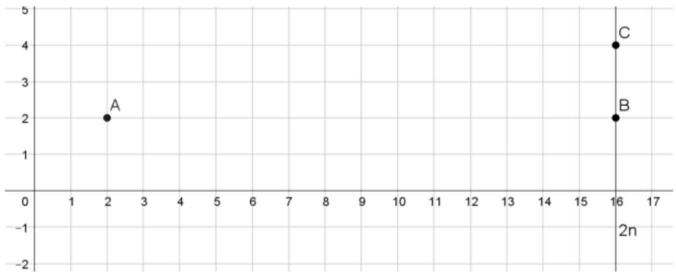
$$p = \frac{1}{2^{2n-2}} \binom{2n-1}{n} = \frac{1}{2} \frac{1}{4^n} \binom{2n}{n}$$

Proof of Proposition 10

Using Proposition 7 with $\xi = 1,$ we find:

$$E_{2n} = \frac{1}{2^{2n-2}}\left(\binom{2n-2}{n-1} + \binom{2n-2}{n-2}\right) = \frac{1}{2^{2n-2}}\binom{2n-1}{n}$$

which proves Proposition 10. Direct computation shows that, for $n \geq 31,$ $p \leq 0.05$ and for $n \geq 795,$ $p \leq 0.01.$



The total number of paths from 0, which do not touch the $x-$ *axis before time* $2n$ *is equal to the total number of paths from* $A(2,2)$ *to* $B(2n,2)$ *and to* $C(2n,4)$.

Using Stirling's formula, we may easily compute an asymptotic estimate, when $n \to +\infty$:

$$P(B\, resists\ \ for\ at\ least\ 2n\ moves) = \frac{1}{2\sqrt{\pi n}} + O\left(\frac{1}{n^{3/2}}\right) \text{ when } n \to +\infty.$$

We can derive from Proposition 10 the probability that a path never touches the x axis at any time $\leq 2n$:

Proposition 11. - *The probability that a path, starting at 0, reaches the vertical* W_{2n} *without ever touching the* x *axis is:*

$$P(S_{2m} \neq 0, m = 1,\dots,n) = \frac{1}{2^{2n}}\binom{2n}{n}$$

Indeed, there are two groups of paths : those which are constantly above and those which are constantly below the axis, both with same probability. This proves Proposition 11. The proof we presented here is much simpler than the one which can be found in the book by [Kalbfleisch].

Take for instance $n = 1000,$ so we play 2000 games. We find:

$$P(S_{2m} \neq 0, m = 1,\dots,n) = 0.018$$

which means that 98.2 % of the paths have returned to the x axis, at least once, before time 2000. This proportion increases when n increases: it is not true that, in general, S_n tends to $+\infty$ or $-\infty$: we see instead that an increasing proportion of the paths keep returning to the x axis: the fortunes are equal.

We have now a clearer picture of the aspect of most paths. Of course, a small number among them will tend to $+\infty$ or $-\infty$, but the largest proportion (increasingly large when n increases) will "oscillate" : they reach high values, return to 0, reach high negative values, return to 0, and so on.

IV. Operator Theory approach

We have seen two approaches of our problem: one is probabilistic (result of a game), one is by means of a distribution of energy, and we have proved that they were equivalent. We now introduce a third approach, by means of Operator Theory. We will develop it in the Second Part.

Assume that the barrier is at 2ξ. At any time $2n,$ at any place $k = 2\xi - 2, 2\xi - 4, \ldots$ we may have some energy. So we may consider that the distribution of energy on a vertical is represented by a sequence in the Banach space $l_1\left(N^*\right)$ of absolutely summable sequences $x_1, x_2, \ldots, x_n, \ldots$ (numbering starts at 1, not 0), see [BB_Banach]. At the position 1, we have the energy near the barrier, at position 2, the energy one step below, and so on (this is typically a matrix numbering).

Then, if we have such a distribution of energy $f\left(n,k\right)$ at time $2n,$ the next step will be a distribution of energy defined by:

$$f\left(n+1,1\right) = \frac{f\left(n,1\right)}{2} + \frac{f\left(n,2\right)}{4} \quad \text{and} \quad f\left(n+1,k\right) = \frac{f\left(n,k-1\right)}{4} + \frac{f\left(n,k\right)}{2} + \frac{f\left(n,k+1\right)}{4} \quad \text{for } k \geq 2.$$

So we see that passing from the energy at step $2n$ to the energy at step $2n+2$ is the result of the action of linear operator $T,$ acting on an infinite dimensional space, namely $l_1\left(N^*\right)$, space of absolutely summable sequences. More precisely, if $X = \left(x_1, x_2, \ldots, x_n, \ldots\right)$, then:

$$TX = \left(\frac{x_1}{2} + \frac{x_2}{4}, \frac{x_1}{4} + \frac{x_2}{2} + \frac{x_3}{4}, \ldots., \frac{x_{n-1}}{4} + \frac{x_n}{2} + \frac{x_{n+1}}{4}, \ldots\right)$$

Such an operator is represented by an infinite matrix:

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \cdots \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \vdots & 0 & \frac{1}{4} & \ddots \end{pmatrix}$$

The operator T is positive: if all coefficients in X are positive, so are all coefficients in TX. It is a contraction : $\|TX\|_1 \leq \|X\|_1$ for all X ; see [BB_op].

Let $t_{i,j}^{(n)}$ be the coefficient of T^n at the i^{th} row and j^{th} column. A direct computation of this coefficient is not easy. Let us see how to compute it, using the previous paragraph.

If we take $X = (0,...,0,1,0,...)$, with 1 at the j^{th} place, the vector $T^n X$ will be $(t_{1,j}, t_{2,j}, ..., t_{i,j}, ...)$. This vector is, by definition, the vector of energies on the $2n^{th}$ vertical, W_{2n}, with numbering starting at 1 near the barrier. So, in the original coordinates, $t_{1,j}$ is at $2\xi - 2$, $t_{2,j}$ at $2\xi - 4$, $t_{i,j}$ at $2\xi - 2i$.

Taking the vector X as initial energy vector means that we put energy 1 at a point situated at $2j$ below the barrier. If we take this point as origin, as we did, it means that the barrier is at $2\xi = 2j$.

So $t_{i,j}$ is the energy received by the point $A_{2n,2k}$, with $k = \xi - i$, when the barrier is at $y = 2j$, that is, for $i \leq n$, using Corollary 6:

$$t_{i,j}^{(n)} = \binom{2n}{n + 2\xi - 2i} - \binom{2n}{n + 2j - (2\xi - 2i)}$$

When $i > n$, the computation is easy: the $n+1^{st}$ row of this matrix is made of the sequence $\frac{1}{2^{2n}}\left(\binom{2n}{0}, \binom{2n}{1}, ..., \binom{2n}{2n}, 0, ...\right)$; the next, that is $n+2^{nd}$, is made of the same sequence, shifted one step to the right, that is $\frac{1}{2^{2n}}\left(0, \binom{2n}{0}, \binom{2n}{1}, ..., \binom{2n}{2n}, 0, ...\right)$, and so on.

Remark. - We observe that this operator has no eigenvalue. Indeed, the equation defining eigenvalues and eigenvectors is $TX = \lambda X$, which can be written:

$$\frac{x_1}{2}+\frac{x_2}{4}=\lambda x_1$$

and for $k \geq 2$:

$$\frac{x_{k-1}}{4}+\frac{x_k}{2}+\frac{x_{k+1}}{4}=\lambda x_k$$

These equations can be written:

$$\begin{aligned} & x_2=(4\lambda-2)x_1 \\ & x_{k+1}=(4\lambda-2)x_k-x_{k-1},\ k\geq 2 \end{aligned} \quad (1)$$

We have $x_1 \neq 0$ (otherwise all $x_k = 0$), so we may take $x_1 = 1$. Let $\mu = 2\lambda - 1$; system (1) becomes:

$$\begin{aligned} & x_1=1 \\ & x_2=2\mu \\ & x_{k+1}=2\mu x_k-x_{k-1},\ k\geq 2 \end{aligned} \quad (2)$$

Let $S=\sum_{k=1}^{+\infty}|x_k|$; we deduce from (2) $|x_{k+1}|\geq 2|\mu||x_k|-|x_{k-1}|$ and summing the equations:

$S \geq 1+2|\mu|S-S$, or $2S(1-|\mu|)\geq 1$, which implies $-1<\mu<1$. Set $\mu=\cos(\vartheta)$; (2) implies that x_k is the k^{th} Chebycheff polynomial (in μ) of second kind, that is:

$$x_k = U_k(\cos(\vartheta)) = \frac{\sin((k+1)\vartheta)}{\sin(\vartheta)}$$

But $\sin(\vartheta)\neq 0$ since $\mu<1$ and the sequence $a_k=\sin(k\vartheta)$ cannot be summable, for any ϑ. Indeed,

– If ϑ is a rational multiple of π, a_k takes only a finite number of values, and each value is taken infinitely many times, so the sum is infinite ;
– If $\dfrac{\vartheta}{\pi}$ is irrational, there is a subsequence of multiples $k\vartheta$ converging (modulo 2π) to $\dfrac{\pi}{2}$ (Kronecker, 1884), so $\sin(k\vartheta)\to 1$ and the sum of the series is infinite.

We will study a similar operator in Part II, in finite dimension, and determine its eigenvalues and eigenvectors.

V. Gaussian interpretation

We have, using the approximation of the binomial law by the normal law, for fixed ξ_1, ξ_2:

$$P(2\xi_1 \le S_{2n} \le 2\xi_2) = \frac{1}{2^{2n}} \sum_{j=\xi_1}^{\xi_2} \binom{2n}{n+j} \approx \int_{\xi_1}^{\xi_2} \exp\left(-\frac{t^2}{2\sigma^2}\right) \frac{dt}{\sigma\sqrt{2\pi}}, \quad \text{with } \sigma^2 = 2n.$$

We want to make this approximation precise.

Proposition 12. (Chernoff's Inequality) – *For any* n *and any* k, $0 \le k \le n$, *we have:*

$$P(S_n \ge k) \le \exp\left(-\frac{k^2}{2n}\right)$$

Proof of Proposition 12

We know that $E(S_n) = 0$ and $\operatorname{var}(S_n) = n$. Using Markov's Inequality $P(X \ge \varepsilon) \le \frac{E(X)}{\varepsilon}$, we write, for any $\lambda > 0$:

$$P(S_n \ge x) = P\left(e^{\lambda S_n} \ge e^{\lambda x}\right) \le e^{-\lambda x} E\left(e^{\lambda S_n}\right)$$

We have also:

$$e^{-\lambda x} E\left(e^{\lambda S_n}\right) = e^{-\lambda x} E\left(\prod_1^n e^{\lambda X_k}\right) = e^{-\lambda x}\left(E e^{\lambda X_1}\right)^n$$

But:

$$E\left(e^{\lambda X_1}\right) = \frac{e^{-\lambda} + e^{\lambda}}{2} \le e^{\lambda^2/2} \tag{1}$$

Indeed,

$$\frac{e^{-\lambda} + e^{\lambda}}{2} = \sum_{k=0}^{+\infty} \frac{\lambda^{2k}}{(2k)!}, \quad e^{\lambda^2/2} = \sum_{k=0}^{+\infty} \frac{\lambda^{2k}}{2^k k!}$$

and $2^k k! \le (2k)!$ which proves (1). We deduce from (1), for any λ:

$$P(S_n \ge x) \le e^{-\lambda x} e^{n\lambda^2/2}$$

and if we take $\lambda = \frac{x}{n}$, we obtain the required estimate. This proves Proposition 12.

Proposition 13.- *For all* $\xi_1 < \xi_2$ *and all* n, *we have the estimate:*

$$\left| \frac{1}{2^{2n}} \sum_{k=\xi_1+1}^{\xi_2} \binom{2n}{n+k} - \int_{2\xi_1}^{2\xi_2} \exp\left(\frac{-t^2}{2\sigma^2}\right) \frac{dt}{\sigma\sqrt{2\pi}} \right| \leq \sqrt{\frac{2}{n}}$$

Proof of Proposition 13

It follows from Berry-Esseen Theorem [Berry-Esseen], which may be stated as follows:

For all x and all n:

$$\left| P\left(\frac{S_n}{\sqrt{n}} \leq x\right) - \int_{-\infty}^{x} \exp\left(\frac{-t^2}{2}\right) \frac{dt}{\sqrt{2\pi}} \right| \leq \frac{1}{\sqrt{n}}$$

which we write under the form:

$$\left| P\left(S_{2n} \leq x\sqrt{2n}\right) - \int_{-\infty}^{x\sqrt{2n}} \exp\left(\frac{-t^2}{2\sigma^2}\right) \frac{dt}{\sigma\sqrt{2\pi}} \right| \leq \frac{1}{\sqrt{2n}}, \text{ with } \sigma = \sqrt{2n}$$

or, with $\xi_1 = \frac{x\sqrt{2n}}{2}$:

$$\left| P\left(S_{2n} \leq 2\xi_1\right) - \int_{-\infty}^{2\xi_1} \exp\left(\frac{-t^2}{2\sigma^2}\right) \frac{dt}{\sigma\sqrt{2\pi}} \right| \leq \frac{1}{\sqrt{2n}}$$

We know that:

$$P\left(S_{2n} \leq 2\xi_1\right) = \frac{1}{2^{2n}} \sum_{k \leq \xi_1} \binom{2n}{n+k}$$

Therefore:

$$\left| \frac{1}{2^{2n}} \sum_{k \leq \xi_1} \binom{2n}{n+k} - \int_{-\infty}^{2\xi_1} \exp\left(\frac{-t^2}{2\sigma^2}\right) \frac{dt}{\sigma\sqrt{2\pi}} \right| \leq \frac{1}{\sqrt{2n}} \qquad (1)$$

and also with ξ_2 :

$$\left| \frac{1}{2^{2n}} \sum_{k \le \xi_2} \binom{2n}{n+k} - \int_{-\infty}^{2\xi_2} \exp\left(\frac{-t^2}{2\sigma^2}\right) \frac{dt}{\sigma\sqrt{2\pi}} \right| \le \frac{1}{\sqrt{2n}} \tag{2}$$

Taking the difference, we obtain the statement of Proposition 13.

Proposition 13 has an interpretation, namely that the energy, on any vertical W_{2n}, between the levels $2\xi_1$ and $2\xi_2$, may be viewed as a gaussian integral between these two levels, the variance of the law being the distance between 0 and the vertical (this distance is $2n$). The error in this approximation is smaller than $\sqrt{\frac{2}{n}}$.

We immediately deduce an estimate for the sum $\frac{1}{2^{2n}} \sum_{k=-\xi}^{\xi} \binom{2n}{n+k}$, valid for all n :

Proposition 14. - *For all ξ and n, we have the estimate:*

$$E_{2n} \le \frac{2\xi}{\sqrt{\pi n}} + \sqrt{\frac{2}{n}}$$

Proof of Proposition 14

Indeed, from Proposition 13:

$$E_{2n} \le \int_{-2\xi}^{2\xi} \exp\left(\frac{-t^2}{2\sigma^2}\right) \frac{dt}{\sigma\sqrt{2\pi}} + \sqrt{\frac{2}{n}} = \int_{-\frac{2\xi}{\sqrt{2n}}}^{\frac{2\xi}{\sqrt{2n}}} \exp\left(\frac{-t^2}{2}\right) \frac{dt}{\sqrt{2\pi}} + \sqrt{\frac{2}{n}} \le \frac{2\xi}{\sqrt{\pi n}} + \sqrt{\frac{2}{n}}$$

which proves Proposition 14.

We now turn to lower estimates for $P(S_n > k)$, in terms of Gaussian integrals.

Proposition 15. - *If* $n > 32\pi e$ and $k < \sqrt{n}$, *we have, with* $c = \frac{1}{4\sqrt{2\pi}}$*:*

$$P(S_n \ge k) \ge c \exp\left(-\frac{k^2}{2n}\right)$$

Proof of Proposition 15

We write Berry-Essen Theorem under the form:

For all x and all n:

$$\left| P\left(\frac{S_n}{\sqrt{n}} > x\right) - \int_x^{+\infty} \exp\left(\frac{-t^2}{2}\right)\frac{dt}{\sqrt{2\pi}} \right| \leq \frac{1}{\sqrt{n}}$$

With $x = \dfrac{k}{\sqrt{n}}$, it gives:

$$P\left(S_n > k\right) \geq \int_{\frac{k}{\sqrt{n}}}^{+\infty} \exp\left(\frac{-t^2}{2}\right)\frac{dt}{\sqrt{2\pi}} - \frac{1}{\sqrt{n}}$$

Let $f\left(x\right)$ be the density of Gauss Law and $F\left(x\right)$ be the repartition function; we have the estimate, for all $x > 0$ ([Komatsu]):

$$F\left(x\right) > \frac{2f\left(x\right)}{\sqrt{x^2+4}+x}$$

which gives here:

$$\int_{\frac{k}{\sqrt{n}}}^{+\infty} \exp\left(\frac{-t^2}{2}\right)\frac{dt}{\sqrt{2\pi}} > \frac{2f\left(\frac{k}{\sqrt{n}}\right)}{\sqrt{\frac{k^2}{n}+4}+\frac{k}{\sqrt{n}}}$$

But, if $k \leq \sqrt{n}$ then $\sqrt{\dfrac{k^2}{n}+4}+\dfrac{k}{\sqrt{n}} \leq \sqrt{5}+1 < 4$ and:

$$\int_{\frac{k}{\sqrt{n}}}^{+\infty} \exp\left(\frac{-t^2}{2}\right)\frac{dt}{\sqrt{2\pi}} > \frac{1}{2\sqrt{2\pi}}\exp\left(-\frac{k^2}{2n}\right)$$

Moreover, $\dfrac{1}{\sqrt{n}} < \dfrac{1}{4\sqrt{2\pi}}\exp\left(-\dfrac{k^2}{2n}\right)$ is satisfied since:

$$\frac{1}{\sqrt{n}} < \frac{1}{4\sqrt{2\pi}}\exp\left(-\frac{1}{2}\right) \leq \frac{1}{4\sqrt{2\pi}}\exp\left(-\frac{k^2}{2n}\right)$$

which is realized, since we assumed $n > 32\pi e$.

So we obtain:

$$P(S_n > k) \geq \int_{\frac{k}{\sqrt{n}}}^{+\infty} \exp\left(\frac{-t^2}{2}\right)\frac{dt}{\sqrt{2\pi}} - \frac{1}{\sqrt{n}} > \frac{1}{2\sqrt{2\pi}}\exp\left(-\frac{k^2}{2n}\right) - \frac{1}{4\sqrt{2\pi}}\exp\left(-\frac{k^2}{2n}\right) = \frac{1}{4\sqrt{2\pi}}\exp\left(-\frac{k^2}{2n}\right)$$

which proves Proposition 15.

For $\xi = 0$, we find the estimate $E_{2n} \leq \sqrt{\frac{2}{n}}$, whereas a direct application of Stirling's formula gives $E_{2n} \leq \frac{1}{\sqrt{\pi n}}$, so the estimate in Proposition 15 is not best possible.

Corollary 16. – *If the initial fortune of B is 2ξ, the probability that the game lasts at least until time $2n$ satisfies the asymptotic estimate :*

$$E_{2n} \leq \frac{2\xi}{\sqrt{\pi n}} + \sqrt{\frac{2}{n}}$$

This is an immediate consequence of Proposition 15.

The setting in terms of Gaussian integrals is much easier to handle, since these integrals are simpler to manipulate than binomial sums. Let us give a complete reinterpretation of the previous paragraph: energy absorption in case of a barrier at ξ.

In this continuous setting, there is no need to differentiate between the odd and even cases, which is also a simplification.

The symmetric of a point $A_{n,t}$ with respect to the barrier $y = \xi$ is $A_{n,2\xi-t}$. We have:

Proposition 17. - *The density of energy sent by 0 to the point $A_{n,t}$, taking into account the annihilation by the barrier, is :*

$$f_n(t) = \frac{1}{\sigma\sqrt{2\pi}}\left(\exp\left(-\frac{t^2}{2\sigma^2}\right) - \exp\left(-\frac{(2\xi - t)^2}{2\sigma^2}\right)\right), \textit{ for } t \leq \xi \textit{, 0 if } t > \xi$$

with $\sigma = \sqrt{n}$.

Proof of Proposition 17

This is a mere rephrasing of the previous results, but we see that the function is simply the difference of two gaussian functions with same variance.

From Proposition 17, we easily deduce the amount of energy on each vertical:

Corollary 18. - *At each step n, the energy left is:*

$$E_n = \int_{-\xi/\sqrt{n}}^{\xi/\sqrt{n}} e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi n}}$$

Proof of Corollary 18

Indeed, this follows from the formula:

$$E_n = \int_{-\infty}^{\xi} e^{-\frac{t^2}{2n}} \frac{dt}{\sqrt{2\pi n}} - \int_{-\infty}^{\xi} e^{-\frac{(2\xi - t)^2}{2n}} \frac{dt}{\sqrt{2\pi n}}$$

We deduce from Corollary 18 the asymptotic estimate:

$$E_n \sim \sqrt{\frac{2}{\pi n}}\, \xi$$

We also obtain the profile of energy, on the vertical W_n, that is the position of the point of maximal energy:

Proposition 19. - *The point of maximal energy on the vertical W_n is the unique $t < 0$ solution of the equation :*

$$\frac{t}{t - 2\xi} = \exp\left(\frac{-2\xi(\xi - t)}{n}\right)$$

If ξ is fixed and $n \to +\infty$, it satisfies:

$$t \approx -\frac{1}{2}\sqrt{\xi^2 + 4n} + \frac{3\xi}{2}$$

which shows that $t \to -\infty$ when $n \to +\infty$.

Proof of Proposition 19

We have to find the maximum of the function f_n. The derivative is:

$$f_n'(t) = \frac{1}{\sigma\sqrt{2\pi}}\left(-\frac{t}{\sigma^2}\exp\left(-\frac{t^2}{2\sigma^2}\right)+\frac{t-2\xi}{\sigma^2}\exp\left(-\frac{(t-2\xi)^2}{2\sigma^2}\right)\right)$$

So, the condition $f_n' = 0$ is equivalent to:

$$\frac{t}{t-2\xi} = \exp\left(\frac{t^2}{2\sigma^2}-\frac{(t-2\xi)^2}{2\sigma^2}\right) = \exp\left(\frac{-2\xi(\xi-t)}{n}\right) \tag{1}$$

Since the right hand side of (1) is positive, we must have $t<0$. Consider the function $h(t) = \frac{t}{t-2\xi}$; the derivative is $h'(t) = \frac{-2\xi}{(t-2\xi)^2} < 0$, so the function is decreasing, has the limit 1 at $-\infty$ and takes the value 0 at $t=0$. The function $g(t) = \exp\left(\frac{-2\xi(\xi-t)}{n}\right)$ is increasing, has limit 0 when $t \to -\infty$, takes the value $\exp\left(\frac{-2\xi^2}{n}\right) > 0$ at $t=0$. Therefore, a unique solution $t<0$ of equation (1) exists.

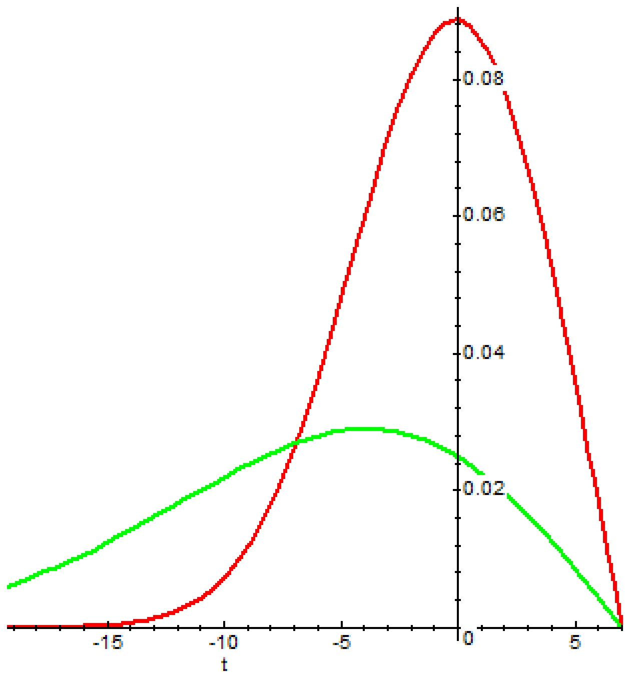
When $n \to +\infty$, we have the rough estimate:

$$\frac{t}{t-2\xi} \sim 1 - \frac{2\xi(\xi-t)}{n}$$

that is:

$$t \approx -\frac{1}{2}\sqrt{\xi^2+4n}+\frac{3\xi}{2}$$

which implies that $t_n \to -\infty$ and proves Proposition 19.



Graph of $f_n(t)$ *for* $\xi = 7$, $n = 20$ *(red),* $n = 100$ *(green)*

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