POLYNOMIALS WITH COMPLEX COEFFICIENTS :

SIZE OF THE FACTORS, REPARTITION OF THE ZEROS.

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Abstract. – We relate the size of the factors of a polynomial with the repartition of its zeros. First, we show that a polynomial with zeros on the unit circle always has a factor which is exponentially large. Then we give a symbolic formula, valid in the distribution sense, which allows one to reconstruct a polynomial from the repartition function of its zeros. From this formula we deduce a reciprocal to a well-known result of Erdös and Turan.

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We deal here with polynomials $P(z) = \sum_0^n a_j z^j$, with complex coefficients, normalized with leading coefficient 1. We write such a polynomial under the form

$$P(z) \;=\; \prod_1^n (z - z_j).$$

A *factor* of P is a polynomial $Q = \prod_J (z - z_j)$, where J is any subset of $\{1, \ldots, n\}$. We are interested in relating the size of coefficients in the factors of P and the repartition of the zeros of P. Upper bounds for the size of the factors were given by the author in [1] and [2] ; we deal here with lower bounds.

In the first part of the present paper, using a result of Erdös-Turan [3], we show that any polynomial with zeros on the unit circle has a factor which is exponentially large.

In the second part, we give a symbolic formula, valid in the distribution sense, which allows to reconstruct the polynomial from the repartition function of its zeros.

1. Size of Factors.

We define the L_∞-norm :

$$\|P\|_\infty \;=\; \max_{\theta \in [0, 2\pi]} |P(e^{i\theta})|.$$

We have :

Theorem 1. – *Let P be a polynomial of degree n, with all zeros on the unit circle. Then P has a factor Q with*

$$\|Q\|_\infty \;\geq\; e^{\delta n - 1} ,$$

where δ is an absolute constant : $\delta \sim 0.00196$.

Proof. – We may assume that $\|P\|_\infty = |P(1)|$. Let $z_j = e^{i\theta_j}$, $j = 1, \ldots, n$, be the roots of P ; we may order them :

$$0 \;<\; |\theta_1| \;\leq\; \ldots \;\leq\; |\theta_n| \;\leq\; \pi,$$

which implies

$$|1 - z_1| \;\leq\; \ldots \;\leq\; |1 - z_n|. \tag{1}$$

Let $Q_k = (z - z_{k+1}) \cdots (z - z_n)$, $C_k = |Q_k(1)|/|P(1)|$, and $C = \max_k C_k$. Then, for all $k = 1, \ldots, n$,

$$\frac{1}{C} \;\leq\; |1 - z_1| \cdots |1 - z_k|, \tag{2}$$

which implies by (1) :

$$\frac{1}{C} \;\leq\; |1 - z_k|^k ,$$

or

$$\frac{1}{C^{1/k}} \;\leq\; 2 \sin \frac{|\theta_k|}{2} \;\leq\; |\theta_k|. \tag{3}$$

We now use a well-known result due to Erdös-Turan [3], which asserts that, if the ratio $\|P\|_\infty / \sqrt{|a_0 a_n|}$ is not too large, the roots are uniformly distributed in different angles with vertex at O. Precisely, if $N_{\alpha,\beta}$ is the number of roots with $\arg z_j \in [\alpha, \beta]$, then :

$$\left| N_{\alpha,\beta} - \frac{\beta - \alpha}{2\pi} n \right| \;\leq\; c \sqrt{n \log \frac{\|P\|_\infty}{\sqrt{|a_0 a_n|}}} \;;$$

Erdös-Turan obtained $c = 16$; a value which was improved later by Ganelius [4] :

$$c \;=\; \sqrt{2\pi / c'} \;, \qquad \text{with } c' \;=\; \int_1^\infty \frac{\log t}{1 + t^2}\, dt,$$

that is $c \sim 2.6190$.

By our ordering of the zeros, the sector $(|\theta_k|, 2\pi - |\theta_k|)$ contains $n-k$ zeros. So

$$\left| \frac{n-k}{n} - \frac{2\pi - 2|\theta_k|}{2\pi} \right| \ \leq \ c\sqrt{\frac{\log \|P\|_\infty}{n}} \ ,$$

which implies

$$|\theta_k| \ \leq \ \frac{\pi k}{n} + c\pi\sqrt{\frac{\log \|P\|_\infty}{n}}. \qquad (4)$$

Using (3), we get

$$\frac{1}{C^{1/k}} \ \leq \ \frac{\pi k}{n} + c\pi\sqrt{\frac{\log \|P\|_\infty}{n}},$$

or

$$\frac{1}{C} \ \leq \ \left(\frac{\pi k}{n} + c\pi\sqrt{\frac{\log \|P\|_\infty}{n}} \right)^k . \qquad (5)$$

Let now λ be the positive root of the equation

$$\lambda^2 \ = \ c^2 e\pi(1+\lambda).$$

We choose $k = \left[\frac{n}{e\pi(1+\lambda)} \right]$. We consider two cases :

a). $\|P\|_\infty \ \leq \ exp\left\{ \frac{\lambda^2 n}{c^2\pi^2 e^2 (1+\lambda)^2} \right\}$.

Then :

$$c\sqrt{\frac{\log \|P\|_\infty}{n}} \ \leq \ \frac{\lambda}{e\pi(1+\lambda)} \ ,$$

and by (5),

$$\frac{1}{C} \ \leq \ \left(\frac{\pi k}{n} + \frac{\lambda}{e(1+\lambda)} \right)^k \ \leq \ e^{-k}.$$

Therefore :

$$C \ \geq \ e^k \ \geq \ exp\{ \frac{n}{e\pi(1+\lambda)} - 1 \} \ ,$$

which shows that there is a factor Q of P with

$$\|Q\|_\infty \ \geq \ exp\{ \frac{n}{e\pi(1+\lambda)} - 1 \} \|P\|_\infty \ \geq \ exp\{ \frac{n}{e\pi(1+\lambda)} - 1 \} \ .$$

b). $\|P\|_\infty \ \geq \ exp\left\{ \frac{\lambda^2 n}{c^2\pi^2 e^2 (1+\lambda)^2} \right\}$.

In this case, taking $Q = P$, we find a factor with

$$\|Q\|_\infty \ \geq \ \exp\{ \frac{\lambda^2 n}{c^2\pi^2 e^2 (1+\lambda)^2} \}.$$

The choice of λ then gives the result.

We observe that the distinction between cases a) and b) cannot be avoided. In some cases (for instance $P = 1 - z^n$), the factor Q with largest norm is a true factor of P, in other cases (for instance $P = (1+z)^n$), it is P itself. The first occurs when $\|P\|_\infty$ is not too large, the second when $\|P\|_\infty$ is quite large.

In terms of order of magnitude, our result is best possible, but the constant δ in the statement of the Theorem is not sharp : we rely on the result of Erdös-Turan, in which the constant c, even after Ganelius' improvement, was not sharp. It would be reasonable to beleive that the extreme case is that of $P = 1 - z^n$, with

$$Q \;=\; \prod_{n/6}^{5n/6} (z - e^{2ij\pi/n}),$$

that is, the product of those factors with $|1 - e^{i\theta_j}| \geq 1$, giving the estimate

$$\prod_{n/6}^{5n/6} 2\sin(j\pi/n).$$

2. Reconstructing a polynomial from the repartition of its zeros.

The result we establish now allows us to reconstruct the polynomial from the repartition function of its zeros. It gives a converse to the result of Erdös and Turan : if $\|P\|_\infty/\sqrt{|a_0 a_n|}$ is large, the roots are not uniformly distributed : some sectors receive more than some others, of same size.

We normalize the polynomial as before.

Let $\mu(x,\theta)$ be the number of zeros of P in the sector $\{|z| \leq x \;, |\arg(z)| \leq |\theta|\}$.

We first establish a symbolic formula.

Theorem 2. – *Let P be a polynomial written as in* (1), *with* $P(1) \neq 0$. *Then :*

$$\log|P(1)| \;=\; \int_0^{+\infty}\int_0^{\pi} \frac{\partial^2\mu}{\partial x\partial\theta}\,\log|1 - xe^{i\theta}|\,d\theta dx. \tag{2}$$

We first explain the meaning to be given to this formula.

Since $P(1) \neq 0$, there is a small neighborhood of $\{1\}$, in the complex plane, with no zero of P : $V_\varepsilon = \{z \;;\; |1-z| \leq \varepsilon\}$. Let $\Omega = \mathbb{R}^2_+ \setminus V_\varepsilon^+$, where $\mathbb{R}^2_+ = \{z \;; Im(z) > 0\}$, $V_\varepsilon^+ = \{z \in V_\varepsilon \;; Im(z) > 0\}$. This is an open set, depending of course of the polynomial.

The function $|1 - xe^{i\theta}|^2 = 1 + x^2 - 2x\cos\theta$ does not vanish in Ω, so $\log|1 - xe^{i\theta}|$ is C^∞ in Ω, and also is $C^\infty(\bar\Omega)$.

We observe that $\mu(x,\theta) = 0$ on V_ε, and is a measurable bounded function on $\bar\Omega$. Let R be large enough, so that all roots z_j satisfy $|z_j| \leq R$. Then, for any θ, $\mu(x,\theta)$ is constant for $x > R$.

Being bounded and measurable, the function $\mu(x,\theta)$ defines a distribution in the space $\mathcal{D}'(\Omega)$, and so has a derivative, in the distribution sense, $\frac{\partial\mu}{\partial x}(x,\theta)$. This derivative is compactly supported in Ω, by what we just said. Also, obviously, $\frac{\partial^2\mu}{\partial x\partial\theta}$ is compactly supported in Ω, and therefore belongs to $\mathcal{E}'(\bar\Omega)$, space of distributions on $\mathbb{R}^2$, with compact support $K \subset \bar\Omega$.

There is a canonical duality between $\mathcal{E}'(\bar\Omega)$ and $\mathcal{E}(\bar\Omega) = C^\infty(\bar\Omega)$, and by this duality, the action :

$$\langle \frac{\partial^2\mu}{\partial x\partial\theta}, \log|1 - xe^{i\theta}|\,\rangle \tag{3}$$

makes sense. Formula (2) is the symbolic notation for (3).

We now turn to the proof of the Theorem.

First, we observe that we may assume that all zeros lie in the upper half- plane. Indeed, $|P(1)| = \prod|1 - z_j| = \prod|1 - \bar z_j|$, so we may replace z_j by its conjugate if necessary.

We may also assume that P has no zero at O, since a zero at O does not affect $P(1)$, nor $\frac{\partial^2\mu}{\partial x\partial\theta}$.

Finally, we may assume that P has no zero on the real axis. Indeed, let $z_1, \ldots, z_p$ be the zeros on the real axis, $z_{p+1}, \ldots, z_n$ the others. For $k \geq 1$, let P_k be the polynomial with zeros $\lambda_1 = z_1 e^{i/k}, \ldots, \lambda_p = z_p e^{i/k}, z_{p+1}, \ldots, z_n$. Then P_k has no zero on the real axis. Obviously, when $k \to \infty$, $|P_k(1)| \to |P(1)|$. Let μ_k be the corresponding function for P_k. We will see that

$$\frac{\partial^2 \mu_k}{\partial x \partial \theta} \to \frac{\partial^2 \mu}{\partial x \partial \theta}, \quad \text{when } k \to \infty, \quad \text{in } \mathcal{E}'(\bar{\Omega}). \tag{4}$$

To prove (4), its enough to do it when P is a monomial, $P = z - \lambda$, with λ real, say $\lambda > 0$. Then λ is replaced by $\lambda e^{i/k}$, and $\mu_k = 1$ if $x \geq \lambda$ and $\theta \geq 1/k$, 0 otherwise.

Let $\phi \in \mathcal{E}(\bar{\Omega})$. Then :

$$\begin{aligned}
\langle \frac{\partial^2 \mu_k}{\partial x \partial \theta}, \phi \rangle &= \langle \mu_k, \frac{\partial^2 \phi}{\partial x \partial \theta} \rangle \\
&= \int_\lambda^{+\infty} \int_{1/k}^{\pi} \frac{\partial^2 \phi}{\partial x \partial \theta} \, dx d\theta \\
&= - \int_{1/k}^{\pi} \frac{\partial \phi}{\partial \theta}(\lambda, \theta) d\theta \\
&= - \phi(\lambda, \pi) + \phi(\lambda, 1/k)
\end{aligned}$$

and when $k \to \infty$, this last quantity tends to

$$-\phi(\lambda, \pi) + \phi(\lambda, 0) = \langle \frac{\partial^2 \mu}{\partial x \partial \theta}, \phi \rangle .$$

So we now prove formula (2) when all zeros satisfy $Im(z_j) > 0$.

Since $|P(1)| = \prod |1 - z_j|$, we have, denoting by δ_z the Dirac measure at the point z,

$$\log |P(1)| = \sum_j \log |1 - z_j| = \sum_j \langle \delta_{z_j}, \log |1 - z| \rangle , \tag{5}$$

in the duality $\mathcal{E}'(\Omega), \mathcal{E}(\Omega)$.

For $\zeta \in \mathbb{C}$ with $Im(\zeta) > 0$, set $\arg \zeta = \alpha$ and consider the sector function :

$$S_\zeta(z) = 1 \text{ if } \{|z| \geq |\zeta| \text{ and } \arg(z) \geq \alpha\} , = 0 \text{ otherwise.}$$

This is the equivalent, in the complex plane, of the Heaviside function on the line, and it has a Dirac measure as a derivative. Indeed, let's compute $\frac{\partial^2 S_\zeta}{\partial x \partial \theta}$ in $\mathcal{D}'(\Omega)$. If $\phi \in \mathcal{D}(\Omega)$, we have :

$$\begin{aligned}
\langle \frac{\partial^2 S_\zeta}{\partial x \partial \theta}, \phi \rangle &= \langle S_\zeta, \frac{\partial^2 \phi}{\partial x \partial \theta} \rangle \\
&= \int_0^\infty \int_0^\pi S_\zeta \frac{\partial^2 \phi}{\partial x \partial \theta} \, dx d\theta \\
&= \int_{|\zeta|}^\infty \int_\alpha^\pi \frac{\partial^2 \phi}{\partial x \partial \theta} \, dx d\theta \\
&= \phi(|\zeta|, \alpha) - \phi(|\zeta|, \pi).
\end{aligned}$$

But $\phi(|\zeta|, \pi) = 0$ since $\phi \in \mathcal{D}(\Omega)$, so we obtain :

$$\frac{\partial^2 S_\zeta}{\partial x \partial \theta} = \delta_\zeta ,$$

as we announced.

Let's come back to the polynomial. We have :

$$\mu(x,\theta) \;=\; \sum_1^n S_{z_j}(x,\theta),$$

which gives :

$$\frac{\partial^2 \mu}{\partial x \partial \theta} \;=\; \sum_1^n \delta_{z_j} \;,$$

substituting in formula (5) gives the result.

Theorem 2 can be translated into a result using only the function $\mu(x,\theta)$ itself, and not its derivatives anymore. Let R be, as before, the radius of any disk, centered at O, containing all the zeros of P. Then :

Theorem 3. – *Let P be a polynomial written as in (1), with $P(1) \neq 0$. Then :*

$$\begin{aligned}\log|P(1)| \;=\; & \int_0^R \int_0^\pi \mu(x,\theta)(1-x^2)\frac{\sin\theta}{|1-xe^{i\theta}|^4}\, dx d\theta \\ & - \int_0^\pi \mu(R,\theta)\frac{R\sin\theta}{|1-Re^{i\theta}|^2}\, d\theta - \int_0^R \frac{\mu(x,0)}{1-x}\, dx - \int_0^R \frac{\mu(x,\pi)}{1+x}\, dx \\ & - \mu(R,0)\log|1-R| + n\log(1+R).\end{aligned}$$

Proof. – The result is deduced from Theorem 2 after two integrations by parts. In order to justify them, we observe that the distribution $\frac{\partial^2\mu}{\partial x \partial\theta}$ has its support in $D(O,R)$; the integration would be justified if, instead of $\log|1-xe^{i\theta}|$ we had a function $\phi(x,\theta)$ with compact support in $D(O,R)\cap\Omega$.

So, for $\varepsilon > 0$, we take two functions $g_\varepsilon(x)$ and $h_\varepsilon(\theta)$, with $g_\varepsilon \in \mathcal{D}(]0,R[)$, $h_\varepsilon \in \mathcal{D}(]0,\pi[)$, $0 \leq g_\varepsilon \leq 1$, $0 \leq h_\varepsilon \leq 1$, and $g_\varepsilon = 1$ on $[\varepsilon, R-\varepsilon]$, $h_\varepsilon = 1$ on $[\varepsilon, \pi-\varepsilon]$. We then replace $\log|1-xe^{i\theta}|$ by $g_\varepsilon(x)h_\varepsilon(\theta)\log|1-xe^{i\theta}|$, and we can now perform the integration by parts. We finally let $\varepsilon \to 0$.

The meaning of Theorem 3 is especially simple for polynomials having all their zeros on the unit circle. For such a polynomial, we put $\nu(\theta) = \mu(1,\theta)$, $0 \leq \theta \leq \pi$: this is the number of zeros satisfying $|\arg z| \leq \theta$. Then :

Theorem 4. – *Let $P(z) = \prod_1^n (z-z_j)$ be a polynomial with all zeros on the unit circle and $P(1) \neq 0$. Then :*

$$\log|P(1)| \;=\; -\frac{1}{2}\int_0^\pi \nu(\theta)\cot(\theta/2)\, d\theta + n\log 2.$$

This formula implies a reciprocal to the result of Erdös and Turan mentioned above. Indeed, it shows that if $|P(1)|$ is large, not too many zeros can be too close to 1, and thus there is an angular sector which receives less zeros than it would, if the zeros were equidistributed. Precisely, we have :

Proposition 5. – *Let P be as in Theorem 4. Then, for every $\alpha \in [0,\pi]$,*

$$\nu(\alpha) \;\leq\; \frac{\log|P(1)| - n\log 2}{\log\sin(\alpha/2)} \;.$$

Proof. – We just write, using Theorem 4 :

$$\nu(\alpha)\int_\alpha^\pi \cot(\theta/2)\, d\theta \;\leq\; \int_\alpha^\pi \nu(\theta)\cot(\theta/2)\, d\theta \;\leq\; 2\log\frac{2^n}{|P(1)|} \;.$$

If the roots were equidistributed, the number in the sector $|\theta| \leq \alpha$ would be $\alpha n/\pi$. But we have

$$\frac{\log(2^n/|P(1)|)}{\log 1/\sin(\alpha/2)} \;<\; \frac{\alpha n}{\pi} \;,$$

if, for given α, $|P(1)|$ is large enough.

Let's take an example. Let $P(z)$ satisfy $\|P\|_\infty \geq 2^{3n/4}$. Then Proposition 5 shows that the number of zeros in the sector $|\theta| \leq \pi/3$ is smaller than $n/4$; in case of equirepartition, it would be $n/3$.

Let's give a second application of Theorem 4 :

Proposition 6. – *Assume that P is as in Theorem 4, the roots written in the order of $|\theta_k|$ increasing. Then, for all $k = 1, \ldots, n$,*

$$\frac{|P(1)|^{1/k}}{2^{n/k}} \leq \sin(|\theta_k|/2) .$$

Proof. – We observe that $\nu(\theta_k) = k$, for all $k = 1, \ldots, n$. So we have :

$$\begin{aligned}
\int_0^\pi \nu(\theta) \cot(\theta/2) d\theta &\geq \int_{|\theta_k|}^\pi \nu(\theta) \cot(\theta/2) d\theta \\
&\geq k \int_{|\theta_k|}^\pi \cot(\theta/2) d\theta \\
&\geq -2k \log \sin(|\theta_k|/2),
\end{aligned}$$

and therefore,

$$\log |P(1)| \leq k \log \sin(|\theta_k|/2) + n \log 2,$$

which gives our formula.

This Proposition shows that if $|P(1)|$ is large, $|\theta_k|/2$ must be close to $\pi/2$, or $|\theta_k|$ close to π, which means that most roots are close to -1.

A formula, similar to that of Theorem 4, holds for polynomials having all their zeros in the interval $[-1, 1[$ of the real axis. For $-1 \leq x < 1$, let $\mu(x)$ be the number of zeros in the interval $[x, 1[$. Then :

$$\log |P(1)| = -\int_{-1}^1 \frac{\mu(x)}{1-x} dx + n \log 2.$$

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