A Minimization Problem
connected with Generalized Jensen's Inequality

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Let $P = a_0 + a_1 z + a_2 z^2 + \cdots$ be a polynomial with complex coefficients, and let d, $0 < d < 1$, $k \in \mathbb{N}$. We say that P has concentration d at degrees at most k if :

$$\sum_0^k |a_j| \ \geq \ d \sum_{j \geq 0} |a_j| \tag{1}$$

Other ways of measuring such a concentration can be expressed. For instance,

$$(\sum_0^k |a_j|^2)^{1/2} \ \geq \ d(\sum_{j \geq 0} |a_j|^2)^{1/2} \tag{2}$$

or :

$$\sum_0^k |a_j| \ \geq \ d\|P\|_\infty \tag{3}$$

where $\|P\|_\infty = \max_\theta |P(e^{i\theta})|$.

The last one is of course more general, since both (1) and (2) imply (3).

This concept was originally introduced by P. Enflo and the author [1], who proved, for a polynomial satisfying (3), a generalized Jensen's Inequality :

There exists a constant $C(d,k)$ such that, for any polynomial satisfying (3),

$$\int_0^{2\pi} \log(\frac{|P(e^{i\theta})|}{\|P\|_\infty}) \frac{d\theta}{2\pi} \ \geq \ C(d,k) \tag{4}$$

(actually the proof is given in [1] only under assumption (2) ; it was given under assumption (3) by the present author in [2], but there is no conceptual difference.)

The precise value of $C(d,k)$ is unknown. Asymptotic estimates, when $k \to +\infty$, were given by the present author in [2], where was proved that, asymptotically, $C(d,k) \geq -2k$, and that the best constant $\mathcal{C}(d,k)$ satisfies, for $d = 1/2$, the estimate $\mathcal{C}(1/2,k) \leq -2k \log 2$.

After normalization, and denoting $|P|_1 = \sum |a_j|$, the problem related to condition (1) becomes :

$$\inf\{\int_0^{2\pi} \log(\frac{|P(e^{i\theta})|}{|P|_1}) \frac{d\theta}{2\pi} \ ; \ P \text{ satisfies } (1)\} \tag{5}$$

This problem was completely solved by A. Rigler, S. Trimble and R.S. Varga [4] for the class of the Hurwitz polynomials. These polynomials have real, positive coefficients, all roots are either real negative, or pairwise conjugate, with negative real parts.

In the special case $d = 1/2$, the constant they find is precisely $-2k \log 2$; the polynomials for which it is reached being of the form $(\frac{z+1}{2})^{2k}$. But outside this class of polynomials, nothing is known about the precise value of $C(d,k)$, even for small values of k.

Here, we deal with the problem connected with inequality (3), and solve it completely for $k = 1$. We recall that for $k = 0$, the result reduces to the classical Jensen's Inequality, so the infimum is just d. The natural setting for this problem is not that of polynomials, but that of H^∞ functions, for which (3) also makes sense. So, finally, what we study is :

$$\inf\{\int_0^{2\pi} \log(\frac{|f(e^{i\theta})|}{\|f\|_\infty}) \frac{d\theta}{2\pi} \ ; \ f \in \ H^\infty \text{ and } f \text{ satisfies } (3)\} \tag{6}$$

Theorem 1. *– For $k = 1$, the solution of Problem 6 is the unique number $c < 0$, solution of the equation :*

$$e^c \ (1 - 2c) \ = \ d$$

There are no functions for which the infimum is actually attained. A sequence of functions F_n realizing the infimum better and better, that is satisfying

$$\|F_n\|_\infty \ = \ 1 \ ; \quad \int \log |F_n| \to c, \ n \to \infty$$

is of the following type : the F_n's are outer functions, and if they are written under the form :

$$F_n(z) \ = \ exp \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} u_n(t) \frac{dt}{2\pi}$$

then u_n is real ≤ 0, $\int_0^{2\pi} u_n(t) \frac{dt}{2\pi} = c$, and the u_n's are more and more concentrated near 0 : for every $\varepsilon > 0$, $\int_{|t|>\varepsilon} u_n(t) dt \to 0$, when $n \to +\infty$

Proof of Theorem 1.– We are going to make heavy use of the canonical decomposition of any function in H^p (here H^∞) :

$$f \ = \ B.S.F \ , \tag{7}$$

where B is a Blaschke factor, S is a singular function, and F an outer function (with, here, $F \in H^\infty$). We refer, for instance, to the book by Hoffmann [3] for the basic facts about this decomposition.

For an analytic function $f = a_0 + a_1 z + a_2 z^2 + \cdots$, we put $\pi(f) = a_0 + a_1 z$.

Our proof is divided into three parts. First, we remove the Blaschke product in the decomposition (7), then the singular part, and finally we deal with the outer part.

1) First step : removal of the Blaschke product.

This is obtained by means of the following :

Proposition 2. – *The Infimum in Problem (6) is the same if we restrict ourselves to functions f which have no Blaschke factor in their canonical decomposition (7).*

The proof of Proposition 2 itself will be done in several steps, making repeated use of the following Lemma :

Lemma 3. – *Let $f = b_0 + b_1 z + \cdots$, and $B_0 = \frac{\bar a}{|a|}\frac{a-z}{1-\bar a z}$, with $|a|<1$, be a Blaschke factor. If*

$$|\pi(fB_0)|_1 \ \geq \ d,$$

then, either $|b_0| \geq d/2$, or $|\pi(f)|_1 \geq d$.

Proof of Lemma 3.– We have

$$\begin{aligned}|\pi(fB_0)|_1 &= |b_0 a| + |b_1|a| - \frac{b_0 \bar a}{|a|}(1-|a|^2)| \\ &\leq |b_0||a| + |b_1||a| + |b_0|(1-|a|^2)\end{aligned}$$

For the simplicity of notations, we may therefore assume that a, b_0, b_1 are real positive. Put :

$$\phi(a) \ = \ -b_0 a^2 + a(b_0+b_1) + b_0$$

Then $\phi(a)$ reaches its maximum for $a_0 = (b_0+b_1)/2b_0$. Two cases may occur :

- either $a_0 \geq 1$. Since $\phi(a)$ is strictly increasing between 0 and 1, the maximum of ϕ on [0,1] is reached at $a=1$. So we get :

$$b_0 + b_1 \ \geq \ d.$$

- or $a_0 \leq 1$. This means that $(b_0+b_1)/2b_0 < 1$, or $b_1 < b_0$.

The maximum value of ϕ, obtained for $a = a_0$, is therefore :

$$\frac{(b_0+b_1)^2}{4b_0} + b_0 \ \leq \ 2b_0$$

so $b_0 \geq d/2$, and Lemma 3 is proved.

We now prove Proposition 2. Let $B = \prod_j \frac{\bar a_j}{|a_j|}\frac{a_j - z}{1-\bar a_j z}$ be a Blaschke product (with $\sum(1-|a_j|) < \infty$). We put :

$$B_j(z) \ = \ \frac{\bar a_j}{|a_j|}\frac{a_j - z}{1-\bar a_j z}$$

Then $|B_j(e^{i\theta})| = 1$ for every θ. We look at $f.B$ and assume that :

$$|\pi(f.B)|_1 \ \geq \ d\ \|f.B\|_\infty \ = \ d\ \|f\|_\infty$$

We put $f_1 = f\prod_{j\geq 2} B_j$, so $f = f_1.B_1$. By Lemma 3, either :

a) $|\pi(f)|_1 \geq d\|f\|_\infty$, or

b) $|f_1(0)| \geq \frac{d}{2}\|f\|_\infty$.

In the second case, classical Jensen's Inequality gives :

$$\int_0^{2\pi} \log\frac{|f_1(e^{i\theta})|}{\|f\|_\infty}\frac{d\theta}{2\pi} \ \geq \ \log\frac{d}{2}$$

and it will follow from Proposition 6 below that the infimum in Problem (6) cannot be achieved by f_1. So we are left with case a), and we start again with $f_2 = f.B_3.B_4 \ldots$.

Let now $f_n = f.B_n.B_{n+1}\ldots$. Cases a) and b) are defined as previously. If at any step we fall into case b), we get the estimate $\log \frac{d}{2}$, which is greater than that of Proposition 6 below, and therefore f_n (and f) is not suitable for the infimum in (6).So we are left with the case where all f_n's fall into case a).

Put $g = S.F$ in the canonical decomposition (7). Then $|f| = |g|$ a.e., and obvious computations (using the fact that $\sum_{j\geq n}(1-|a_j|) \to 0$ when $n\to\infty$) show that $|\pi(f_n)|_1 \to |\pi(g)|_1$. So the estimate given by g in (6) is the same as the estimate given by f, and we are left with a function without Blaschke product. So, Proposition 2 is proved (admitting temporarily Proposition 6 below).

2) Second Step : Removal of the singular function.

The argument is of the same nature as previously, though computations are of course different.

Proposition 4. – *The infimum in Problem 6 is the same if we restrict ourselves to outer functions.*

Proof. – We need a Lemma, similar to Lemma 3 :

Lemma 5. – *Let S be a singular function, and $f = b_0 + b_1 z + \cdots$ in H^∞. If $|\pi(f.S)|_1 \geq d$, then either $|b_0| \geq d/2$, or $|\pi(f)|_1 \geq d$*

Proof of Lemma 5.– We write a singular function :

$$S(z) \;=\; exp\{-\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta)\}$$

where μ is a positive measure, singular with respect to Lebesgue measure (see e.g. Hoffmann [3], p.66). We know that $|S(e^{i\theta})| = 1$ a.e.

We put $c_j = \int_0^{2\pi} e^{ij\theta} d\mu(\theta)$; they are the Fourier coefficients of μ. Then :

$$S(0) \;=\; exp\{-\int d\mu(\theta)\} \;=\; e^{-c_0}\;,$$

$$S'(0) \;=\; -2c_1 e^{-c_0}\;.$$

So :

$$\begin{aligned}\pi(f.S) \;&=\; b_0 e^{-c_0} + z(-2c_1 e^{-c_0} b_0 + b_1 e^{-c_0})\;,\\ |\pi(f.S)|_1 \;&=\; e^{-c_0}|b_0| + e^{-c_0}|-2c_1 b_0 + b_1|\\ &\leq\; e^{-c_0}[|b_0| + 2|c_1||b_0| + |b_1|]\end{aligned}$$

Again, we may now assume b_0, b_1, c_1 to be real positive (c_0 is automatically real positive). Since $|c_1| \leq c_0$, our assumption implies :

$$d \;\leq\; e^{-c_0}(b_0 + 2b_0 c_0 + b_1) \tag{8}$$

With $t = c_0$, we put :

$$\phi(t) \;=\; e^{-t}(b_0 + b_1 + 2b_0 t).$$

Then $\phi'(t) = -(2b_0 t + b_1 - b_0)e^{-t}$, and ϕ takes its maximum at $(b_0 - b_1)/2b_0$. We have two cases :

- either $b_0 - b_1 \geq 0$,

Then ϕ reaches its maximum on $[0,\infty[$ at the point $(b_0-b_1)/2b_0$. The value of this maximum is $2b_0 exp((b_1 - b_0)/2b_0) \;\leq\; 2b_0$, and therefore $b_0 \leq d/2$.

- or $b_0 - b_1 \leq 0$,

Then ϕ reaches its maximum on $[0,\infty[$ at the point $t = 0$. The value of this maximum is $b_0 + b_1$, so $b_0 + b_1 \geq d$.

This proves our Lemma. Proposition 4 now follows easily : let $f = S.F$, where S is singular and F outer. Then if :

$$|\pi(S.F)| \ \geq \ d\|S.F\|_\infty \ = \ d\|F\|_\infty$$

then either $|F(0)| \geq \frac{d}{2}\|F\|_\infty$, so $\int \log(|F|/\|F\|_\infty) \geq \log(d/2)$ and the infimum will not be attained at F by Proposition 6 below, or

$$|\pi(F)|_1 \ \geq \ d\|F\|_\infty,$$

and we have removed the singular part S in g.

3) Step 3 : The minimization problem for outer functions.

We now study Problem (6), assuming F to be in H^∞ and *outer*. Then F can be written :

$$F(z) \ = \ exp\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \ \log k(\theta) \ \frac{d\theta}{2\pi}$$

where $k(\theta)$ is real, >0, and $\int|\log k(\theta)| < \infty$.

We may assume $\|F\|_\infty = 1$. Since $|F(e^{i\theta})| = k(\theta)$ a.e., we know that $k(\theta) \leq 1$ a.e., so $\log k(\theta) \leq 0$ a.e.

Put $u(\theta) = \log k(\theta)$. Then u is a real function, integrable, and ≤ 0 a.e. We now compute the first two coefficients of F :

$$F'(z) \ = \ F(z) \ \int_0^{2\pi} \frac{2e^{i\theta}}{(e^{i\theta}-z)^2} \ u(\theta) \ \frac{d\theta}{2\pi}$$

We put $c_j = \int_0^{2\pi} e{-}ij\theta u(\theta)\frac{d\theta}{2\pi}$. So we get :

$$F(0) \ = \ exp\int u(\theta)\frac{d\theta}{2\pi} \ = \ e^{c_0}$$
$$F'(0) \ = \ 2e^{c_0}\int e^{-i\theta}u(\theta)\frac{d\theta}{2\pi} \ = \ 2c_1e^{c_0}$$

We know that, for an outer function, Jensen's Inequality is actually an equality ([4], p.62), so, since $\|F\|_\infty = 1$, we get :

$$\int_0^{2\pi} \log|F(e^{i\theta})|\frac{d\theta}{2\pi} \ = \ c_0$$

Our problem may now be rephrased :

"Find the minimum of c_0, assuming $e^{c_0}(1+2|c_1|) \geq d$, where c_0, c_1, are Fourier coefficients of a real, negative, integrable function. " (9)

Since $e^{c_0} \geq d/(1+2|c_1|)$, the minimum of c_0 will be obtained by giving to $|c_1|$ the largest possible value. But :

$$|c_1| \ = \ |\int e^{-i\theta}u(\theta)\frac{d\theta}{2\pi}| \ \leq \ -\int u(\theta)\frac{d\theta}{2\pi}$$

Fix now $\varepsilon > 0$. There are functions $u(\theta)$ such that $|c_1| \geq (1-\varepsilon) - c_0$. Indeed, a function $u < 0$, with support in $[-\eta, \eta]$, for $\eta > 0$ small enough, will have this property.

So, for each $\varepsilon > 0$, the solution of our problem is among the c_0 for which $e^{c_0}(1-2(1-\varepsilon)c_0) \geq d$. Letting $\varepsilon \to 0$, we find that the solution of our problem is the smallest value of $c_0 < 0$ for which :

$$e^{c_0}(1-2c_0) \ \geq \ d$$

that is, the unique solution of the equation

$$e^c(1-2c) \ = \ d \tag{10}$$

This finishes the study in Step 3. In order to prove our Theorem, all we have to do now is to compare the solution of equation (10) to the estimate which we obtained in Propositions 2 and 4 :

Proposition 6. – *The solution of equation (10) is strictly smaller than* $\log(d/2)$.

Proof of Proposition 6.– All we have to show is that :

$$e^{\log(d/2)}(1-2\log(d/2)) \;>\; d$$

and this follows easily from the estimate $\log(d/2) < -1/2$.

This concludes the proof of Theorem 1. Numerically, for $d = 1/2$, one finds $c_0 = -2.4773$, and this value is smaller than the value $-2\log 2$ obtained in [4] for Problem (5).

Similar reasoning can be made in the case $k > 1$, and Steps 1, 2 can be carried over. The main difficulty is Problem (9), which can be stated for more general k's : we have not been able to find a description of the solution, valid for all values of k.

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