AN OPERATOR, ON A SEPARABLE HILBERT SPACE, WITH ALL POLYNOMIALS HYPERCYCLIC

by Bernard BEAUZAMY

Abstract. – We construct an operator T, on a separable Hilbert space, with one hypercyclic point x_0, and such that for any polynomial p, with complex coefficients, the point $p(T)x_0$ is also hypercyclic.

Institut de Calcul Mathématique
U.F.R. de Mathématiques
Université de Paris 7
2, place Jussieu
75 251 Paris Cedex 05
FRANCE

Let x_0 be a point in a Banach space E, and T be a linear operator on E. The orbit of x_0 under T is just the set of iterates

$$F_{x_0} \quad = \quad \{x_0, Tx_0, T^2x_0, \ldots\}.$$

The point x_0 is said to be *cyclic* for T if the vector space generated by F_{x_0} is dense in E, and *hypercyclic* if F_{x_0} itself is dense in E.

The invariant subspace problem, solved negatively by P. Enflo in Banach spaces (see P. Enflo [5]) and still unsolved in Hilbert spaces, can of course be rephrased as : Let T be an operator ; does there exist (beside 0) a point x_0 which is not cyclic ? In Enflo's example, all non-zero points are cyclic.

So one is naturally led to an investigation of the regularity of the orbits of a linear operator. Trying to find points for which the orbits is regular (meaning, for instance, that $\|T^n x\| \to \infty$, when $n \to \infty$) was done in our book [4], chap. 3. Here, conversely, we concentrate on irregular orbits : those of hypercyclic points, and try to construct operators with as many hypercyclic points as possible.

The first result in this direction was obtained by S. Rolewicz [8], who constructed on l_p $(1 \leq p < \infty)$ or c_0 an operator with one hypercyclic point. Of course, its iterates are also hypercyclic, but if one considers for instance $(x_0 + Tx_0)/2$, nothing says that this vector is still hypercyclic. Indeed, the construction can be modified in order to provide also a finite number of such vectors, but only a finite number.

The question was raised by P. Halmos [6] : can one produce an operator, in a separable Hilbert space, for which the set of hypercyclic points would contain a vector space ? We solve this question here. We don't prove that all points are cyclic, so our example might still have invariant subspaces. We don't know if this is the case or not.

As it is the case for the invariant subspace problem, things are more advanced in Banach spaces, and C. Read [7] has produced an example of an operator, on the space l_1, for which all non-zero points are hypercyclic. Previously, an example of an operator, with a slightly weaker property, called super- cyclicity, was constructed by the author [1] (for every non-zero point x_0, the half-lines generated by the orbit are dense in the whole space). Such operators, of course, have no invariant subspaces.

Our construction originates in the ideas introduced by P. Enflo to construct a Banach space and an operator on it with no non-trivial invariant subspaces [5]. But here we have a Hilbert space setting, and things become harder. A previous result in the same direction, also in a Hilbert space, was obtained by the author in [2], where an operator was constructed, with one hypercyclic point, and such that for any polynomial p with rational coefficients the point $p(T)x_0$ is also hypercyclic. Going from polynomials with rational coefficients to all polynomials is far from being as trivial as it may seem.

Our result had a preliminary announcement in [3].

Theorem A. – *There is a separable complex Hilbert space, an operator T on it, with an hypercyclic point x_0, such that for any polynomial p, with complex coefficients, the point $p(T)x_0$ is also hypercyclic.*

In fact, our construction provides a much stronger information. We denote by l^2_w the weighted l_2 space defined by :

$$l^2_w \quad = \quad \{(a_j)_{j\geq 0}\, ; \sum_{j\geq 0}(j+1)|a_j|^2 \; < \; +\infty\},$$

endowed with the norm $|(a_j)_{j\geq 0}|_w \; = \; (\sum_{j\geq 0}(j+1)|a_j|^2)^{1/2}$.

Theorem B. *– There is a separable Hilbert space H, completion of the polynomials in one variable x for a norm $\|.\| \leq |.|_w$, such that the multiplication by x is continuous on it and, for this operator, all non-zero elements of l^2_w are hypercyclic.*

The construction of this example will occupy the rest of this section, and will be divided into several steps.

§1. Enumeration of the triples.

We first enumerate the triples $(q_j, q'_j, \varepsilon_j)_{j\geq 1}$, where :
- q_j, q'_j are polynomial with rational coefficients (that is : both real and imaginary parts rational),
- q_j has always " 1 " as the first non-zero coefficient (the degrees being written in increasing order),
- ε_j is of the form $1/2^l$, $l \geq 1$.

Doing this enumeration, we require, for all $j \geq 1$:
a) $d^o\ q_j < j$, $d^o\ q'_j < j$,
b) $\varepsilon_j \geq 2^{-j}$,
c) if $n_1 < \cdots < n_k < \cdots$ are integers such that :

$$\begin{cases} q_{n_1} = q_{n_2} = \cdots = q_{n_k} = \cdots \\ q'_{n_1} = q'_{n_2} = \cdots = q'_{n_k} = \cdots \end{cases}$$

then $\varepsilon_{n_1} > \varepsilon_{n_2} > \cdots > \varepsilon_{n_k} > \cdots$.

These requirements can easily be met the following way : in $\mathbb{R}^3$, we write an enumeration of the polynomials with rational coefficients on the " x " axis and on the " y " axis, taking into account condition a). On the " z " axis, we put 2^{-l} at $z = l$, for $l = 1, 2, \ldots$. Then, for each k, we enumerate entirely the set $\{(x + y + z) \leq k\}$, before enumerating $\{(x + y + z) \leq k + 1\}$, and we realize this by enumerating the set $\{(x + y) \leq k'\}$, for increasing $k' \leq k$.

We consider the norm $|(a_j)_{j\geq 0}|_w$ as a norm on the space of polynomials, and, if $p = \sum_{i\geq 0} a_i x^i$, we define : $|p|_w = (\sum_{i\geq 0}(i+1)|a_i|^2)^{1/2}$.

The space l^2_w thus defined is a Hilbert space, an algebra, and multiplication by x has norm $\sqrt{2}$.

We now define the *systems*. For every $j \geq 1$, we put :

$$\bar{\jmath} = \inf\{j' \ ; \ |q_j - q_{j'}|_w < 1/2, \ q'_j = q'_{j'} \ , \ \varepsilon_j = \varepsilon_{j'}\}$$

We then say that j belongs to the *system* of the integer $\bar{\jmath}$. We observe that here all systems are defined at once, and not inductively, contrarily to what we did in [2].

The enumeration of systems will be made with greek letters, $\nu = 1, 2, \ldots$, so, for every j, the value of $\bar{\jmath}$ will be a greek letter, eg $\bar{\jmath} = \nu$.

We observe that, by the definition of the systems and the normalization we have chosen, the index of the first non-zero coefficient in q_j depends only on $\bar{\jmath}$. We call it m_ν, if $\bar{\jmath} = \nu$, so we write :

$$q_j = x^{m_\nu} + b^{(j)}_{m_\nu+1} x^{m_\nu+1} + \ldots \qquad (1)$$

We finally introduce the following notations :

$$\theta_j = |q_j|_w \ , \ \ \theta'_j = |q'_j|_w \ , \ \ \theta^*_n = \max_{j\leq n} \theta_j \ , \ \ {\theta'}^*_n = \max_{j\leq n} \theta'_j.$$

Our construction will be totally determined by a sequence of integers $(N_j)_{j\geq 0}$, strictly increasing, which will be chosen by induction. If $\bar{\jmath} = \nu$, we put $l_j = x^{N_\nu}$.

2. The norm $\|.\|_n$.

We define $\|.\|_{(0)} = \|.\|_w$. Fix now an integer $n \geq 1$. For any polynomial p, we look at all representations of the form :

$$p \; = \; r \; + \; \sum_{j=1}^{n} \sum_{\alpha} a_{j,\alpha} x^{\alpha} (l_j q_j - q_j') \; , \tag{2}$$

where r is a polynomial, $a_{j,\alpha}$ are complex numbers ($j = 1, \ldots, n$, $\alpha \in \mathbb{N}$), and we define :

$$|p|^2_{(n)} \; = \; \inf\{|r|^2_w + \sum_{\nu \geq 1} \sum_{\alpha \geq 0} 4^{\alpha} \varepsilon^2_{\nu} (|\sum_{j,\bar{j}=\nu} a_{j,\alpha} q_j|^2_w (\theta^2_{\nu} + 1)^{-1} + |\sum_{j,\bar{j}=\nu} a_{j,\alpha}|^2)\} \tag{3}$$

where the infimum is taken over all representations of the form (2).

In order to simplify this expression, we take the following notations :

$$A \; = \; \sum_{j=1}^{n} \sum_{\alpha} a_{j,\alpha} x^{\alpha} (l_j q_j - q_j') \; ,$$

$$[A]^2_{(n)} \; = \; \sum_{\nu \geq 1} \sum_{\alpha \geq 0} 4^{\alpha} \varepsilon^2_{\nu} (|\sum_{j,\bar{j}=\nu} a_{j,\alpha} q_j|^2_w (\theta^2_{\nu} + 1)^{-1} + |\sum_{j,\bar{j}=\nu} a_{j,\alpha}|^2) \; .$$

Quite clearly, we have :

$$|p_1 + p_2|_{(n)} \; \leq \; |p_1|_{(n)} + |p_2|_{(n)} \; ,$$

$$|\lambda p|_{(n)} \; = \; |\lambda| |p|_{(n)} \; , \qquad \text{for } \lambda \in \mathbb{C}$$

and $p = 0$ implies $|p|_{(n)} = 0$. The converse of this implication comes up only at the end of the construction. Despite this fact, we will speak of the "norm" $|.|_{(n)}$, but we keep in mind that it is only a quasi-norm. The following properties of the norm $|.|_{(n)}$ will be used :

Proposition 1. – *For all $n \geq 1$:*

a)

$$|.|_{(n)} \; \leq \; |.|_{(n-1)} \; \leq \cdots \leq |.|_w \; ,$$

b)

$$|(l_j q_j - q_j')|_{(n)} \; \leq \sqrt{2} \; \varepsilon_j \; , \qquad j = 1, \ldots, n$$

c)

$$|l_j p|_{(n)} \; \leq \; 2^{N_{\nu}} \; |p|_{(n)} \; , \qquad \text{if } \bar{j} = \nu \; , \; j \leq n$$

d)

$$|xp|_{(n)} \; \leq \; 2|p|_{(n)} \; ,$$

e) The norm $|.|_{(n)}$ is hilbertian.

Proof. – a) is obvious, b) follows from the representation of $l_j q_j - q_j'$ with $r = 0$ and all the $a_{i,\alpha} = 0$ except $a_{j,0} = 1$. c) follows from the representation obtained by replacing x^{α} by $x^{\alpha} + N_{\nu}$.

To see e), we observe that, for all p_1, p_2,

$$2(|p_1|^2_{(n)} \; + \; |p_2|^2_{(n)} \; \geq \; |p_1 + p_2|^2_{(n)} \; + \; |p_1 - p_2|^2_{(n)}$$

and the converse inequality follows after the change of variables $u = p_1 + p_2$, $v = p_1 - p_2$.

Remark. – We could make the norm $|.|_{(n)}$ equivalent (with constants depending on n) to $|.|_w$ by adding to $[A]_n^2$ the term $\sum_{j\geq 1}\sum_{\alpha\geq 0} 4^\alpha \varepsilon_j^2$. This would change nothing to our construction.

We also need the following obvious :

Lemma 2. – *For all polynomials p_1, p_2, every η, $0<\eta<1$, if $C\geq (1-\eta)/\eta$,*

$$|p_1+p_2|_w^2 + |p_2|_w^2 \;\geq\; (1-\eta)|p_1|_w^2$$

The proof is left to the reader.

§3. Study of $|1|_{(n)}$.

Proposition 3. – *If the sequence $(N_j)_{j\geq 0}$ grows fast enough, we have*

$$|1|_{(n)} \;\geq\; 1/2\,, \qquad \text{for every } n\geq 1.$$

Proof of Proposition 3. – We know that $|1|_{(n)} \leq |1|_w \;=\; 1$. Therefore, we can find a representation of 1, of the form (2),

$$1 \;=\; 1 \;-\; A \;+\; A \qquad (4),$$

which gives the estimate

$$\mathcal{C}_n^2 \;=\; |1-A|_w^2 \;+\; [A]_{(n)}^2 \qquad (5)$$

with

$$\mathcal{C}_n^2 \;\leq\; 4 \qquad (6)$$

We will show that $\mathcal{C}_n^2 \;\geq\; 1/4$, and this will prove our Proposition. In order to do so, we first need a control upon the high degree terms (that is α large) in A.

Lemma 4. – *Set*

$$K \;=\; 4\log_2(2^{4\nu}(\sqrt{N_\nu+1}\sqrt{\theta_\nu^{*2}+1} \;+\; \theta_\nu^{'*}))$$

The representation obtained from (4) by keeping in A, for all j, only the terms with :

$$\begin{cases} \alpha\leq K_{\mu-1} & \text{if } \bar{\jmath}<\mu \\ \alpha\leq K_\mu & \text{if } \bar{\jmath}=\mu \end{cases}$$

gives an estimate $\mathcal{C'}_n^2$ with

$$\mathcal{C}_n^2 \;\geq\; \frac{1-8^{-\mu}}{1+8^{-\mu}}\,\mathcal{C'}_n^2$$

Proof of Lemma 4. – Set $K=K_\mu$. We write :

$$A' \;=\; \sum_{j=1}^{n}\sum_{\alpha\leq K} a_{j,\alpha}x^\alpha(l_jq_j-q_j')\;,$$

$$A'' \;=\; A \;-A'\;,\quad I" \;=\; |A''|_w\;,$$

$$I''_\nu \;=\; |\sum_{\bar{\jmath}=\nu}\sum_{\alpha>K} a_{j,\alpha}x^\alpha(l_jq_j-q_j')|_w\;,$$

so $I'' \leq \sum_\nu I''_\nu$. We will now estimate I''_ν.

$$I''_\nu \leq \sum_{\alpha>K} \sqrt{\alpha+1}(|\sum_{\bar{j}=\nu} a_{j,\alpha} l_j q_j|_w + |\sum_{\bar{j}=\nu} a_{j,\alpha} q'_j|_w).$$

But $l_j = x^{N_j}$, $q'_j = q'_\nu$ if $\bar{j}=\nu$. So :

$$\begin{aligned} I''_\nu &\leq \sum_{\alpha>K} \sqrt{\alpha+1}(\sqrt{N_\nu+1}|\sum_{\bar{j}=\nu} a_{j,\alpha} q_j|_w + \theta'_\nu |\sum_{\bar{j}=\nu} a_{j,\alpha}|) \\ &\leq \frac{1}{\varepsilon_\nu}\sqrt{N_\nu+1}(\sum_{\alpha>K}(\alpha+1)4^{-\alpha})^{1/2}(\sum_\alpha 4^\alpha |\sum_{\bar{j}=\nu} a_{j,\alpha} q_j|_w^2 \varepsilon_\nu^2)^{1/2} \\ &\quad + \theta'_\nu \frac{1}{\varepsilon_\nu}(\sum_{\alpha>K}(\alpha+1)4^{-\alpha})^{1/2}(\sum_\alpha 4^\alpha |\sum_{\bar{j}=\nu} a_{j,\alpha}|^2 \varepsilon_\nu^2)^{1/2} \\ &\leq 2^\nu \sqrt{N_\nu+1}\, 2^{-K/2}\sqrt{\theta_\nu^2+1}\, \mathcal{C}_n + 2^{\nu-K/2}\, \theta'_\nu\, \mathcal{C}_n \\ &\leq 2^{\nu-K/2}(\sqrt{N_\nu+1}\sqrt{\theta_\nu^2+1}+\theta'_\nu)\mathcal{C}_n \end{aligned}$$

So :

$$\begin{aligned} \sum_{\nu=1}^{\mu-1} I''_\nu &\leq 2^{-K_{\mu-1}} 2^{\mu-1}(\sqrt{N_{\mu-1}+1}\sqrt{\theta_{\mu-1}^{*2}+1}+\theta_{\mu-1}^{'*})\mathcal{C}_n \\ &\leq 8^{-\mu}\mathcal{C}_n/2 \end{aligned}$$

by the choice of $K_{\mu-1}$.

Now, from Lemma 2, with $\eta = 8^{-\mu}$:

$$|1-A'-A''|_w^2 + 8^\mu |A''|_w^2 \geq (1-8^{-\mu})|1-A'|_w^2$$

and so

$$\mathcal{C}_n^2 \geq \frac{1-8^{-\mu})}{1+8^{-\mu}}\, \mathcal{C}'_n$$

as stated.

So now, instead of (4), we have a representation :

$$1 = 1 - A' + A' \tag{9}$$

with :

$$A' = \sum_{\bar{j}<\mu}\sum_{\alpha\leq K_{\mu-1}} a_{j,\alpha} x^\alpha (l_j q_j - q'_j) + \sum_{\bar{j}=\mu}^{n}\sum_{\alpha\leq K_\mu} a_{j,\alpha} x^\alpha (l_j q_j - q'_j) \tag{10}$$

We put $s_j = \sum_{\alpha\leq K_{\mu-1}} a_{j,\alpha} x^\alpha$, for $\bar{j}<\mu$, and $s_j = \sum_{\alpha\leq K_\mu} a_{j,\alpha} x^\alpha$, for $\bar{j}=\mu$. For a polynomial $p=\sum c_j x^j$ and $k\in \mathbb{N}$, we put

$$p|^k = \sum_{j\leq k} c_j x^j\ ,\quad p|_k = \sum_{j>k} c_j x^j.$$

We write (9) as

$$1 = 1 - \sum_{\bar{j}<\mu} s_j(l_j q_j - q'_j) + \sum_{\bar{j}=\mu} s_j q'_j - \sum_{\bar{j}=\mu} s_j l_j q_j + A' \tag{11}$$

We make the following induction hypothesis : if $\mathcal{C}_n^2 \leq 4$, then :

$$|\sum_{\bar{j}=\nu} s_j q'_j|^0| \leq 8^{-\nu}\ ,\quad \text{for } \nu<\mu \tag{12}$$

and now we prove it for $\nu=\mu$.

Assume that this is false. Then : $|\sum_{\bar{j}=\mu} s_j q_j'|^0| \; > \; 8^{-\mu}$ and therefore :

$$|\sum_{\bar{j}=\mu} s_j|^0| \; > \; \frac{1}{8^\mu \theta_\mu'} \tag{13}$$

From (6) and Lemma 5 follow that $\mathcal{C}_n'^2 \; \leq \; 8$. Therefore :

$$\sum_\alpha 4^\alpha |\sum_{\bar{j}=\mu} a_{j,\alpha}|^2 \varepsilon_\mu^2 \; \leq \; 8$$

$$\sum_\alpha 4^\alpha |\sum_{\bar{j}=\mu} a_{j,\alpha}|^2 \; \leq \; 2^{2\mu+3} \tag{14}$$

Let $D \in \mathbb{N}$. We have, since $d^o \; q_\mu' < \mu$:

$$\begin{aligned} &|\sum_{\bar{j}=\mu} s_j q_j'|_D|_w \; = \; |(q_\mu' \sum_{\bar{j}=\mu} s_j)|_D|_w \\ &= |q_\mu'(\sum_{\bar{j}=\mu} s_j|_{D-\mu})|_D|_w \; \leq |q_\mu'(\sum_{\bar{j}=\mu} s_j|_{D-\mu})|_w \\ &\leq \; \theta_\mu' |(\sum_{\bar{j}=\mu} s_j)|_{D-\mu}|_w \; = \; \theta_\mu'(\sum_{\alpha > D-\mu} |\sum_{\bar{j}=\mu} a_{j,\alpha}|^2 (\alpha+1)^{1/2}) \\ &\leq \theta_\mu'(D-\mu) \; 4^{\mu-D} \; 2^{2\mu+3} \; \leq \; 1/64 \end{aligned}$$

by a proper choice of $D = D_\mu$, independent (of course !) of N_μ.

Now, put $p_0 \; = \; 1 - \sum_{\bar{j}<\mu} s_j(l_j q_j - q_j')$. Applying again Lemma 2 with $\eta = 1/8, \; C = 8$, we get :

$$|p_0 + \sum_{\bar{j}=\mu} s_j q_j'|^D + \sum_{\bar{j}=\mu} s_j q_j'|_D - \sum_{\bar{j}=\mu} s_j l_j q_j|_w + 1/8$$

$$\geq \quad (1-1/8)|p_0 + \sum_{\bar{j}=\mu} s_j q_j'|^D - \sum_{\bar{j}=\mu} s_j l_j q_j|_w \; ,$$

and therefore :

$$\mathcal{C}_n'^2 + 1/8 \; \geq \; (1-1/8)|p_0 + \sum_{\bar{j}=\mu} s_j q_j'|^D - \sum_{\bar{j}=\mu} s_j l_j q_j|_w.$$

The degree of p_0 is at most $K_{\mu-1}N_{\mu-1} + \mu - 1$. So, if

$$N_\mu \; > \; \max(D_\mu, K_{\mu-1}N_{\mu-1} + \mu - 1) \; ,$$

$$|p_0 + \sum_{\bar{j}=\mu} s_j q_j'|^D - \sum_{\bar{j}=\mu} s_j l_j q_j|_w^2 \; = \; |p_0 + \sum_{\bar{j}=\mu} s_j q_j'|^D|_w + |\sum_{\bar{j}=\mu} s_j l_j q_j|_w^2$$

$$\geq \; |\sum_{\bar{j}=\mu} s_j l_j q_j|_w^2 \; = \; |x^{N_\mu} \sum_{\bar{j}=\mu} s_j q_j|_w^2.$$

But q_j, $\bar{j} = \mu$, starts with x^{m_μ}. So :

$$|x^{N_\mu} \sum_{\bar{j}=\mu} s_j q_j|_w \; \geq \; |x^{N_\mu + m_\mu}|_w / (8^\mu \theta_\mu') \; , \qquad \text{by (13)}$$

$$\geq \frac{1}{8^\mu \theta_\mu'} \; \sqrt{N_\mu + 1} \; > \; 16 \; ,$$

by a proper choice of N_μ, and this contradicts (12).

Now, we look at (11) once again. We have :

$$\begin{aligned}
\mathcal{C}_n'^2 &\geq |1-\sum_{\bar{\jmath}<\mu} s_j(l_jq_j-q_j') \;+\; \sum_{\bar{\jmath}=\mu} s_jq_j' \;-\; \sum_{\bar{\jmath}=\mu} s_jl_jq_j|_w \\
&\geq |(1-\sum_{\bar{\jmath}<\mu} s_j(l_jq_j-q_j') \;+\; \sum_{\bar{\jmath}=\mu} s_jq_j')|^0|_w \\
&\geq |1-(\sum_{\bar{\jmath}\leq\mu} s_jq_j')|^0|_w \\
&\geq 1-\sum_{\nu\geq 1} 8^{-\nu} \;\geq\; 3/4
\end{aligned}$$

and this proves Proposition 3.

§4. The final norm.

Let now $\|p\| \;=\; \lim |p|_{(n)}$, for any polynomial p. We have the following properties of the limit "norm" $\|p\|$:

Proposition 5.

a) $\|.\| \;\leq\; |.|_w$

b) $\|l_jq_j-q_j'\| \;\leq \sqrt{2}\;\varepsilon_j$, for all $j\geq 1$

c) $\|l_jp\| \;\leq\; 2^{N_\nu}\|p\|$, for all ν, all j, $if\ \bar{\jmath}=\nu$

d) $\|xp\| \;\leq 2\|p\|$

e) $\|1\| \;\geq\; 1/2$

f) The norm $\|.\|$ is hilbertian.

Let now H be the completion of the space of polynomials with complex coefficients, under the norm $\|.\|$. It follows from d) that the operator T of multiplication by x is continuous on H, and satisfies $\|T\| \;\leq 2$.

Theorem 6. – *In the space H, all elements of l_w^2 (except 0) are hypercyclic for T.*This means : for every $\varepsilon>0$, every q inl_w^2, every q' in H, there is a $N\geq 1$ with :

$$\|T^Nq-q'\| \;\leq \varepsilon \tag{15}$$

Proof of Theorem 6. – We may assume that the first non-zero coefficient is 1 : indeed, if this coefficient is c, we prove that :

$$\|T^N(q/c)-q'/c\| \;\leq \varepsilon/|c|.$$

Now, we observe that it's enough to prove (15) when q' has rational coefficients, because there is such a q" with $\|q\text{"}-q'\| \;\leq\; |q\text{"}-q'|_w \;\leq\; \varepsilon/2$, and if $\|T^Nq-q\text{"}\| \;\leq\; \varepsilon/2$, then $\|T^Nq-q'\| \;\leq\; \varepsilon$. We may also assume, of course, that ε is of the form $1/2^l$, $l\geq 1$.

So there is a sequence $(n_j)_{j\geq 0}$ of integers in the enumeration with $q_{n_j}\to q$ in l_w^2, $q_{n_j}'=q'$, $\varepsilon_{n_j}=\varepsilon$, for all j. We may finally assume that $|q_{n_j}-q_{n_1}|_w \;<\; 1/2$, so $\overline{n_j} \;\leq\; n_1$, for $j\geq 1$. Let $\|.\|_{op}$ denote the operator norm from $\|.\|$ into itself. By Proposition 5, c), we have :

$$\|l_{n_j}\|_{op} \;\leq\; 2^{N_{n_1}} .$$

Therefore,

$$\begin{aligned}
\|l_{n_j}q-q'\| &\leq\; \|l_{n_j}q_{n_j}-q'\|+\|l_{n_j}\|_{op}\|q_{n_j}-q\| \\
&\leq\; \sqrt{2}\varepsilon/4+2^{N_{n_1}}\|q_{n_j}-q\|
\end{aligned}$$

and $q_{n_j}-q\to 0$, so $\|l_{n_j}q-q'\| \;\leq\; \varepsilon/2$, for j large enough, and Theorem 6 is proved.

The fact that $\|.\|$ is a norm on the space of polynomials follows immediately from Theorem 6 and Proposition 3. Indeed, for every p, there is a l such that $\|lp-1\|<1/4$, so $\|lp\| \;>\; 1/4$, and $\|p\| \;>\; \|l\|_{op}/4$.

Remark. – We observe that our construction has the following property, which we may call "central action" :

The l_j which acts on q_j (that is, satisfying for instance $\|l_j q_j - 1\| < \varepsilon$) depends only on $\bar{j}$ and not on j itself. For instance, for a given q, the same x^{N_ν} satisfies $\|x^{N_\nu} q_j - 1\| < 1/2$ if $|q_j - q|_w < 1/2$.

This property holds because the "systems" are computed with respect to the norm $|.|_w$ and not in the final norm. As we will see, such a simple description is excluded if one wants to construct an operator with all vectors hypercyclic, and, in this respect, our example has the strongest possible property.

Indeed, assume that for every $\varepsilon > 0$, every q, there is a polynomial l such that if $\|q' - q\| < \varepsilon$, then $\|lq - 1\| < \varepsilon$. Then $\|l(q - q')\| < 2\varepsilon$, and $\|l\|_{op} \leq 2$.

Now, let p_n be a sequence of almost eigenvectors, with respect to some λ, $\lambda \in \sigma(T)$. So we have $\|p_n\| = 1$, and $(x - \lambda)p_n \to 0$. Let l_n be the polynomials satisfying $\|l_n p_n - 1\| < \varepsilon$. By the previous computation, $\|l_n\| \leq 2$. But :

$$\|l_n(x - \lambda)p_n - (x - \lambda)\| \;\leq\; \varepsilon\|x - \lambda\|_{op}$$

and since $\|l\|_{op}$ is bounded, $l_n(x - \lambda)p_n \to 0$, thus $\|x - \lambda\| \leq \varepsilon\|x - \lambda\|_{op}$; a contradiction if originally ε was chosen small enough.

References

[1] Beauzamy, B. : Un opérateur sans sous-espace invariant non- trivial : simplification de l'exemple de P. Enflo. *Integral Eq. and Operator Theory*, vol. 8, 1985, p. 1–70.

[2] Beauzamy, B. : An operator, on a separable Hilbert space, with many hypercyclic vectors. *Studia Math.* 1987, pp.71–78.

[3] Beauzamy, B. : Un opérateur, sur l'espace de Hilbert, dont tous les polynômes sont hypercycliques. *Note C.R.A.S.*, Paris, t. 303, Série I, n. 18, 1986, p. 923–927.

[4] Beauzamy, B. : Introduction to Operator Theory and Invariant Subspaces. *North-Holland*, Mathematical Library, 1988.

[5] Enflo, P. : On the invariant subspace problem in Banach spaces. *Acta Math.* , 1987.

[6] Halmos, P. : A Hilbert space problem book. *Graduate Texts in Mathematics*, 2nd edition.

[7] Read, C. : The invariant subspace problem for a class of Banach spaces, 2: hypercyclic operators. *Israël J. of Maths*, 63 (1988), pp.1–40.

[8] Rolewicz, S. : On Orbits of elements. *Studia Math.*, 32, 1969, p. 17–22.