Correction to the paper "Estimates for $H^2$-functions ..."

by Bernard Beauzamy

As pointed out by [], Lemma 2.6 is not correct, so the proof of Proposition 2.5 has to be slightly modified, and the statement of Theorem 2.7 (derived the same way) contains slightly different estimates.

**Proposition 2.5.** – *If* $\delta_k(f) < (1-|\alpha|)/8$, *then*

$$\delta_{k-1}(g) \;\leq\; \frac{4}{1-|\alpha|}\,\delta_k(f)\ .$$

**Proof**. – First, we write the Taylor expansion of $f$ :

$$f(z) \;=\; \alpha b_0 + (-b_0+\alpha b_1)z + \cdots + (-b_{j-1}+\alpha b_j)z^j + \cdots \tag{2.13}$$

We write $d$ instead of $\mathrm{cf}_k(f)$, and $\delta$ instead of $\delta_k(f)$. We deduce from (2.13) :

$$\sum_{k+1}^{\infty} |-b_{j-1}+\alpha b_j|^2 \;=\; \delta^2\big(|\alpha b_0|^2 + \sum_{0}^{k-1}|-b_j+\alpha b_{j+1}|^2\big). \tag{2.14}$$

But :

$$\begin{aligned}
|b_k| \;&\leq\; \sum_0^\infty |\alpha|^j|-b_{k+j}+\alpha b_{k+j+1}| \\
&\leq (\sum_0^\infty |\alpha|^{2j})^{1/2}\,(\sum_0^\infty |-b_{k+j}+\alpha b_{k+j+1}|^2)^{1/2} \\
&\leq \left(\frac{1}{1-|\alpha|^2}\right)^{1/2} (\sum_0^\infty |-b_{k+j}+\alpha b_{k+j+1}|^2)^{1/2}
\end{aligned}$$

So we deduce from (2.14) :

$$(1-|\alpha|^2)|b_k|^2 \;\leq\; \sum_{k+1}^\infty |-b_{j-1}+\alpha b_j|^2 \;\leq\; 3\delta^2\sum_0^k |b_j|^2\ ,$$

and this implies

$$|b_k|^2 \;\leq\; \frac{3\delta^2}{1-|\alpha|^2-3\delta^2}\sum_0^{k-1}|b_j|^2\ . \tag{2.15}$$

We also have

$$(\sum_{k+1}^\infty |-b_{j-1}+\alpha b_j|^2)^{1/2} \;\geq\; (1-|\alpha|)(\sum_{k+1}^\infty |b_j|^2)^{1/2}\ ,$$

which implies

$$\sum_{k+1}^\infty |b_j|^2 \;\leq\; \frac{3\delta^2}{(1-|\alpha|)^2}\sum_0^k |b_j|^2\ . \tag{2.16}$$

Using (2.15), we deduce from (2.16)

$$\sum_{k+1}^{\infty} |b_j|^2 \leq \frac{3\delta^2}{(1-|\alpha|)^2}\left(1+\frac{3\delta^2}{1-|\alpha|^2-3\delta^2}\right)\sum_{0}^{k-1}|b_j|^2 . \tag{2.17}$$

Using (2.15) once again, we finally obtain

$$\begin{aligned}\sum_{k}^{\infty} |b_j|^2 &\leq \left(\frac{3\delta^2}{(1-|\alpha|)^2}\left(1+\frac{3\delta^2}{1-|\alpha|^2-3\delta^2}\right)+\frac{3\delta^2}{1-|\alpha|^2-3\delta^2}\right)\sum_{0}^{k-1}|b_j|^2\\ &\leq \frac{6\delta^2}{(1-|\alpha|)(1-|\alpha|^2-3\delta^2}\sum_{0}^{k-1}|b_j|^2 ,\end{aligned}$$

from which the Proposition follows immediately.

We can now prove :

**Theorem 2.7.** – *Let $f$ be a function in $H^2$, with the zeros written in increasing order :*

$$|\alpha_1| \leq |\alpha_2| \leq |\alpha_3| \leq \cdots$$

*Then the $k+1$-st zero $\alpha_{k+1}$ satisfies :*

$$|\alpha_{k+1}| \geq 1-4\delta^{1/(k+1)} ,$$

*with $\delta=\delta_k(f)$.*

**Proof**. – The case $k=0$ is left to the reader, and we assume $k\geq 1$. We write

$$f = (\alpha_1 - z)\cdots(\alpha_k - z)g.$$

We first observe that

$$|\alpha_{k+1}| \geq \mathrm{cf}_0(g) . \tag{2.18}$$

Indeed, $\alpha_{k+1}$ is the first zero of $g$. Jensen's formula gives :

$$|g(0)| \prod_{\substack{n\geq k+1\\ |\alpha_n|\leq 1}} \frac{1}{|\alpha_n|} \leq M(g) \leq |g|_2 ;$$

but since

$$\prod_{\substack{n\geq k+1\\ |\alpha_n|\leq 1}} \frac{1}{|\alpha_n|} \geq \frac{1}{|\alpha_{k+1}|} ,$$

we deduce

$$|\alpha_{k+1}| \geq \frac{|g(0)|}{|g|_2} ,$$

as we claimed.

Since $\mathrm{cf}_0^2(g)=1/(1+\delta_0^2(g))$, we deduce from (2.18)

$$1-|\alpha_{k+1}| < \delta_0^2(g). \tag{2.19}$$

.

We consider two cases :

**Case 0**. – $|\alpha_k| \ \geq \ 1 - 4\delta^{1/(k+1)}$ .

Then a fortiori $|\alpha_{k+1}|$ satifies the same estimate, and the theorem is proved, or

**Case 1**. – $|\alpha_k| \ < \ 1 - 4\delta^{1/(k+1)}$ .

We now consider this last case. Then also $|\alpha_1|, \cdots, |\alpha_k|$ satisfy this estimate, which implies

$$\delta \ < \ \frac{(1-|\alpha_k|)^{k+1}}{4^{k+1}} \ . \tag{2.20}$$

Set now $f_1 = f\,,\ f_2 = (\alpha_2 - z)\cdots(\alpha_k - z)g\,,\ \ldots,\ f_k \ = \ (\alpha_k - z)g\,,\ f_{k+1} = g\,.$

Since $\delta < (1-|\alpha_1|)/8$, Proposition 2.5 implies

$$\delta_{k-1}(f_2) \ < \ \frac{4\delta}{1-|\alpha_1|} \ ,$$

and by (2.20),

$$\frac{4\delta}{1-|\alpha_1|} \ < \ \frac{1-|\alpha_2|}{8} \ .$$

Therefore, Proposition 2.5 gives

$$\delta_{k-2}(f_3) \ < \ \frac{4\delta_{k-1}(f_2)}{1-|\alpha_2|} \ < \ \frac{4^2\delta}{(1-|\alpha_1|)(1-|\alpha_2|)} \ .$$

Since for every $j = 1, \ldots, k$, condition (2.20) implies :

$$\frac{4^{j-1}\delta}{(1-|\alpha_1|)\cdots(1-|\alpha_{j-1}|)} \ < \ \frac{1-|\alpha_j|}{8} \ ,$$

Proposition 2.5 gives

$$\delta_{k-j}(f_{j+1}) \ < \ \frac{4\delta_{k-j+1}(f_j)}{1-|\alpha_j|} \ < \ \frac{4^j\delta}{(1-|\alpha_1|)\cdots(1-|\alpha_j|)} \ .$$

Finally, for $j = k$, we get

$$\delta_0(g) \ < \ \frac{4^k\delta}{(1-|\alpha_1|)\cdots(1-|\alpha_k|)} \ \leq \ \frac{4^k\delta}{(1-|\alpha_k|)^k} \ . \tag{2.21}$$

Taking (2.20) into account once again gives :

$$\delta_0(g) \ < \ \delta^{1/(k+1)} \ ,$$

and by (2.19),

$$1 - |\alpha_{k+1}| \ < \ \delta^{2/(k+1)} \ < \ 4\delta^{1/(k+1)} \ ,$$

and the Theorem is proved.

From these estimates one can easily deduce an asymptotic behavior of $|\alpha_{k+1}|$ when $d$ is close to 1 :

**Corollary 2.8.** – *When* $d \to 1^-$, *the* $k+1$*-st zero of* $f$ *satisfies :*

$$1 - |\alpha_{k+1}| \ \sim \ 4\big(2(1-d)\big)^{1/2(k+1)} \ .$$

We have investigated, so far, the structure of the set $\{f = 0\}$. We now turn to the set $\{|f| < \varepsilon\}$.