ON THE LEADING COEFFICIENTS OF REAL MANY-VARIABLE POLYNOMIALS

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Abstract. – For an homogeneous polynomial P in N variables, $x_1, \cdots, x_N$, of degree k, the leading terms are those which contain only one variable, raised to the power k. If $0 \leq P \leq 1$ when all variables satisfy $0 \leq x_j \leq 1$, how large can the leading coefficients be ? Estimates were given in [1] by Aron - Beauzamy - Enflo ; we improve these estimates in general and solve the problem completely for $k = 2$ and 3.

Symbolic Computation (MAPLE on a *Digital* DecStation 5000) was heavily used at two levels : first in order to get a preliminary intuition on the concepts discussed here, and second, as symbolic manipulation on polynomials, in most proofs.

Numerical analysis was made on a Connection Machine CM2, using the hypercube representation obtained by Beauzamy - Frot - Millour [2].

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Let

$$P(x_1,\dots,x_N) \;=\; \sum_{|\alpha|=k} a_\alpha x_1^{\alpha_1}\cdots x_N^{\alpha_N} \;,$$

with $\alpha=(\alpha_1,\dots,\alpha_N)$, $|\alpha|=\alpha_1+\cdots+\alpha_N$, be an homogeneous polynomial of degree k in N variables $x_1,\dots,x_N$.

As already done by Aron - Beauzamy - Enflo in [1], among all coefficients a_α, we distinguish the *leading* ones, denoted by a_l $(l=1,\dots,N)$: a_l is the coefficient of the sole variable x_l, raised to the power k. So the *leading terms* are those which contain just one variable, raised to the power k (this terminology is of course inspired by the one-variable situation). All other terms contain at least two variables, and the polynomial can be written

$$P(x_1,\dots,x_N) \;=\; \sum_{l=1}^{N} a_l x_l^k + \sum_{|\beta|=k} a_\beta x_1^{\beta_1}\cdots x_N^{\beta_N} \;, \tag{1}$$

where in the last term all β's have at least two non-zero components.

The question raised in [1] is the following : if we know $\sum_{l=1}^{N}|a_l|$, can we find a lower bound for $\max\limits_{0\le x_j\le 1}|P(x_1,\cdots,x_N)|$?

The reason for this question, explained in [1], is that such a result allows to decrease the number of terms in the polynomial one needs to consider : in order to find a lower bound for $\max\limits_{0\le x_j\le 1}|P(x_1,\cdots,x_N)|$ (a quantity which depends on <u>all terms</u> in P), one needs only to consider the $a_l's$, which represent only N terms.

In [1] was shown that

$$\sum_{1}^{N}|a_l|\le C_k \max_{0\le x_j\le 1}|P(x_1,\cdots,x_N)| \tag{3}$$

with $C_k\le 4k^2$, and that the best C_k must satisfy $C_k\ge k$.

We will obtain estimates for C_k from below by an iterative procedure (at each step, replacing the variable by a previously obtained polynomial), and this iterative procedure will require $0\le P\le 1$ when all variables satisfy $0\le x_j\le 1$ (and not just $|P|\le 1$). For this technical reason, we investigate :

$$B_k \;=\; \sup\left\{\sum_{l=1}^{N}|a_l|\;;\; P \text{ as in (1)},\; 0\le P\le 1 \text{ if } 0\le x_j\le 1,\; j=1,\dots,N\;;\; N=1,2,\dots\right\} \tag{4}$$

and we want to find a lower bound for B_k. We write $|P|_{lead}$ instead of $\sum_{l=1}^{N}|a_l|$.

Our main result is :

Theorem 1. – *For $k\ge 2$, the following estimates hold :*

$B_2\ge 4$, $B_3\ge 9$, *and for* $k\ge 3$,

$$B_k\ge k^{\log 6/\log 3}\;.$$

Each of these estimates requires the production of a corresponding polynomial. The techniques are different in each case.

Proposition 2. – *The polynomial in N variables, homogeneous of degree 2 :*

$$P(x_1,\dots,x_N) \;=\; A\left(\frac{x_1^2+\cdots+x_N^2}{N}-\frac{2}{N(N-1)}\sum_{i<j}x_ix_j\right),$$

with $A=4(N-1)/N$ if N is even, $A=4N/(N+1)$ if N is odd, satisfies $0\le P\le 1$ if all x_j's satisfy $0\le x_j\le 1$.

Proof of Proposition 2.

– Let's first show that $P\ge 0$, that is :

$$\frac{2}{N(N-1)}\sum_{i<j}x_ix_j \;\le\; \frac{1}{N}(x_1^2+\cdots+x_N^2)$$

or

$$\frac{1}{N(N-1)}\left(\sum_{i\ne j}x_ix_j+\sum_i x_i^2\right) \;\le\; \left(\frac{1}{N}+\frac{1}{N(N-1)}\right)\sum_i x_i^2\ ,$$

which is equivalent to

$$\frac{1}{N}\left(\sum_i x_i\right)^2\le\sum_i x_i^2,$$

a consequence of Hölder's inequality.

– Let's now show that $P\le 1$. Since P is a convex function of each x_j, it's enough to show it when $x_j=0$ or 1.

Let's assume that K of the x_j's are 1, $N-K$ are 0.

The condition $P\le 1$ reads

$$A\left(\frac{K}{N}-\frac{2}{N(N-1)}\frac{K(K-1)}{2}\right) \;\le\; 1,$$

or

$$A\,\frac{K(N-K)}{N(N-1)} \;\le\; 1\ .$$

The maximum of $\dfrac{K(N-K)}{N(N-1)}$ is reached for $K\sim N/2$. More precisely, if N is even, $N=2M$, it is obtained for $K=M$, and gives $\dfrac{N}{4(N-1)}$, and if $N=2M+1$, it is obtained for $K=M$, and gives $\dfrac{N+1}{4N}$. The values of A follow.

This gives the required estimate for B_2 in Theorem 1. We observe that the maximum is not obtained for a fixed number of variables, but letting $N\to+\infty$.

We now turn to the case of degree 3 :

Proposition 3. – *The polynomial*

$$P(x_1,\dots,x_N) \;=\; A\sum_1^N x_i^3 - B\sum_{i\ne j}x_i^2x_j + C\sum_{i<j<k}x_ix_jx_k\ , \tag{5}$$

with

$$\begin{aligned} A &= \frac{1}{N^3}(3N-4)^2, \\ B &= \frac{8}{N^3}(3N-6), \\ C &= \frac{96}{N^3}, \end{aligned}$$

satisfies $0 \leq P \leq 1$ *when all* x_j *'s satisfy* $0 \leq x_j \leq 1$.

Assuming this result, we see that the estimate for B_3 follows : indeed, when $N \to +\infty$, $|P|_{lead} \to 9$. But here again, the maximum is not reached for any prescribed number of variables.

Proof of Proposition 3.

We take P under the form (5) and compute the values of A, B, C, so it has the required properties. First, we study the case where K of the variables x_j take the value 1, and $N-K$ take the value 0 ($0 \leq K \leq N$). We set

$$\varphi(K) \;=\; P(\underbrace{1,\ldots,1}_{K\ times},0,\ldots,0)$$

and so :

$$\varphi(K) \;=\; AK - BK(K-1) + \frac{C}{6}K(K-1)(K-2). \tag{6}$$

We will choose A, B, C, such that $0 \leq \varphi(K) \leq 1$ for $K = 0,\ldots,N$, and with A as large as possible since $|P|_{lead} = AN$.

The polynomial $\varphi(x)$ must vanish at 0, must satisfy $0 \leq \varphi(x) \leq 1$ if $x \in [0,N]$, and we want $\varphi(1)$ to be as large as possible. Therefore, we will require φ to have a double zero α, $0 \leq \alpha \leq N$ (which ensures $\varphi(x) \geq 0$, $0 \leq x \leq N$), and we prescribe $\varphi(x)$ to be of the form

$$\varphi(x) \;=\; \gamma x(x-\alpha)^2, \tag{7}$$

where $0 \leq \alpha \leq N$, and $\gamma > 0$ have to be chosen. Then clearly $\varphi(K) \geq 0$ for all $K \geq 0$, and by a result of Choi - Lam - Reznick ([3], theorem 3.7), since deg $P \leq 3$, this implies that $P \geq 0$ when $x_j \geq 0$.

We have

$$\varphi'(x) \;=\; \gamma\,(x-\alpha)(3x-\alpha),$$

and so, in order to impose $\varphi(x) \leq 1$, $0 \leq x \leq N$, all we have to require is $\varphi(\alpha/3) \leq 1$, $\varphi(N) \leq 1$, that is

$$\begin{cases} 2\gamma\alpha^3/27 \;\leq\; 1 \\ \gamma N(N-\alpha)^2 \;\leq\; 1 \end{cases}$$

We put $\alpha = \lambda N$ $(0 < \lambda < 1)$, and we obtain

$$\begin{cases} 4\gamma\lambda^3 N^3/27 \;\leq\; 1 \\ \gamma N^3(1-\lambda)^2 \;\leq\; 1 \end{cases} \tag{8}$$

But $\varphi(1) = \gamma(1-\alpha)^2 = \gamma(1-\lambda N)^2$ is an increasing function of γ. So $\varphi(1)$ will be maximal if both inequalities in (8) are equalities. Solving in λ, MAPLE gets

$$\left(\frac{\lambda}{3}\right)^3 \;=\; \left(\frac{1-\lambda}{2}\right)^2$$

and finds the solutions $\lambda = 3/4,\ 3,\ 3$.

This gives $\alpha = 3N/4$, $\gamma = 16/N^3$, and

$$\varphi(x) \;=\; \frac{16}{N^3}\, x(x - \frac{3}{4}N)^2.$$

The values of A, B, C, are now deduced from the system :

$$\varphi(1) \;=\; A\;,\quad \varphi(2) \;=\; 2A - 2B\;,\quad C \;=\; 6\gamma,$$

easily solved by MAPLE.

We now show that $P \le 1$ when all x_j are ≤ 1.

For this, we first observe that P can be written as :

$$P(x_1, \ldots, x_N) \;=\; \frac{9}{N} \sum_1^N x_i^3 - \frac{24}{N^2} \left(\sum_1^N x_i^2 \right) \left(\sum_1^N x_i \right) + \frac{16}{N^3} \left(\sum_1^N x_i \right)^3 . \tag{9}$$

To simplify our notation, we put

$$m_1 \;=\; \frac{1}{N} \sum_1^N x_i\;,\quad m_2 \;=\; \frac{1}{N} \sum_1^N x_i^2\;,\quad m_3 \;=\; \frac{1}{N} \sum_1^N x_i^3\;,$$

and P becomes

$$P \;=\; 9m_1 - 24m_1m_2 + 16m_3.$$

We have :

Lemma 4. – *Let P be a symmetric polynomial of degree 3, written as :*

$$P \;=\; am_3 + bm_1m_2 + cm_1^3,$$

where $a + b + c \le 1$, $3a + b \ge 0$, $2b + 3c \ge 0$. Then, if $P \ge 0$ when all $x_i \ge 0$, P automatically satisfies $P \ge 1$ when $0 \le x_i \le 1$.

Proof of Lemma 4. – We set $x_i = 1 - t_i$. If $x_i \le 1$, $t_i \ge 0$, and with the notation

$$\mu_1 \;=\; \frac{1}{N} \sum_1^N t_i\;,\quad \mu_2 \;=\; \frac{1}{N} \sum_1^N t_i^2\;,\quad \mu_3 \;=\; \frac{1}{N} \sum_1^N t_i^3,$$

P becomes :

$$P \;=\; -(a\mu_3 + b\mu_1\mu_2 + c\mu_1^3) + a + b + c - 3\mu_1(a + b + c) + \mu_2(3a + b) + \mu_1^2(2b + 3c).$$

Since $a\mu_3 + b\mu_1\mu_2 + c\mu_1^3 \ge 0$, the condition $P \le 1$ will be satisfied as soon as :

$$a + b + c - 3\mu_1(a + b + c) + \mu_2(3a + b) + \mu_1^2(2b + 3c) \le 1.$$

But $\mu_2 \le \mu_1$, $3a + b \ge 0$, so this inequality holds if

$$a + b + c - \mu_1(1 - \mu_1)(2b + 3c) \le 1,$$

which is satisfied by assumption. This proves Lemma 4, and finishes the proof of the Theorem in the case $k = 3$.

Remark. – If we put

$$Q(x_1,\ldots,x_{2N}) \;=\; P(x_1,\ldots,x_N) - P(x_{N+1},\ldots,x_{2N}),$$

we find that $|Q(x_1,\ldots,x_{2N})| \le 1$ if $0 \le x_j \le 1$, and $|Q|_{lead} \to 18$ when $N \to +\infty$.

So the best constant C_3 in the inequality

$$|Q|_{lead} \le C_k \sup_{0\le x_j\le 1} |Q(x_1,\cdots,x_N)|$$

satisfies $C_3 \ge 18$; Theorem 1.2 in [1] shows that $C_k \le 4k^2$.

Can this construction of P, with large leading coefficients, be carried over for $k>3$? We don't know. Following the same pattern would require :

– finding a polynomial $\varphi(x)$ of degree K, in one variable, with $0 \le \varphi(x) \le 1$ if $0 \le x \le N$, and $\varphi(1)$ as large as possible,

– identifying the many-variable polynomial $P(x_1,\ldots,x_N)$, homogeneous of degree k, with first term $A\sum_1^N x_i^k$, such that

$$P(\underbrace{1,\ldots,1}_{j\ times},0,\ldots,0) \;=\; \varphi(j),$$

for $j=0,\ldots,N$,

– Proving that $0\le P\le 1$ if all x_j's satisfy $0 \le x_j \le 1$.

The first two steps are not very hard to perform, but the last one -proving that $0 \le P \le 1$- does not seem within our reach at present. Of course, the result of Choi- Lam-Reznick we have used is not valid for $k \ge 3$, but this is not the main point : our proof, for $k=3$, only uses this result for simplicity (our original proof did not). The main point is that no tool is presently known, ensuring that a many- variable polynomial, of degree $k>3$, satisfies $0\le P\le 1$ when all x_i satisfy $0 \le x_i \le 1$.

Since this problem cannot be solved, we have two possibilities. The first one is to build P, with $0 \le P \le 1$, by some iterative procedure from a known polynomial : this will lead to the estimate for B_k in Theorem 1. These estimates are not in k^2 as we would like, but they are better than anything previously known.

The second one will be to change the norm, and replace

$$\sup_{0\le x_j\le 1} |P(x_1,\ldots,x_N)|$$

by the quantity

$$\sup_{x_j=0,1} |P(x_1,\ldots,x_N)|,$$

which will be discussed at the end of the paper.

We now turn to the iterative procedure in order to estimate B_k.

Proposition 5. – *Assume we can find a polynomial P_0, with N_0 variables, homogeneous of degree k_0, with the properties :*

– all leading coefficients are 1,

– if all x_j satisfy $0 \le x_j \le 1$, then $0 \le P_0 \le 1$.

Then $B_k \geq k^{\log N_0/\log k_0}$, *for all* k *of the form* $k = k_0^j$, $j \in \mathbb{N}$.

Proof of Proposition 5.– By assumption $|P_0|_{lead} = N$ and $0 \leq P_0 \leq 1$ if all $x_j \in [0,1]$. Set $P_1 = P_0$, and

$$P_2 \;=\; P_0\left(P_1(x_1,\dots,x_{N_0}), P_1(x_{N_O+1},\dots,x_{2N_0}),\dots,P_1(x_{(N_0-1)N_0+1},\dots,x_{N_0^2})\right).$$

So P_2 has N_0^2 variables, $|P_2|_{lead} = N^2$, deg $P_2 = k_0^2$, and $0 \leq P_2 \leq 1$ if $x_i \in [0,1]$.

Assume P_{j-1} has been defined, with N_0^{j-1} variables, $\deg P_{j-1} = k^{j-1}$, and $0 \leq P_{j-1} \leq 1$ if $x_i \in [0,1]$. Set :

$$P_j \;=\; P_0\left(P_{j-1}(x_1,\dots,x_{N_O^{j-1}}),\dots,P_{j-1}(x_{(N_0-1)N_0^{j-1}+1},\dots,x_{N_0^j})\right).$$

So P_j has N_0^j variables, $|P_j|_{lead} = N_0^j$, deg $P_j = k_0^j$, and $0 \leq P_j \leq 1$ if $x_i \in [0,1]$.

Set $k = \deg P_j$. Then :

$$N_0^j \;\leq\; B_k.$$

But $k = k_0^j$, $j = \log k/\log k_0$, and

$$B_k \geq N_0^{\log k/\log k_0} \;=\; k^{\log N_0/\log k_0},$$

as we announced. This proves Proposition 5.

We observe that, in order to be applied, this inductive procedure requires a polynomial with leading coefficients all equal to 1, and this is not the case of the ones we have exhibited so far.

So we will prove :

Proposition 6. – *The polynomial in 6 variables, with 56 terms :*

$$P_0(x_1,\dots,x_6) \;=\; \sum_1^6 x_i^3 - \frac{1}{2}\sum_{i\neq j} x_i^2 x_j + \frac{1}{2}\sum_{i<j<k} x_i x_j x_k \tag{10}$$

satisfies $0 \leq P_0 \leq 1$ *if* $x_i \in [0,1]$.

This proposition, producing a polynomial of degree 3 with 6 variables, gives the estimate $k^{\log 6/\log 3}$ in Theorem 1. It improves upon the estimate $k^{\log 3/\log 2}$, obtained by A.Tonge from the consideration of the polynomial

$$P_0(x,y,z) \;=\; x^2 + y^2 + z^2 - xy - yz - zx,$$

which also satisfies $0 \leq P_0 \leq 1$ if $x,y,z \in [0,1]$.

Before proving Propostion 6, we will state :

Proposition 7. – *The polynomial* P_0 *defined in (10) is, among all polynomials of degree 3 with leading coefficients 1, with 6 variables, the only one which may satisfy* $0 \leq P_0 \leq 1$ *if all* $x_i \in [0,1]$. *There is no such polynomial with 7 variables.*

Proof of Proposition 7.– We consider any degree 3 polynomial with N variables, of the form :

$$P \;=\; \sum_1^N x_i^3 - C_1 \sum_{i<j} x_i^2 x_j + C_2 \sum_{i<j<k} x_i x_j x_k.$$

Taking K of the variables x_i to be 1, $N-K$ to be 0, we obtain the set of conditions

$$0 \;\leq\; K - K(K-1)C_1 + \frac{K(K-1)(K-2)}{6}C_2 \;\leq\; 1,$$

which can be written, for $K \geq 2$:

$$\frac{1}{K} \;\leq\; C_1 - \frac{K-2}{6} C_2 \;\leq\; \frac{1}{K-1} \;. \tag{11}$$

Taking successively $K = 2$, 3, 4, we get

$$\begin{aligned} \frac{1}{2} \;&\leq\; C_1 \;\leq\; 1, \\ C_2 \;&\geq\; 6(C_1 - \frac{1}{2}) \;\geq\; 0, \\ C_2 \;&\geq\; 1/2. \end{aligned}$$

The left-hand side conditions in (11) can be written

$$C_1 \;\geq\; \frac{K-2}{6} C_2 + \frac{1}{K} \;, \tag{12}$$

and since $C_2 \geq 1/2$, the strongest one will be the one with highest K.

The right-hand side gives :

$$C_1 \;\leq\; \frac{K-2}{6} C_2 + \frac{1}{K-1}, \tag{13}$$

and the conditions for $K \geq 4$ are weaker than those for $K = 4$, and so we keep these for $K = 3$, 4, that is

$$C_1 \;\leq\; \frac{1}{6} C_2 + \frac{1}{2} \tag{14}$$

$$C_1 \;\leq\; \frac{1}{3} C_2 + \frac{1}{3} \;. \tag{15}$$

This implies that no 7-variable polynomial may exist. Indeed, we would have by (12)

$$C_1 \;\geq\; \frac{5}{6} C_2 + \frac{1}{7},$$

and by (15)

$$\frac{5}{6} C_2 + \frac{1}{7} \;\leq\; \frac{1}{3} C_2 + \frac{1}{3},$$

which gives $C_2 \leq 8/21$, contradicting $C_2 \geq 1/2$.

This also implies the uniqueness for $K = 6$. Indeed, (12) gives :

$$C_1 \;\geq\; \frac{2}{3} C_2 + \frac{1}{6},$$

and compatibility with (14), (15) implies $C_2 = 1/2$.

Coming back to (11), we find

$$\frac{1}{K} \;\leq\; C_1 - \frac{K-2}{12} \leq \frac{1}{K-1} \;,$$

and for $K = 4$, this gives $C_1 \leq 1/2$, and finally $C_1 = 1/2$, which proves Proposition 7.

We now prove Proposition 6.

1. – To show that $P_0 \geq 0$ if $x_j \geq 0$, by the Theorem of Choi - Lam - Reznick [3] already cited, it is enough to do it when K of the variables are equal to 1, $6-K$ equal to 0 $(K = 0, \ldots, 6)$. Set

$$\varphi(K) \;=\; P(\underbrace{1, \ldots, 1}_{K \ times}, \underbrace{0, \ldots, 0}_{6-K \ times}).$$

Then

$$\begin{aligned} \varphi(K) \;&=\; K - \frac{K(K-1)}{2} + \frac{K(K-1)(K-2)}{12} \\ &=\; \frac{1}{12} K(K-4)(K-5), \end{aligned}$$

and $\varphi(K) \geq 0$ for $K = 0, \ldots, 6$.

2. – To show that $P_0 \leq 1$ if $x_j \in [0,1]$, we write P_0 under symmetric form :

$$\begin{aligned} P_0 &= 10\left(\frac{1}{6}\sum_1^6 x_i^3\right) - 27\left(\frac{1}{6}\sum_1^2 x_i^2\right)\left(\frac{1}{6}\sum_1^6 x_i\right) + 18\left(\frac{1}{6}\sum_1^6 x_i\right)^3 \\ &= 10m_3 - 27m_1m_2 + 18m_1^3, \end{aligned}$$

and we apply Lemma 4 again.

This concludes the proof of Proposition 7, and that of Theorem 1.

Remark. – In a preliminary version of this paper, the proof of Proposition 5 was obtained by symbolic manipulation, the following way : Maple computes the 6 partial derivatives (which have degree 2), and the differences $\dfrac{\partial P}{\partial x_i} - \dfrac{\partial P}{\partial x_j}$. The entire system of differences is then solved. Then one studies the boundary cases $x_j = 0, 1$. The proof presented here is of course much simpler, but there is no evidence it exists for degree 5 and above.

We now investigate similar concepts for the quantity

$$\{P\}_{0,1} = \max_{x_j=0,1} |P(x_1, \dots, x_N)|$$

and define D_k as the smallest constant such that

$$|P|_{lead} \leq D_k \max_{x_j=0,1} |P(x_1, \dots, x_N)|$$

holds for all polynomials P, homogeneous of degree k, in many variables $x_1, \dots, x_N$.

First, the proof of Theorem 1.2 in [1] shows that

$$D_k \leq 4k^2.$$

We are going to prove :

Proposition 8. – *For every $k \geq 1$, $D_k \geq 2k^2$.*

Proof of Proposition 8. – We consider P under the form :

$$\begin{aligned} P &= A_0(\frac{1}{N}\sum_1^N x_i^k) + A_1(\frac{1}{N}\sum_1^N x_i^{k-1})(\frac{1}{N}\sum_1^N x_i) + A_2(\frac{1}{N}\sum_1^N x_i^{k-2})(\frac{1}{N}\sum_1^N x_i)^2 \\ &\quad + \cdots + A_j(\frac{1}{N}\sum_1^N x_i^{k-j})(\frac{1}{N}\sum_1^N x_i)^j + \cdots + A_{k-1}(\frac{1}{N}\sum_1^N x_i)^k \\ &= A_0 m_k + A_1 m_{k-1} m_1 + \cdots + A_j m_{k-j} m_1^j + \cdots + A_{k-1} m_1^k, \end{aligned}$$

with our previous notation.

The coefficient of x_1^k is

$$\frac{A_0}{N} + \frac{A_1}{N^2} + \cdots + \frac{A_{k-1}}{N^k} \ ,$$

and therefore

$$|P|_{lead} = A_0 + \frac{A_1}{N} + \cdots + \frac{A_{k-1}}{N^{k-1}} \ \to \ A_0 \ ,$$

when $N \to +\infty$.

If M of the variables take the value 1, and the other $N-M$ the value 0, we have :

$$P(\underbrace{1,\ldots,1}_{M\ times},\ \underbrace{0,\ldots,0}_{N-M\ times}\ ) = A_0\frac{M}{N} + A_1(\frac{M}{N})^2 + \cdots + A_{k-1}(\frac{M}{N})^k\ ,$$

and so, if we set

$$f(x)\ =\ A_0x + A_1x^2 + \cdots + A_{k-1}x^k,$$

we want $0 \le f(x) \le 1$ if $0 \le x \le 1$, and A_0 maximal.

But $A_0 = f'(0)$, and the solution of this problem is given by the Chebyshev polynomial T_k (see Rivlin [4]).

So we take $f(x)\ =\ \dfrac{(-1)^{k-1}T_k(2x-1)+1}{2}$, and since $-1 \le T_k \le 1$, we have $0 \le f \le 1$ on [0,1]. Also, $f(0)=0$, and $f'(0) = T_k'(-1) = k^2$ (see Rivlin [4], p. 105).

Finally, we set

$$Q(x_1,\ldots,x_{2N})\ =\ P(x_1,\ldots,x_N) - P(x_{N+1},\ldots,x_{2N}),$$

and we obtain the announced estimate.

We observe that the coefficients $A_0,\ldots,A_{k-1}$ can be explicitly computed from the coefficients of the Chebyshev polynomial. In fact, P can be written as

$$P(x_1,\ldots,x_N)\ =\ \frac{1}{N}\sum_{i=1}^{N}\frac{x_i^{k+1}}{m_1}\,f(m_1/x_i).$$

The quantity $\{P\}_{0,1}$ is of course much easier to compute than any of the existing norms ; however, it is not a norm : $\{x^2y - xy^2\}_{0,1} = 0$.

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