# ESTIMATES FOR $H^2$ FUNCTIONS WITH CONCENTRATION AT LOW DEGREES AND APPLICATIONS TO COMPLEX SYMBOLIC COMPUTATION

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**Abstract**. – Many results are well-known for $H^2$ functions, such as canonical decomposition, repartition of the zeros, and so on. We show that the concept of *concentration at low degrees* plays a central rôle in quantitative versions of these results, and we show how they can be applied in complex symbolic computation.

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## 0. Introduction.

In Symbolic Computation, most systems, such as *MACSYMA*, have nice features allowing to find the roots of a polynomial. The coefficients may be real or complex, algorithms such as "ALLROOTS" allow to give a list, with prescribed accuracy, of all the roots of the polynomial, and these algorithms work quite well when the degree is not too high.

Things are not so satisfactory when we deal with complex functions which are not just polynomials, such as functions which are analytic inside the open unit disk $D$. These functions may be for instance bounded almost everywhere on the unit circle (these are $H^\infty$ functions), or in the classical $L_2$ space on the circle (these are $H^2$ functions). Such a function has an infinite Taylor expansion

$$f(z) \;=\; \sum_0^\infty a_j z^j \;, \quad \text{with} \; \sum_0^\infty |a_j|^2 < \infty,$$

and they arise quite naturally, for instance in signal analysis.

But no matter whether the function is defined by a Taylor series, or by a single term (such as $\exp\{-(1+z)/(1-z)\}$), symbolic computation does not allow us, so far, to find its zeros. Indeed, there can be infinitely many, consisting in a sequence converging to the unit circle. So a first step is to know how many zeros there are in each disk $D(r)$, centered at 0, with radius $r<1$.

But a function in $H^2$ may have an arbitrary large number of zeros in (for instance) $D(1/2)$, or in any $D(r)$, $0<r<1$. So we need to find an *a priori* classification of the $H^2$ functions, that is to build a sequence of classes $\mathcal{C}_n$ of functions, very simply defined (so anyone can decide at first glance to which class a given function belongs), and with the following property : if $f$ is in $\mathcal{C}_n$, it has at most $n$ zeros in $D(1/2)$ (for instance). Then, when such a classification has been constructed, one can expect to build algorithms allowing us to find all the zeros in $D(1/2)$.

The key concept for such a classification is that of *concentration at low degrees* for an $H^2$ function, a concept due to P. Enflo and the author, and which has already proved to be useful in various areas, such as Jensen's inequality (Beauzamy [2], Rigler–Trimble–Varga [8]), products of polynomials in one variable (Beauzamy- Enflo [1]) or in many variables (Beauzamy–Bombieri–Enflo–Montgomery [3]).

Let's look at the converse problem : we want to eliminate all the points $z$ in the open disk where $|f(z)|<\varepsilon$. Of course, the set of zeros has Lebesgue measure 0, but this is useless for the computer, which does not understand Lebesgue measure. On the real line, it would eliminate a few intervals, and say that on what remains the function is large, say $|f(t)| \geq 1/10$, except on $I_1, \ldots, I_k$. This is also what we do in the complex domain, but the intervals will be replaced by small circles, and here again a theoretical problem arises from the fact there can be infinitely many.

So the key notion for this study will be :

*There is a function $\phi(\varepsilon)$, with $\phi(\varepsilon) \to 0$ when $\varepsilon \to 0$, such that the set $\{|f(z)|<\varepsilon\}$ can be covered by a union of disks, with sum of radii $<\phi(\varepsilon)$.*

But such a statement cannot hold in general for $H^2$ functions, if we take no precaution : just the functions $f_n(z)=z^n$, $n=1,2,\ldots$ become smaller and smaller on $D(1/2)$, though $\|f_n\|_2=1$ for all $n$. So here again we need a classification of $H^2$ functions, and one function $\phi(\varepsilon)$ in each class. The classification will be the same as before, and will depend on the *concentration at low degrees*.

Let $f = \sum_{j\geq} a_j z^j$, with $\sum_j |a_j|^2 < \infty$ be a function in $H^2$. Let $k \in \mathbb{N}$, $0 < d \leq 1$. We say that $f$ has concentration $d$ at degree $k$ if :

$$\Big(\sum_0^k |a_j|^2\Big)^{1/2} \ \geq\ d\,\Big(\sum_0^\infty |a_j|^2\Big)^{1/2} . \tag{0.1}$$

This concept was introduced by Per Enflo and the author in [1], where it was used in order to obtain, for products of polynomials, estimates from below independent of the degrees.

In § 1, we study the canonical decomposition : if $f$ has concentration $d$ at degree $k$, what can be said about its Blaschke term, singular factor, outer factor ? We see that for the singular and outer parts, concentration at degree $k$ implies concentration at degree 0. We deduce that these two functions can be bounded from below, in any disk $D(r)$ $(0 < r < 1)$, by a number which depends only on $d$, $k$, $r$.

In § 2, we study the zeros of $H^2$ functions with concentration at low degrees. It's well-known that the zeros $\alpha_n$ of any $H^2$ function must satisfy $\sum 1-|\alpha_n| < \infty$, so if there are infinitely many, they must converge to the unit circle. But no special speed is required, except for this condition. If we assume concentration $d$ at degree $k$ for the function, we will be able to give a minimal speed of convergence, and find a bound for the number of zeros in any disk $D(r)$, depending only on $d$, $k$, $r$.

Let again $f$ be a function with concentration $d$ at degree $k$. In § 3, we compute the measure of the set where $\{|f(z)| < \varepsilon\}$, and we give estimates depending only on the concentration data. In § 4, we show that this set can be covered by a reunion of disks, with sum of radii tending to zero when $\varepsilon \to 0$ : this extends an old result of Henri Cartan.

Let's now introduce some norms which we will need. First,

$$\|f\|_2 \ =\ \left(\int_0^{2\pi} |f(e^{i\theta})|^2\,\frac{d\theta}{2\pi}\right)^{1/2} , \tag{0.2}$$

is the $H^2$ norm, which is the same as :

$$|f|_2 \ =\ \left(\sum_0^\infty |a_j|^2\right)^{1/2} .$$

We also introduce, for a polynomial $P(z) = \sum_{j\geq 0} b_j z^j$,

$$\begin{aligned} \|P\|_\infty &= \sup_{\theta\in\Pi} |P(e^{i\theta})|, \\ |P|_1 &= \sum_j |b_j|, \\ |P|_\infty &= \max_j |b_j|. \end{aligned}$$

Finally, for $f \in H^2$ written as before, we define the partial sums by :

$$s_k(f) \ =\ \sum_0^k a_j z^j \ \ ,\ k = 0, 1, 2, \ldots \tag{0.3}$$

So concentration $d$ at degree $k$ (measured in the $L_2$ norm) means that :

$$|s_k(f)|_2 \ \geq\ d\,|f|_2 . \tag{0.4}$$

We finally introduce :

$$\mathrm{cf}_k(f) \ =\ \frac{\|s_k(f)\|_2}{\|f\|_2} .$$

The number $\mathrm{cf}_k(f)$ is called the *concentration factor* of $f$ at degree $k$ (measured using the $L_2$ norm).

1. The canonical decomposition.

As it is well-known (see for instance Garnett [6]), any function in $H^1$ (and therefore in $H^2$) has a canonical factorization

$$f = m \cdot F, \tag{1.1}$$

where $m$ is inner and $F$ is outer. The inner factor $m$ can itself be factored into :

$$m = B \cdot S, \tag{1.2}$$

where $B$ is a Blaschke product and $S$ a singular inner function.

The following very simple proposition shows what are the concentrations for each factor separately.

**Proposition 1.1.** – *Assume that $f$ has concentration $d$ at degree $k$. Then :*

$$\|s_k(m)\|_\infty \geq d, \tag{1.3}$$

$$\|s_k(F)\|_2 \geq d\,\|F\|_2 . \tag{1.4}$$

**Proof**. – We use a very simple lemma :

**Lemma 1.2.** – *For any functions $g$, $h$ in $H^2$*

$$\|s_k(gh)\|_2 \leq \|s_k(g) \cdot s_k(h)\|_2 , \tag{1.5}$$

*and also*

$$\|s_k(gh)\|_2 \leq \|g \cdot s_k(h)\|_2 . \tag{1.6}$$

Both estimates are obvious, since the coefficients in $s_k(gh)$ depend only on those of $s_k(g)$ and $s_k(h)$, and since the norm $|\cdot|_2$ is monotone unconditional : for any $f$, $|s_k(f)|_2 \leq |f|_2$.

We now prove Proposition 1.1. Assume $\|f\|_2 = 1$, $\|s_k(f)\|_2 \geq d$. Then :

$$d \leq \|s_k(f)\|_2 = \|s_k(mF)\|_2 \leq \|s_k(m) \cdot F\|_2 \leq \|s_k(m)\|_\infty \cdot \|F\|_2 = \|s_k(m)\|_\infty ,$$

since $\|F\|_2 = \|f\|_2 = 1$.

Inequality (1.4) is proved the same way, using the fact that $\|m\|_\infty = 1$.

The same applies to any factor of the inner function : if $m_1$ is such a factor, similar reasoning gives immediately :

$$\|s_k(m_1)\|_\infty \geq d.$$

**Corollary 1.3.** – *If $f$ has concentration $d$ at degree $k$,*

$$|s_k(m)|_2 \geq \frac{d}{\sqrt{k+1}} . \tag{1.7}$$

**Proof**. – Indeed,

$$d \leq \|s_k(m)\|_\infty \leq |s_k(m)|_1 \leq \sqrt{k+1}\,|s_k(m)|_2 .$$

**Corollary 1.4.** – *Let*

$$B(z) = \prod_1^n \frac{\alpha_j - z}{1 - \bar{\alpha}_j z}$$

*be a finite Blaschke product. Then, for every $k \geq n$,*

$$\|s_k(B)\|_\infty \geq 1.$$

**Proof**. – Set $f = \prod_1^n (\alpha_j - z)$. Then $f$ has concentration 1 at degree $k$, for every $k \geq n$, and its canonical factorization is $f = B \cdot F$, with $F = \prod_1^n (1 - \bar{\alpha}_j z)$.

**Remark**. – This result does not hold for $k < n$. Indeed, by a result of Carathéodory (Garnett [6], th. 2.1, p. 6 (see the proof)), for any polynomial $P$ of degree $k$, satisfying $\|P\|_\infty \leq 1$, one can find a Blaschke product $B$, consisting in a product of $k+1$ terms, with $s_k(B) = P$. So there are Blaschke products with arbitrary small first partial sums.

We now see that, despite its very simple proof, estimate (1.7) is best possible :

**Proposition 1.5.** – *For every $k$, there is a function $f$ in $H^2$, with concentration $d \geq 1/\sqrt{3}$ at degree $k$, and such that in its canonical factorization the inner factor $m$ satisfies :*

$$|s_k(m)|_2 \;=\; \frac{1}{\sqrt{k+1}}\;.$$

**Proof**. – We consider two polynomials :

$$P(z) \;=\; \frac{1}{k+1}\sum_0^k z^j$$

$$Q(z) \;=\; \frac{1}{\sqrt{k+1}}\sum_0^k z^j\;.$$

Since their roots are on the unit circle, they are both outer functions. Moreover $|Q|_2 = 1$. Set $F = Q$.

The polynomial $P$ satisfies $\|P\|_\infty = 1$. By Carathéodory's theorem (cited above), we can find a Blaschke product $B$, which coefficients up to the $k$-th match those of $P$, that is which satisfies :

$$s_k(B) \;=\; P.$$

We set $f = B \cdot F$, and we have :

$$s_k(f) \;=\; s_k(B\cdot F) \;=\; s_k(P\cdot Q),$$

and so :

$$\begin{aligned}
|s_k(f)|_2 \;&=\; \frac{1}{(k+1)^{3/2}}|s_k\big((\sum_0^k z^j)^2\big)|_2 \\
&=\; \frac{1}{(k+1)^{3/2}}|\sum_{n\leq k}\sum_{i+j=n} z^n|_2 \\
&=\; \frac{1}{(k+1)^{3/2}}\big(\sum_{n=0}^{k+1} n^2\big)^{1/2} \\
&\geq\; \frac{1}{(k+1)^{3/2}}\left(\frac{(k+1)^3}{3}\right)^{1/2} \;=\; \frac{1}{\sqrt{3}}\;.
\end{aligned}$$

Since conversely $|f|_2 = |Q|_2 = 1$, we see that $f$ has concentration $1/\sqrt{3}$ at degree $k$, and that

$$|s_k(B)|_2 \;=\; |P|_2 \;=\; \frac{1}{\sqrt{k+1}}\;,$$

which proves our claim.

**Remark**. – Define $B(f)$, $S(f)$, $m(f)$ as the Blaschke part, singular part, inner part of the function $f$. We have shown that :

$$\inf\{|s_k(m(f))|_2 \ ; \ f \text{ has concentration } 1/\sqrt{3} \text{ at degree } k\} \ = \ \frac{1}{\sqrt{k+1}} \ ,$$

and that :

$$\inf\{|s_k(B(f))|_2 \ ; \ f \text{ has concentration } 1/\sqrt{3} \text{ at degree } k\} \ = \ \frac{1}{\sqrt{k+1}} \ ,$$

but for the singular part, we only know that $|s_k(S)|_2 \geq d/\sqrt{k+1}$. So we can ask the following question : what is

$$\inf\{|s_k(S(f))|_2 \ ; \ f \text{ has concentration } d \text{ at degree } k\} \ \ ?$$

We now turn to a deeper study of the outer part $F$. Of course, in general, concentration $d$ at degree $k$ does not imply concentration at any lower degree : $z^3$ has concentration 1 at degree 3, and concentration 0 at degree 0, 1, 2. But for an outer function, concentration $d$ at degree $k$ automatically implies some concentration at degree 0 :

**Proposition 1.6.** – *If $F$ is an outer function, satisfying $\|F\|_2 = 1$, $\|s_k(F)\|_2 \geq d$, then :*

$$|F(0)| \ \geq \ \frac{d^2}{e^2 3^{2(k+1)}} \ .$$

**Proof**. – We write $F$ as

$$F(z) \ = \ \exp \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log h(\theta) \, \frac{d\theta}{2\pi} \ ,$$

where $h \geq 0$ and $\log h \in L_1$. We know that $|F(e^{i\theta})| = h(\theta)$ *a.e.*. So we get :

$$\begin{aligned} F(0) \ &= \ \exp \int_0^{2\pi} \log h(e^{i\theta}) \, \frac{d\theta}{2\pi} \\ &= \ \exp \int_0^{2\pi} \log |F(e^{i\theta})| \, \frac{d\theta}{2\pi} \ . \end{aligned}$$

By Corollary 3.3 below, since $F$ has concentration $d$ at degree $k$,

$$\int_0^{2\pi} \log |F(e^{i\theta})| \, \frac{d\theta}{2\pi} \ \geq \ \log \left(\frac{d}{e3^k}\right)^2 \ .$$

Since $F$ is outer, Jensen's inequality is an equality, and we have :

$$\int_0^{2\pi} \log |F(e^{i\theta})| \, \frac{d\theta}{2\pi} \ = \ \log |F(0)|,$$

and the result follows.

**Lemma 1.7.** – *Assume that $F$ is outer and satisfies $\|F\|_2 = 1$, $|F(0)| \geq \varepsilon$. Then, in every disk of radius $r < 1$, we have :*

$$|F(z)| \ \geq \ \left(\frac{\varepsilon}{\sqrt{e}}\right)^{(1+r)/(1-r)} \ .$$

**Proof** of Lemma 1.7. – Write $F$ as before, and set $u(t) = \log h(t)$. Let $P_r$ be the Poisson kernel. Then, for $z = re^{i\theta}$, we have :

$$\begin{aligned} |F(z)| \ &= \ \exp \int_0^{2\pi} P_r(\theta-t) u(t) \, \frac{dt}{2\pi} \\ &= \ \exp \int_{u \leq 0} P_r(\theta-t) u(t) \, \frac{dt}{2\pi} \cdot \exp \int_{u \geq 0} P_r(\theta-t) u(t) \, \frac{dt}{2\pi} \ . \end{aligned}$$

But

$$\exp \int_{u \geq 0} P_r(\theta - t) u(t) \, \frac{dt}{2\pi} \;\geq\; 1,$$

so

$$|F(z)| \;\geq\; \exp \int_{u \leq 0} P_r(\theta - t) u(t) \, \frac{dt}{2\pi} \; .$$

Since

$$P_r(\theta - t) \;=\; \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} \;\leq\; \frac{1 + r}{1 - r} \; ,$$

we get

$$|F(z)| \;\geq\; \exp\{ \frac{1 + r}{1 - r} \int_{u \leq 0} u(t) \, \frac{dt}{2\pi} \} \; . \tag{1.8}$$

But we have also :

$$\varepsilon \;\leq\; |F(0)| \;=\; \exp \int_0^{2\pi} u(t) \, \frac{dt}{2\pi} \;=\; \exp \int_{u \leq 0} u \cdot \exp \int_{u \geq 0} u \; ,$$

and

$$\begin{aligned} \int_{u \geq 0} u \;&=\; \int_{u \geq 0} \log h(t) \, \frac{dt}{2\pi} \;=\; \frac{1}{2} \int_{u \geq 0} \log h(t)^2 \, \frac{dt}{2\pi} \\ &=\; \frac{1}{2} \int_{u \geq 0} \log |F(e^{it})|^2 \, \frac{dt}{2\pi} \;\leq\; \frac{1}{2} \int |F|^2 \;=\; \frac{1}{2} \; , \end{aligned}$$

so

$$\exp \int_{u \leq 0} u \;\geq\; \frac{\varepsilon}{\sqrt{e}} \; .$$

Coming back to (1.8), we find :

$$|F(z)| \;\geq\; \exp\{ \frac{1 + r}{1 - r} \log \frac{\varepsilon}{\sqrt{e}} \} \; ,$$

if $|z| = r$, and also of course if $|z| \leq r$. This proves the lemma.

An outer function never vanishes in the open unit disk. However, even if we require $\|F\|_2 = 1$, there is no universal lower bound for $|F(0)|$ : for every $\varepsilon > 0$, we can find an outer function $F$ with $\|F\|_2 = 1$ and $|F(0)| = \varepsilon$ : just choose the function $h$ with $\int h^2 = 1$ and $\int \log h = \log \varepsilon$. But if we prescribe a concentration at degree $k$, we get :

**Theorem 1.8.** – *Assume that $F$ is outer and satisfies*

$$\|F\|_2 \;=\; 1 \; , \qquad \|s_k(F)\|_2 \;\geq\; d.$$

*Then, in every disk $D(r)$ with $r < 1$, it satisfies :*

$$|F(z)| \;\geq\; \left( \frac{d^2}{e^{5/2} 3^{2k}} \right)^{(1+r)/(1-r)} \; .$$

**Proof**. – This is now an obvious consequence of Proposition 1.6 and Lemma 1.7.

We observe that no bound can be given in the whole unit disk. Indeed, an outer function can vanish on the unit circle : $1 + z$ has concentration $1/\sqrt{2}$ at degree 0.

The estimates given by Theorem 1.8 are rather sharp. The order of magnitude in $r$ and $k$ cannot be improved.

Indeed, consider first the outer function :

$$F(z) \;=\; \exp \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log h(\theta)\, \frac{d\theta}{2\pi},$$

with $h(\theta)=1$, except if $\theta \in [-\mu,\mu]$, where $h(\theta)=\varepsilon$. The numbers $\mu$ and $\varepsilon$ are related by the condition $(\mu/\pi)\log\varepsilon = \log d$, so when $\varepsilon \to 0$, $\mu \to 0$. If $\varepsilon \sim 0$, $\|F\|_2 = 1$. Moreover :

$$F(0) \;=\; \exp\{\frac{\mu}{\pi}\log\varepsilon\} \;=\; d,$$

so $F$ has concentration $d$ at degree $0$.

If $t \sim 0$,

$$P_r(t) \;=\; \frac{1-r^2}{1-2r\cos t + r^2} \;\sim\; \frac{1+r}{1-r}\ ,$$

so for $\mu \sim 0$, if $z = r$,

$$|F(z)| \;=\; \exp \int_{-\mu}^{\mu} \frac{1-r^2}{1-2r\cos\theta + r^2} \log\varepsilon\, \frac{d\theta}{2\pi} \;\sim\; \exp\{\frac{\mu}{\pi}\frac{1+r}{1-r}\log\varepsilon\} \;=\; d^{(1+r)/(1-r)}.$$

This shows that the estimate for $r$ is best possible.

Next, consider

$$F(z) \;=\; \frac{(1+z)^{2k+1}}{\sqrt{\binom{4k+2}{2k+1}}}\ .$$

This is an outer function, with $\|F\|_2 = 1$, $\|s_k(F)\|_2 = 1/2$.

For $z = -1/2$, we find, with $n = 2k+1$,

$$F(z) \;=\; \frac{1}{2^n}\frac{1}{\sqrt{\binom{2n}{n}}} \;\sim\; \frac{(\pi n)^{1/4}}{2^{2n}} \;\sim\; \frac{(2\pi k)^{1/4}}{4^k}\ ,$$

which shows that the order of magnitude in $k$ is indeed exponential.

We now turn to a similar study for the singular functions. We first need an analogue of Proposition 1.6.

**Proposition 1.9.** – *If $S(z)$ is a singular function with $|s_k(S)|_2 \geq d$, then*

$$|S(0)| \;\geq\; \frac{d^2}{2^{6k}}.$$

**Proof**. – We write

$$S(z) \;=\; \exp\left\{-\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, d\mu(\theta)\right\}\ ,$$

where $\mu$ is a positive measure, singular with respect to Lebesgue measure. Let $(\mu_j)_{j\in\mathbb{Z}}$ be the Fourier coefficients of $\mu$. Then $S(0) = e^{-\mu_0}$, and since

$$\frac{e^{i\theta}+z}{e^{i\theta}-z} \;=\; 1 + 2\sum_1^\infty e^{-in\theta} z^n\ ,$$

we get

$$S(z) \;=\; e^{-\mu_0} e^{-2\sum_1^\infty \mu_n z^n}\ .$$

We now compute the Taylor series of $\exp\{-2\sum_{n\geq 1}\mu_n z^n\}$. Using L. Comtet [5], we write :

$$\exp\left\{\sum_{n=1}^{\infty}\frac{x_n t^n}{n!}\right\} \ = \ 1+\sum_{n\geq 1} Y_n(x_1,x_2,\ldots)\frac{t^n}{n!} \ ,$$

where $Y_0=1$, and, for $n\geq 1$,

$$Y_n \ = \ \sum_{l=1}^{n} B_{n,l}(x_1,x_2,\ldots),$$

with

$$B_{n,l}(x_1,x_2,\ldots) \ = \ \sum_{\substack{c_1+2c_2+3c_3+\cdots=n\\ c_1+c_2+c_3+\cdots=l}} \frac{n!}{c_1!c_2!\cdots(1!)^{c_1}(2!)^{c_2}\cdots}\, x_1^{c_1}x_2^{c_2}\cdots$$

Here we have $x_n=-2\mu_n n!$, and $|\mu_j|\leq\mu_0$, since the measure is positive. So we find :

$$|B_{n,l}(x_1,x_2,\ldots)| \ \leq \ 2^l\mu_0^l B_{n,l}(1!,2!,\ldots).$$

But

$$B_{n,l}(1!,2!,\ldots) \ = \ \binom{n-1}{l-1}\frac{n!}{l!} \ .$$

So, for the $k$-th partial sum of $S$, we obtain :

$$\begin{aligned}|s_k(S)|_1 \ &\leq e^{-\mu_0}\big(1+\sum_{n=1}^{k}\sum_{l=1}^{n}\binom{n-1}{l-1}\frac{2^l\mu_0^l}{l!}\big)\\ &\leq e^{-\mu_0}\big(1+\sum_{l=1}^{k}(\sum_{n=l}^{k}\binom{n-1}{l-1})\frac{2^l\mu_0^l}{l!}\big).\end{aligned}$$

But $\sum_{n=l}^{k}\binom{n-1}{l-1}\leq 2^k$, and

$$\sum_{l=0}^{k}\frac{t^l}{l!} \ \leq \ 4^k e^{t/4} \ .$$

Therefore :

$$|s_k(S)|_1 \ \leq \ e^{-\mu_0}2^{3k}e^{\mu_0/2} \ = \ e^{-\mu_0/2}2^{3k} \ .$$

The assumption $|s_k(S)|_2\geq d$ implies :

$$d \ \leq \ 2^{3k}e^{-\mu_0/2} \ ,$$

from which follows :

$$|S(0)| \ = \ e^{-\mu_0} \ \geq \ \frac{d^2}{2^{6k}} \ .$$

We also have an analogue of Lemma 1.7 :

**Lemma 1.10.** – *If $S$ is a singular function and if $|S(0)|\geq\varepsilon$, then in every disk $D(r)$ with $r<1$,*

$$S(z) \ \geq \ \varepsilon^{\frac{1+r}{1-r}} \ .$$

**Proof**. – Indeed, as in the proof of Lemma 1.7,

$$\begin{aligned}|S(z)| \ &= \ \exp\{-\int P_r(\theta-t)\,d\mu(t)\}\\ &\geq \ \exp\{-\frac{1+r}{1-r}\int d\mu(t)\} \ = \ \exp\{-\frac{1+r}{1-r}\mu_0\} \ .\end{aligned}$$

Since $-\mu_0\geq\log\varepsilon$, the lemma follows.

As previously, we deduce immediately from Proposition 1.9 and Lemma 1.10 :

**Theorem 1.11.** – *Assume that $S$ is a singular function with*

$$\|s_k(S)\|_2 \ \geq \ d.$$

*Then in every disk $D(r)$ with $r<1$ it satisfies :*

$$|S(z)| \ \geq \ \big(\frac{d^2}{2^{6k}}\big)^{(1+r)/(1-r)} \ .$$

Here again, the estimates cannot hold in the whole unit disk. For instance, if $\mu$ is the Dirac measure $\delta_0$, we obtain the singular function

$$S(z) = \exp\{-\frac{1+z}{1-z}\} \ ,$$

and if $z = re^{i\theta}$, $|S(z)| = \exp\{-P_r(\theta)\}$. If $z$ is real, say $z = r$,

$$|S(z)| \ = \ \exp\{-\frac{1+r}{1-r}\} \ ,$$

which tends to 0 when $r \to 1^-$. So, though $|S(e^{i\theta})| = 1$ *a.e.*, there are points at which $|S(e^{i\theta})| = 0$. The geometry of the set where $|S(z)|$ is small will be studied in § 3.

Of course, no result such as theorem 1.8 or theorem 1.11 may hold for Blaschke products, since their zeros are precisely inside the unit disk. But if we assume some concentration at low degrees, at lot can be said about their repartition. This topic will be studied in the coming paragraph.

## 2. The zeros of $H^2$ functions with concentration at low degrees.

Let, as before, $f = \sum_0^\infty a_j z^j$, with $\sum |a_j|^2 < \infty$, be a function in $H^2$. The number of zeros of $f$ inside any disk $D(0,r)$, centered at $0$, with radius $r<1$, is of course finite, but it can be arbitrary large : any Blaschke product with prescribed zeros in this disk provides such a function, with moreover $\|f\|_\infty = 1$.

Let $(\alpha_n)_{n\geq 1}$ be the enumeration of the zeros of $f$ in the open unit disk, written in increasing order of moduli

$$|\alpha_1| \ \leq \ |\alpha_2| \ \leq \ |\alpha_3| \ \leq \ \cdots,$$

each of them being repeated according to its multiplicity. The sequence of zeros must satisfy $\sum_n 1-|\alpha_n| < \infty$, but, as we already said, this condition is the only one valid in general. It does not allow, of course, any prediction on how large the $n$-th zero is.

For entire functions, estimates do exist for the rate of growth of the zeros to infinity, depending on the order and type of the function (see B. Levin [7]).

We will show here that, again, the concept of concentration at low degrees provides a satisfactory description for a scale of growth.

This question was originally studied by Sylvia Chou in [4], where the concentration was measured with the $l_1$ norm :

$$\sum_0^k |a_j| \ \geq \ d \sum_{j\geq 0} |a_j|.$$

She showed that for a polynomial satisfying this estimate, there is a closed disk of radius $R(d,k)>0$, in which it has at most $k+1$ roots. She gave precise lower and upper estimates of this radius, and computed it exactly for the class of Hurwitz polynomials.

Our frame, here, is more general, since the concentration is measured with the $l_2$ norm. Moreover, we compute the number of zeros inside any disk of radius $r < 1$. Our methods are completely different and rely upon the canonical factorization (since here we don't have $\sum |a_j| < \infty$). The estimates in both cases cannot be deduced one from the other.

We now turn to the description of the results. Most of them were originally discussed with S. Dobyinsky and J.-B. Baillon.

**Theorem 2.1.** – *Let $d$, $0 < d \leq 1$, $k \in \mathbb{N}$, $r$, $0 < r < 1$. In the open disk of center O, radius $r$, the number of zeros of any function in $H^2$ with concentration $d$ at degree $k$ is at most :*

$$N(d,k,r) \;=\; \min_{0<x<1} \frac{\log 1/d - k \log x}{\log \frac{1+xr}{x+r}} \;. \tag{2.1}$$

*An upper bound for this number is therefore :*

$$N(d,k,r) \;\leq\; \frac{\log 1/d - k \log r}{\log \frac{1+r^2}{2r}} \;. \tag{2.2}$$

For $r = 0$, $N(d,k,0) = k$, and for $r = 1$, $N(d,k,1) = +\infty$ ; both results are obvious, since $f$ may have O as a multiple zero of order $k$, and since the number of zeros in the unit disk is unbounded : the polynomials $1 + 2z^n$ have concentration $1/\sqrt{5}$ at degree 0 and have $n$ roots inside the open unit disk.

We also observe that for $k = 0$, the result is well-known (see B. Levin [7], th. 5 p. 14), and follows readily from Jensen's formula :

$$|a_0| \prod_{|\alpha_n| \leq 1} \frac{1}{|\alpha_n|} \;\leq\; M(f),$$

where $M(f)$ is Mahler's measure of $f$, that is

$$M(f) \;=\; \exp \int_0^{2\pi} \log |f(e^{i\theta})| \, \frac{d\theta}{2\pi} \;.$$

Indeed, since $M(f) \leq \|f\|_2$, and since $|a_0| \geq d\,\|f\|_2$, we get :

$$\prod_{|\alpha_n| \leq r} \frac{1}{|\alpha_n|} \;\leq\; \frac{1}{d} \;,$$

which implies that the number of zeros inside $D(0,r)$ is at most

$$N(d,0,r) \;=\; \frac{\log 1/d}{\log 1/r} \;, \tag{2.3}$$

which is precisely the result given by formula (2.1) for $k = 0$.

The theorem can be proved by induction on $k$, but this leads to poor estimates, so we prefer to give a direct proof, the idea of which was suggested to us by L. Carleson, who also pointed out that any normalization (instead of the $l_2$ norm) stronger than the Nevanlinna class would lead to results of the same nature (though quantitatively different).

**Proof**. – We may assume that $f$ does not vanish at zero. Indeed, if it does, we can either approximate the function by another one which does not, or divide it by some $z^l$, and then observe that

$$N(d, k-l, r) + l \;\leq\; N(d, k, r).$$

Let

$$0 \;<\; |\alpha_1| \;\leq\; |\alpha_2| \;\leq\; \cdots,$$

be an enumeration of the zeros of $f$ inside the open unit disk. We write the corresponding Blaschke product :

$$B(z) \;=\; \prod_1^\infty \frac{\bar{\alpha}_i}{|\alpha_i|} \frac{\alpha_i - z}{1 - \bar{\alpha}_i z} \;,$$

and the corresponding decomposition :

$$f \;=\; B \cdot S \cdot F \;,$$

where $S$ is a singular function and $F$ an outer function.

Moreover, for fixed $r$, $0 < r < 1$, we further decompose $B$ into :

$$B'(z) \;=\; \prod_{|\alpha_i| < r} \frac{\bar{\alpha}_i}{|\alpha_i|} \frac{\alpha_i - z}{1 - \bar{\alpha}_i z} \;,$$

$$B''(z) \;=\; \prod_{|\alpha_i| \geq r} \frac{\bar{\alpha}_i}{|\alpha_i|} \frac{\alpha_i - z}{1 - \bar{\alpha}_i z} \;.$$

If $f \in H^2$ and $0 < R \leq 1$, we put $f_R(z) = f(Rz)$. Then the Fourier coefficients of $f_R$ are :

$$c_j(f_R) \;=\; R^j c_j(f) \;, \qquad j = 0, 1, \ldots \tag{2.4}$$

To prove the theorem, we may assume $|f|_2 = 1$. Then $|s_k(f)|_2 = d$, and therefore, by (2.4), for any $R$, $0 < R \leq 1$,

$$|s_k(f_R)|_2 \;\geq\; d\, R^k \;. \tag{2.5}$$

From the decomposition :

$$f_R \;=\; B'_R \cdot B''_R \cdot S_R \cdot F_R \;,$$

follows

$$|s_k(f_R)|_2 \;\leq\; |f_R|_2 \;\leq\; \|B'_R\|_\infty \cdot \|B''_R\|_\infty \cdot \|S_R\|_\infty \cdot |F_R|_2 \;. \tag{2.6}$$

But since $B''$ and $S$ are inner functions,

$$\|B''_R\|_\infty \;\leq\; 1 \;, \qquad \|S_R\|_\infty \;\leq\; 1,$$

and

$$|F_R|_2 \;\leq\; |F|_2 \;=\; |f|_2 \;=\; 1.$$

If $|\alpha| \leq r$, $|z| \leq R$, a simple computation shows that :

$$\left| \frac{\alpha - z}{1 - \bar{\alpha} z} \right| \;\leq\; \frac{r + R}{1 + rR} \;. \tag{2.7}$$

Let $N$ be the number of zeros of $f$ inside the open disk $D(0, r)$, that is, the number of terms in $B'$. We deduce from (2.5), (2.6), (2.7) :

$$dR^k \;\leq\; \left( \frac{r + R}{1 + rR} \right)^N \;,$$

that is :

$$N \;\leq\; \frac{\log \frac{1}{d} + k \log \frac{1}{R}}{\log \frac{1 + rR}{r + R}} \;,$$

and since $R$ is arbitrary, we obtain the result.

**Remark**. – In the case $k=0$, the result is best possible. Indeed, if we consider a finite Blaschke product :

$$B(z) \;=\; \left(\frac{r-z}{1-rz}\right)^N \quad ,$$

it satisfies $|B|_2 = 1,\ B(0) = r^N$, and so has concentration $d$ at degree 0 if

$$N \;=\; \frac{\log 1/d}{\log 1/r}\ .$$

There is another proof of the Theorem (giving the same estimate), using the following result (to be proved in Lemma 4.9 below) : for $0 < R < 1$, there is a $z_0$, $|z_0| = R$, such that if we set $z = \frac{w-z_0}{1-\bar z_0 w}$, the function $\tilde f(w) = f(z)$ has concentration $dR^k$ at degree 0, when $f$ has concentration $d$ at degree $k$.

Assuming this, by Jensen's formula as above, the number of zeros of $\tilde f$ in the disk $\{|w| \le r'\}$ is at most

$$N \;=\; \frac{\log 1/d - k\log R}{\log 1/r'}\ .$$

But the zeros of $f$ satisfy $\left|\frac{\alpha_i - z_0}{1-\bar\alpha_i z_0}\right| \le r'$, and so the number of $\alpha_i$'s with $|\alpha_i| \le r$ is at most $N$, if $\frac{r+R}{1+rR} = r'$. Minimizing over $R$ gives the result.

We see that $N(d,k,r)$ is proportional to $k$, which is of course satisfactory. However, the coefficient of $k$ is not sharp. For instance, for $d=1$, $f$ is just a polynomial of degree $k$, so we should have $N(1,k,r) = k$ for every $r$. But we find $N(1,k,r) = \frac{1+r}{1-r}k$ (since $\log x$ and $1/\log\frac{1+xr}{x+r}$ are both increasing functions of $x$, the minimum is a limit when $x \to 1^-$).

In fact, this lack of sharpness is due to the method itself : there is *always* a loss of concentration between the function and the Blaschke product ; we will come back on this later.

What about the correct order of magnitude ? The estimate $N(d,k,r) \ge k$ is obvious, but let's show that $N(d,k,r) \ge \alpha k$, for some $\alpha > 1$, if $d < 1$.

Indeed, consider $P = (z+r)^{\alpha k}$ : it has $\alpha k$ roots of modulus $r$, and we will see that $\mathrm{cf}_k(P) \to 1$ when $k \to \infty$, if $\alpha$ is correctly chosen ($\alpha > 1$, close enough to 1).

First, we observe that

$$\|P\|_2 \;\ge\; \frac{(1+r)^{\alpha k}}{\sqrt{\alpha k + 1}}\ .$$

Also :

$$\begin{aligned} 1 - \mathrm{cf}_k(P) \;&\le\; \frac{1}{\|P\|_2}\Big(\sum_{j=k}^{\alpha k}\binom{\alpha k}{j}^2 r^{2(\alpha k - j)}\Big)^{1/2} \\ &\le\; \frac{1}{\|P\|_2}\binom{\alpha k}{k}\Big(\sum_0^\infty r^{2j}\Big)^{1/2} \\ &\le\; \frac{1}{\|P\|_2}\sqrt{\frac{1}{1-r^2}}\binom{\alpha k}{k} \end{aligned}$$

Using Stirling's formula, we write :

$$\binom{\alpha k}{k} \;\sim\; \left(\frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}}\right)^k \sqrt{\frac{\alpha}{2\pi(\alpha-1)k}}\ .$$

So we see that $\mathrm{cf}_k(P) \to 1$ when $k \to \infty$, if we choose $\alpha$ so that

$$\frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}}\frac{1}{(1+r)^\alpha} \;<\; 1.$$

Setting $\alpha = 1 + 1/n$, this is equivalent to

$$n^{1/(1+n)}(1+\frac{1}{n}) \ < \ 1+r.$$

For $r = 1/2$, this holds for $n \geq 7$, which gives $\alpha = 8/7$. The correct order of magnitude is therefore between $1.14k$ and $3.1k$.

There is no clear comparison between the polynomial $P = (z+\alpha)^n$ and the Blaschke term $B = (\frac{\alpha+z}{1+\bar{\alpha}z})^n$, in terms of concentration at a given degree $k$. It depends strongly on $k$ and $n$. For instance, for $k = 0$, the concentration of $P$ at degree 0 is small when $n$ is large, and that of $B$ can be made arbitrarily close to 1, and, for $k = 1$, $z + \alpha$ has concentration 1 at degree 1 and $\frac{\alpha+z}{1+\alpha z}$ has concentration strictly less than 1.

From our first result, we now deduce lower estimates for the moduli of the zeros of $f$, written as before in increasing order. Of course, for a function with concentration $d$ at degree $k$, the first $k$ zeros may be 0, so such estimates can start only with $|\alpha_{k+1}|$.

**Theorem 2.2** – **Let** $d$**,** $0 < d \leq 1$**, and** $k \in \mathrm{IN}$**.** *Let* $f$ *be an* $H^2$*-function with concentration* $d$ *at degree* $k$*. Then, for every* $j \geq 1$*, its* $k+j$*-th zero satisfies the estimate :*

$$|\alpha_{k+j}| \ \geq \ \frac{jd^{1/j}}{e(k+j)} \ .$$

**Proof**. – For fixed $d$, $k$, we write $N(r)$ instead of $N(d,k,r)$. It is of course an increasing function of $r$. Take any $r$ such that :

$$N(r) \ < \ k+j \ ,$$

this means that $|\alpha_{k+j}| \geq r$. Thus, if $N'(r)$ is any increasing function with $N(r) \leq N'(r)$, any $r$ for which $N'(r) < k+j$ gives a lower bound for $|\alpha_{k+j}|$.

We are looking for the largest $r$ such that

$$\min_{0<x<1} \frac{\log 1/d - k\log x}{\log \frac{1+xr}{x+r}} \ < \ k+j,$$

or –which is the same–, for the largest $r$ such that, for some $x$,

$$\log\frac{1}{d} - k\log x \ < \ (k+j)\ \log\frac{1+xr}{x+r} \ ,$$

which means :

$$\frac{1}{dx^k} \ < \ \left(\frac{1+xr}{r+x}\right)^{k+j} \ . \tag{2.8}$$

This will hold as soon as

$$\frac{1}{dx^k} \ \leq \ \frac{1}{x^{k+j}}\left(\frac{1}{1+r/x}\right)^{k+j} \ ,$$

or for any $\lambda > 0$,

$$\frac{x^{j\lambda}}{d^\lambda} \ \leq \ \left(\frac{1}{1+r/x}\right)^{(k+j)\lambda} \ .$$

But

$$\left(\frac{1}{1+r/x}\right)^{(k+j)\lambda} \ \geq \ 1 - \frac{r(k+j)\lambda}{x} \ .$$

So we see that our inequality will be satisfied as soon as

$$\frac{x^{j\lambda}}{d^\lambda} + \frac{r(k+j)\lambda}{x} \ \leq\ 1 \ .$$

Put $r' = r(k+j)$ and

$$y(x) \ = \ \frac{x^{j\lambda}}{d^\lambda} + \frac{\lambda r'}{x} \ .$$

Then $y$ reaches its minimum for $x = \left(\frac{r'd^\lambda}{j}\right)^{\frac{1}{j\lambda+1}}$, and the value of this minimum is :

$$y \ = \ \frac{r'^{j\lambda/(j\lambda+1)}}{d^{\lambda/(j\lambda+1)}} \left(\lambda j^{1/(j\lambda+1)} + j^{-j\lambda/(j\lambda+1)}\right) .$$

So condition 2.8 will hold if, for some $\lambda > 0$,

$$r' \ \leq\ d^{1/j}\left(\lambda j^{1/(j\lambda+1)} + j^{-j\lambda/(j\lambda+1)}\right)^{-(j\lambda+1)/j\lambda} \ = \ \frac{jd^{1/j}}{(j\lambda+1)^{\frac{j\lambda+1}{j\lambda}}} \ .$$

In order to find the largest value for $r'$, we set $n = 1/(j\lambda)$, and find

$$\min_n j^{-1}\left(1+\frac{1}{n}\right)^{n+1} \ = \ \frac{e}{j} \ .$$

This gives to $r'$ the value $\frac{1}{e} j d^{1/j}$, and proves the theorem.

For $j = 1$, we find

$$|\alpha_{k+1}| \ \geq\ \frac{d}{e(k+1)} \ , \tag{2.9}$$

which we can compare to S. Chou's result :

If $P$ satisfies

$$\sum_0^k |a_j| \ \geq\ d \sum_{j\geq 0} |a_j| \ , \tag{2.10}$$

then

$$|\alpha_{k+1}| \ \geq\ \frac{d}{2e(k+1)} \ . \tag{2.11}$$

The estimate (2.9) seems better. But in fact, if (2.10) holds, we deduce for the $l_2$-norm :

$$(\sum_0^k |a_j|^2)^{1/2} \ \geq\ \frac{d}{\sqrt{k+1}} (\sum_{j\geq 0} |a_j|^2)^{1/2} \ ,$$

and thus by (2.9) :

$$|\alpha_{k+1}| \ \geq\ \frac{d}{e(k+1)^{3/2}} \ ,$$

which is worse than (2.11). So in fact the $l_1$ and the $l_2$ situations correspond to different methods and settings, and any attempt to deduce one from the other will provide bad estimates.

The result (2.9) is quite sharp with respect to $k$, and with respect to $d$ when $d$ is small. Indeed, consider :

$$P(z) \;=\; (z + \frac{d}{k+1})^{k+1} \;.$$

Then $|\alpha_{k+1}| = d/(k+1)$. Let's now compute the concentration of $P$ at degree $k$. We have :

$$|P(e^{i\theta})|^2 \;=\; 1 + 2\frac{d}{k+1}\cos\theta + \frac{d^2}{(k+1)^2} \;,$$

$$\begin{aligned} |e^{i\theta} + \frac{d}{k+1}|^{2(k+1)} \;&=\; (1 + 2\frac{d}{k+1}\cos\theta + \frac{d^2}{(k+1)^2})^{k+1} \\ &\to\; e^{2d\cos\theta} \;,\quad \text{when } k \to \infty. \end{aligned}$$

Therefore, when $k \to \infty$,

$$\begin{aligned} \int_{-\pi}^{\pi} |e^{i\theta} + \frac{d}{k+1}|^{2(k+1)} \frac{d\theta}{2\pi} \;&\to\; \int_{-\pi}^{\pi} e^{2d\cos\theta} \frac{d\theta}{2\pi} \\ &\geq\; 1 + 2d^2 \int_{-\pi}^{\pi} \cos^2\theta \frac{d\theta}{2\pi} \\ &\geq\; 1 + d^2 \;. \end{aligned}$$

Since the degree of $P$ is precisely $k+1$,

$$\frac{|s_k(P)|_2^2}{|P|_2^2} \;=\; \frac{|P|_2^2 - 1}{|P|_2^2} \;=\; 1 - \frac{1}{|P|_2^2} \;\geq\; 1 - \frac{1}{1+d^2} \;=\; \frac{d^2}{1+d^2},$$

so, when $d$ is small,

$$\frac{|s_k(P)|_2}{|P|_2} \;\geq\; \frac{d}{\sqrt{1+d^2}} \;\sim\; d.$$

For $k = 0$, our problem can easily be solved directly. We denote by $\alpha_1(f)$ the first zero of $f$.

**Proposition 2.3.** – *We have :*

$$\min\{|\alpha_1(f)| \;;\; f \text{ has concentration } d \text{ at degree } 0\} \;=\; d,$$

*this minimum is attained for the Blaschke term :*

$$f(z) \;=\; c\frac{d-z}{1-dz} \;,$$

*where* $|c| = 1$.

**Proof**. – We write the canonical decomposition :

$$f \;=\; B \cdot S \cdot F,$$

and $B = B_1 \cdot B_2$, with $B_1$ consisting of a single factor $(\alpha_1 - z)/(1 - \bar{\alpha}_1 z)$, $B_2$ containing all other zeros of $f$. Then :

$$|f(0)| \;=\; |B_1(0)||B_2(0)||S(0)||F(0)|.$$

Since $|B_2(0)| \leq 1$, $|S(0)| \leq 1$, and $|F(0)|/\|F\|_2 \leq 1$, we find $|B_1(0)| \geq d$, which means $|\alpha_1| \geq d$. Equality holds only if $|B_2(0)| = |S(0)| = |F(0)|/\|F\|_2 = 1$, and this implies that all other factors are trivial.

Let's come back to the general setting : $k \geq 0$, and let's study the behavior of $|\alpha_{k+j}|$ when $j$ increases. First, it's quite interesting to observe that the estimate for $|\alpha_{k+2}|$ is in $\sqrt{d}$, thus substantially different from that of $|\alpha_{k+1}|$. This phenomenon has a reality, since the estimate of $|\alpha_{k+1}|$ is sharp and since $\sqrt{d}$ is substantially bigger than $d$ when $d$ is small. But, quite obviously, when $j \to \infty$, the estimates become less and less precise since the given radius does not tend to $1$.

In order to obtain estimates showing that $|\alpha_{k+j}|$ tends to 1 when $j \to \infty$, we have to come back to Theorem 2.1 and use computations of another type.

**Theorem 2.4.** – *Let $f$ be a function in $H^2$, with concentration $d$ at degree $k$. Then, for every $n > k$, $|\alpha_n| = r_n(d,k)$, where*

$$r_n(d,k) \;=\; \inf\{r \;;\; r \text{ satisfies the equation } N(d,k,r) = n\}.$$

The number $r_n$ satisfies for every $n > k$ the estimate

$$r_n \;\geq\; 1 - \frac{4}{n}(\log\frac{1}{d} + k\log 2),$$

and the asymptotic estimate, when $n \to \infty$ :

$$r_n \;\geq\; 1 - \frac{3}{n}(\log\frac{1}{d} + k\log 2).$$

**Proof**. – The first assertion is obvious. To obtain the first estimate, we let $r = 1 - \mu$ in formula (2.1). It becomes :

$$\begin{aligned}
\min_{0<x<1} \frac{\log 1/d - k\log x}{\log\frac{1+xr}{x+r}} &= \min_{0<x<1} \frac{\log 1/d - k\log x}{\log\left(1+\frac{\mu(1-x)}{1+x-\mu}\right)} \\
&\leq \min_{0<x<1} \frac{\log 1/d - k\log x}{\log\left(1+\frac{\mu(1-x)}{1+x}\right)} \\
&\leq \frac{\log 1/d + k\log 2}{\log(1+\frac{\mu}{3})} \\
&\leq \frac{\log 1/d + k\log 2}{\mu/4}\ ,
\end{aligned}$$

since $0 < \mu < 1$. Let's now take

$$\mu \;=\; \frac{4}{n}(\log\frac{1}{d} + k\log 2)\ .$$

We have $1 - |\alpha_n| \leq \mu$, and the estimate is proved. The asymptotic estimate is obtained the same way, observing that, when $\mu \to 0$,

$$\log(1+\mu\frac{1-x}{1+x}) \;\sim\; \mu\,\frac{1-x}{1+x}\ ,$$

and taking $x = 1/2$.

**Remark**. – We conjecture that the correct estimate in Theorem 2.2 is

$$|\alpha_{k+j}| \;\geq\; \frac{j d^{1/j}}{k+j}\ . \tag{2.12}$$

This is compatible with Proposition 2.3 (case $k = 0,\ j = 1$), and would provide estimates showing directly that $|\alpha_{k+j}| \to 1$ when $j \to \infty$. These estimates are also of the type

$$1 - |\alpha_{k+j}| \;\sim\; \frac{1}{j}\log\frac{1}{d}\ ,$$

thus compatible with Theorem 2.4.

All the results of this paragraph, so far, are more or less satisfactory when $d$ is small. Let's now investigate the following question : what happens when $d \to 1$ ? Then the function $f$ "tends" to become a polynomial of degree $k$, so one might expect $|\alpha_{k+1}| \to \infty$. But this is wrong. Consider :

$$P_n \;=\; 1 + \frac{1}{n} z^n \; .$$

The concentration at degree 0 is $(1 + 1/n^2)^{-1/2} \to 1$, when $n \to \infty$, and all the roots have moduli tending to 1. As we will see, this is the correct answer : all roots after the $k+1$-st must get closer and closer to the unit circle, when $d$ approaches 1.

To obtain this result, we need to develop a different approach which does not rely on the canonical factorization : even the conjectural estimates (2.12) do not show it.

This is due to the following phenomenon, which we already pointed out and which is easy to observe : there may be a loss of concentration between the function and its Blaschke factor : we already mentioned a factor $1/\sqrt{k+1}$, when $k \to \infty$, in § 1. But even for small values of $k$, it can be observed. Consider $P = a - z$, with $|a| < 1$ ; it has concentration 1 at degree 1. Its canonical factorization is

$$P \;=\; \frac{a - z}{1 - \bar{a}z} \cdot (1 - \bar{a}z).$$

The Taylor expansion of the Blaschke term is

$$\frac{a - z}{1 - \bar{a}z} \;=\; a - (1 - |a|^2)(z + \bar{a}z^2 + \bar{a}^2 z^3 \cdots),$$

and

$$\mathrm{cf}_1(B) \;=\; \big(|a|^2 + (1 - |a|^2)^2\big)^{1/2},$$

which takes $\sqrt{3/4}$ as minimal value. So there is a polynomial of degree 1 such that $B$ has only concentration $\sqrt{3/4}$ at degree 1. This shows clearly that it's hopeless to use the canonical factorization in order to get results when $d$ is close to 1.

We now introduce new notations. Let $f = \sum_0^\infty a_j z^j$ be, as before, an $H^2$ function. We define :

$$\sigma_k(f) \;=\; \Big(\sum_0^k |a_j|^2\Big)^{1/2} \; , \quad \sigma'_k(f) \;=\; \Big(\sum_{k+1}^\infty |a_j|^2\Big)^{1/2} \; ,$$

$$\delta_k(f) \;=\; \frac{\sigma'_k(f)}{\sigma_k(f)} \; .$$

Let $d_k = \mathrm{cf}_k(f)$ be the concentration factor of $f$ at degree $k$. The numbers $d_k$ and $\delta_k$ are related by the obvious formula :

$$\delta_k^2 \;=\; \frac{1 - d_k^2}{d_k^2} \; ,$$

so if $d_k$ is close to 1, $\delta_k$ is close to 0. Let now $g = \sum_0^\infty b_j z^j \in H^2$, $\alpha \in \mathbb{C}$, and set

$$f \;=\; (\alpha - z) g.$$

**Proposition 2.5.** – *If* $\delta_k(f) < (1 - |\alpha|)/8$, *then*

$$\delta_{k-1}(g) \;\le\; \frac{4}{1 - |\alpha|}\, \delta_k(f) \; .$$

**Proof**. – First, we write the Taylor expansion of $f$ :

$$f(z) \;=\; \alpha b_0 + (-b_0 + \alpha b_1)z + \cdots + (-b_{j-1} + \alpha b_j)z^j + \cdots \tag{2.13}$$

We write $d$ instead of $\mathrm{cf}_k(f)$, and $\delta$ instead of $\delta_k(f)$. We deduce from (2.13) :

$$\sum_{k+1}^{\infty} |-b_{j-1} + \alpha b_j|^2 \;=\; \delta^2 \big(|\alpha b_0|^2 + \sum_{0}^{k-1} |-b_j + \alpha b_{j+1}|^2\big). \tag{2.14}$$

But :

$$\begin{aligned} |b_k| \;&\le\; \sum_0^{\infty} |\alpha|^j |-b_{k+j} + \alpha b_{k+j+1}| \\ &\le (\sum_0^{\infty} |\alpha|^{2j})^{1/2} (\sum_0^{\infty} |-b_{k+j} + \alpha b_{k+j+1}|^2)^{1/2} \\ &\le \left(\frac{1}{1-|\alpha|^2}\right)^{1/2} (\sum_0^{\infty} |-b_{k+j} + \alpha b_{k+j+1}|^2)^{1/2} \end{aligned}$$

So we deduce from (2.14) :

$$(1-|\alpha|^2)|b_k|^2 \;\le\; \sum_{k+1}^{\infty} |-b_{j-1} + \alpha b_j|^2 \;\le\; 3\delta^2 \sum_0^k |b_j|^2 \;,$$

and this implies

$$|b_k|^2 \;\le\; \frac{3\delta^2}{1-|\alpha|^2 - 3\delta^2} \sum_0^{k-1} |b_j|^2 \;. \tag{2.15}$$

We also have

$$(\sum_{k+1}^{\infty} |-b_{j-1} + \alpha b_j|^2)^{1/2} \;\ge\; (1-|\alpha|)(\sum_{k+1}^{\infty} |b_j|^2)^{1/2} \;,$$

which implies

$$\sum_{k+1}^{\infty} |b_j|^2 \;\le\; \frac{3\delta^2}{(1-|\alpha|)^2} \sum_0^k |b_j|^2 \;. \tag{2.16}$$

Using (2.15), we deduce from (2.16)

$$\sum_{k+1}^{\infty} |b_j|^2 \;\le\; \frac{3\delta^2}{(1-|\alpha|)^2} \left(1 + \frac{3\delta^2}{1-|\alpha|^2 - 3\delta^2}\right) \sum_0^{k-1} |b_j|^2 \;. \tag{2.17}$$

Using (2.15) once again, we finally obtain

$$\begin{aligned} \sum_k^{\infty} |b_j|^2 \;&\le\; \left(\frac{3\delta^2}{(1-|\alpha|)^2}\left(1 + \frac{3\delta^2}{1-|\alpha|^2-3\delta^2}\right) + \frac{3\delta^2}{1-|\alpha|^2-3\delta^2}\right) \sum_0^{k-1} |b_j|^2 \\ &\le \frac{6\delta^2}{(1-|\alpha|)(1-|\alpha|^2-3\delta^2)} \sum_0^{k-1} |b_j|^2 \;, \end{aligned}$$

from which the Proposition follows immediately. We thank J.L. Frot for pointing out a mistake in an earlier proof of this Proposition.

We can now prove :

**Theorem 2.6.** – *Let $f$ be a function in $H^2$, with the zeros written in increasing order :*

$$|\alpha_1| \ \leq \ |\alpha_2| \ \leq \ |\alpha_3| \ \leq \ \cdots$$

*Then the $k+1$-st zero $\alpha_{k+1}$ satisfies :*

$$|\alpha_{k+1}| \ \geq \ 1 - 4\delta^{1/(k+1)} \ ,$$

*with $\delta = \delta_k(f)$.*

**Proof**. – The case $k=0$ is left to the reader, and we assume $k \geq 1$. We write

$$f \ = \ (\alpha_1 - z)\cdots(\alpha_k - z)g.$$

We first observe that

$$|\alpha_{k+1}| \geq \mathrm{cf}_0(g) \ . \tag{2.18}$$

Indeed, $\alpha_{k+1}$ is the first zero of $g$. Jensen's formula gives :

$$|g(0)| \prod_{\substack{n \geq k+1 \\ |\alpha_n| \leq 1}} \frac{1}{|\alpha_n|} \ \leq \ M(g) \ \leq \ |g|_2 \ ;$$

but since

$$\prod_{\substack{n \geq k+1 \\ |\alpha_n| \leq 1}} \frac{1}{|\alpha_n|} \ \geq \ \frac{1}{|\alpha_{k+1}|} \ ,$$

we deduce

$$|\alpha_{k+1}| \ \geq \ \frac{|g(0)|}{|g|_2} \ ,$$

as we claimed.

Since $\mathrm{cf}_0^2(g) = 1/(1+\delta_0^2(g))$, we deduce from (2.18)

$$1 - |\alpha_{k+1}| \ < \ \delta_0^2(g). \tag{2.19}$$

We consider two cases :

**Case 0**. – $|\alpha_k| \ \geq \ 1 - 4\delta^{1/(k+1)}$ .

Then a fortiori $|\alpha_{k+1}|$ satifies the same estimate, and the theorem is proved, or

**Case 1**. – $|\alpha_k| \ < \ 1 - 4\delta^{1/(k+1)}$ .

We now consider this last case. Then also $|\alpha_1|, \cdots, |\alpha_k|$ satisfy this estimate, which implies

$$\delta \ < \ \frac{(1-|\alpha_k|)^{k+1}}{4^{k+1}} \ . \tag{2.20}$$

Set now $f_1 = f$, $f_2 = (\alpha_2 - z)\cdots(\alpha_k - z)g$, $\ldots$, $f_k \ = \ (\alpha_k - z)g$, $f_{k+1} = g$.

Since $\delta < (1-|\alpha_1|)/8$, Proposition 2.5 implies

$$\delta_{k-1}(f_2) \ < \ \frac{4\delta}{1-|\alpha_1|} \ ,$$

and by (2.20),

$$\frac{4\delta}{1-|\alpha_1|} \ < \ \frac{1-|\alpha_2|}{8} \ .$$

Therefore, Proposition 2.5 gives

$$\delta_{k-2}(f_3) \ < \ \frac{4\delta_{k-1}(f_2)}{1-|\alpha_2|} \ < \ \frac{4^2\delta}{(1-|\alpha_1|)(1-|\alpha_2|)} \ .$$

Since for every $j=1,\dots,k$, condition (2.20) implies :

$$\frac{4^{j-1}\delta}{(1-|\alpha_1|)\cdots(1-|\alpha_{j-1}|)} \ < \ \frac{1-|\alpha_j|}{8} \ ,$$

Proposition 2.5 gives

$$\delta_{k-j}(f_{j+1}) \ < \ \frac{4\delta_{k-j+1}(f_j)}{1-|\alpha_j|} \ < \ \frac{4^j\delta}{(1-|\alpha_1|)\cdots(1-|\alpha_j|)} \ .$$

Finally, for $j=k$, we get

$$\delta_0(g) \ < \ \frac{4^k\delta}{(1-|\alpha_1|)\cdots(1-|\alpha_k|)} \ \leq \ \frac{4^k\delta}{(1-|\alpha_k|)^k} \ . \tag{2.21}$$

Taking (2.20) into account once again gives :

$$\delta_0(g) \ < \ \delta^{1/(k+1)} \ ,$$

and by (2.19),

$$1-|\alpha_{k+1}| \ < \ \delta^{2/(k+1)} \ < \ 4\delta^{1/(k+1)} \ ,$$

and the Theorem is proved.

From these estimates one can easily deduce an asymptotic behavior of $|\alpha_{k+1}|$ when $d$ is close to 1 :

**Corollary 2.7.** – *When $d\to 1^-$, the $k+1$-st zero of $f$ satisfies :*

$$1-|\alpha_{k+1}| \ \sim \ 4\big(2(1-d)\big)^{1/2(k+1)} \ .$$

We have investigated, so far, the structure of the set $\{f=0\}$. We now turn to the set $\{|f|<\varepsilon\}$.

## III. The measure of the set where an $H^2$ function is small.

Our starting point is a theorem of H. Cartan (see B. Levin [7]), which provides an estimate from below for the modulus of a polynomial, outside a subset of known shape and measure :

**Theorem.** – *Given any $H>0$ and complex numbers $z_1$, $z_2$, $\dots$, $z_n$, there is a system of circles in the complex plane, with the sum of radii equal to $2H$, such that for each point $z$ lying outside these circles, one has the inequality :*

$$|z-z_1|\cdot|z-z_2|\cdots|z-z_n| \ > \ \left(\frac{H}{e}\right)^n \ . \tag{3.1}$$

In other words, if $M$ is the surface measure in the plane, for a polynomial

$$P(z) \ = \ (z-z_1)\cdots(z-z_n),$$

one has :

$$M\{z\ ;|P(z)|\leq\left(\frac{H}{e}\right)^n\} \ \leq 4\pi H^2 \ .$$

This means also that for any $\delta>0$,

$$M\{z\ ;|P(z)|\leq\delta\} \ \leq 4\pi e^2\delta^{2/n} \ . \tag{3.2}$$

This estimate depends on the degree of the polynomial. We are going to extend it to the frame of $H^2$ functions with *concentration at low degree* :

**Theorem 3.1.** – *Let $f$ be a function in the Hardy space $H^2$, satisfying*

$$\|f\|_2 \;=\; 1\;; \qquad (\sum_{j\le k} |a_j|^2)^{1/2} \;\ge\; d. \tag{3.3}$$

*Let $M$ be the surface measure in the plane, and $\bar{D}$ the closed unit disk. Then, for any $\delta > 0$,*

$$M\{z \in \bar{D}\;;\; |f(z)| \le \delta\} \;\le\; \frac{C(d,k)}{\log(1/\delta)}\;,$$

*where*

$$C(d,k) \;=\; \pi \,\log \frac{d^2}{3^{3(k+2)}}\;.$$

**Proof**. – It does not follow at all Cartan's original proof, which was algebraic and highly dependent of the degree. Here instead, we use Jensen's inequality, as we already did in [1], [2]. We start with estimates which improve slightly the ones we obtained in these papers, though they basically follow the same techniques.

**Lemma 3.2.** – *For any $r$, $0 < r \le 1$, for any function $f$ satisfying (3.3), one has :*

$$\int_0^{2\pi} \log|f(re^{i\theta})|\,\frac{d\theta}{2\pi} \;\ge\; \max_{0<\rho<1} \frac{1+\rho}{1-\rho}\left(\log(dr^k\rho^k) - \frac{1}{2}\frac{1+\rho}{1-\rho}\right)\;.$$

**Proof** of Lemma 3.2. – Fix $r$, $0 < r \le 1$, and define $h(z) = f(rz)$. This function is also in $H^2$. Fix also $\rho$, $0 < \rho < 1$. For any $z_0$, with $|z_0| = \rho$, one has, by the classical Jensen's inequality :

$$\int_0^{2\pi} \log\left|h\left(\frac{e^{i\theta}+z_0}{1+\bar{z}_0 e^{i\theta}}\right)\right|\frac{d\theta}{2\pi} \;\ge\; \log|h(z_0)|\;. \tag{3.4}$$

Let $A$ be the left-hand side of (3.4). A change of variable gives :

$$\begin{aligned} A \;&=\; \int_0^{2\pi} \log|h(e^{i\theta})|\frac{1-\rho^2}{|1-\bar{z}_0 e^{i\theta}|^2}\,\frac{d\theta}{2\pi} \\ &\le \frac{1-\rho}{1+\rho}\int_{\log|h|\le 0} \log|h(e^{i\theta})|\,\frac{d\theta}{2\pi} + \frac{1+\rho}{1-\rho}\int_{\log|h|>0} \log|h(e^{i\theta})|\,\frac{d\theta}{2\pi}\;. \end{aligned}$$

But

$$\int_{\log|h|>0} \log|h(e^{i\theta})|\,\frac{d\theta}{2\pi} \;\le\; \frac{1}{2}\int |h|^2\,\frac{d\theta}{2\pi} \;\le\; \frac{1}{2}\int |f|^2\,\frac{d\theta}{2\pi} \;=\; \frac{1}{2}\;.$$

So we get :

$$\int_{\log|h|\le 0} \log|h(e^{i\theta})|\,\frac{d\theta}{2\pi} \;\ge\; \frac{1+\rho}{1-\rho}\left(A - \frac{1}{2}\frac{1+\rho}{1-\rho}\right),$$

and by (3.4) :

$$\int \log|h(e^{i\theta})|\,\frac{d\theta}{2\pi} \;\ge\; \frac{1+\rho}{1-\rho}\left(\log|h(z_0)| - \frac{1}{2}\frac{1+\rho}{1-\rho}\right). \tag{3.5}$$

Let's write $z_0 = \rho e^{it}$. We have :

$$h(z_0) \;=\; f(rz_0) \;=\; f(r\rho e^{it}) \;=\; \sum_{j=0}^{\infty} a_j r^j \rho^j e^{ijt}\;,$$

and so :

$$\left(\int_0^{2\pi} |f(r\rho e^{it})|^2 \, \frac{dt}{2\pi}\right)^{1/2} = \left(\sum_0^{\infty} |a_j|^2 r^{2j} \rho^{2j}\right)^{1/2}$$
$$\geq \left(\sum_0^{k} |a_j|^2 r^{2j} \rho^{2j}\right)^{1/2}$$
$$\geq dr^k \rho^k \; .$$

So we can find a $z_0$, $|z_0| = \rho$, such that :

$$|h(z_0)| \;=\; |f(rz_0)| \;\geq\; dr^k \rho^k \; .$$

Coming back to (3.5), we find

$$\int_0^{2\pi} \log|f(re^{i\theta})| \, \frac{d\theta}{2\pi} \;\geq\; \max_{0<\rho<1} \frac{1+\rho}{1-\rho} \left(\log(dr^k\rho^k) - \frac{1}{2}\frac{1+\rho}{1-\rho}\right),$$

as we announced.

Taking $\rho = 1/3$, we find :

**Corollary 3.3.** – *If $f \in H^2$ has concentration $d$ at degree $k$, for every $r$, $0 < r \leq 1$,*

$$\int_0^{2\pi} \log|f(re^{i\theta})| \, \frac{d\theta}{2\pi} \;\geq\; 2 \log \frac{dr^k}{e 3^k} + \log \|f\|_2 \; .$$

If we assume $f \in H^\infty$ and $\|f\|_\infty = 1$, with $\|s_k(f)\|_2 \geq d$, the above arguments simplify, since $\log|h| \leq 0$, and one gets :

**Lemma 3.4.** – *Assume $f \in H^\infty$, $\|f\|_\infty \leq 1$, $\|s_k(f)\|_2 \geq d$. Then, for every $r$, $0 < r \leq 1$ :*

$$\int_0^{2\pi} \log|f(re^{i\theta})| \, \frac{d\theta}{2\pi} \;\geq\; \max_{0<\rho<1} \left\{\frac{1+\rho}{1-\rho} \log(dr^k\rho^k)\right\}.$$

This estimate will be used in § 4.

**Lemma 3.5.** – *Assume that $f \in H^2$ has concentration $d$ at degree $k$. Then,*

$$\int_0^1 \int_0^{2\pi} \log|f(re^{i\theta})| \, rdrd\theta \;\geq\; \pi \log \frac{d^2}{e^{k+1} 3^{2k}} \; .$$

**Proof** of Lemma 3.5. – We deduce from Corollary 3.3 :

$$\int_0^1 \int_0^{2\pi} \log|f(re^{i\theta})| \, rdrd\theta \;\geq\; 4\pi \int_0^1 r \log\left(\frac{d}{e}\left(\frac{r}{3}\right)^k\right) \, dr \; ,$$

*MACSYMA* computes this last integral and gives the required lower bound.

We now turn to the proof of the Theorem. We put $\phi(x,y) = f(z)$, for $z = x+iy \in \bar{D}$. Then :

$$\int\int_{\bar{D}} \log|\phi(x,y)| \, dxdy \;\geq\; \pi \log \frac{d^2}{e^{k+1} 3^{2k}} \; . \tag{3.7}$$

Let's split this integral into two parts : the part where $|\phi| \leq \delta$ and the part where $|\phi| > \delta$. We have :

$$\int\int_{|\phi|>\delta} \log|\phi(x,y)| \, dxdy \;\leq\; \int\int_{|\phi|\geq 1} \log|\phi(x,y)| \, dxdy$$
$$\leq \frac{1}{2} \int\int_{\bar{D}} |\phi(x,y)|^2 \, dxdy \; .$$

But

$$
\begin{aligned}
\Big(\int\!\!\int_D |\phi(x,y)|^2\, dxdy\Big)^{1/2} &= \left(\int_0^1 \int_0^{2\pi} |f(re^{i\theta})|^2\, rdrd\theta\right)^{1/2} \\
&\le \left(\int_0^1 rdr \int_0^{2\pi} |f(e^{i\theta})|^2\, d\theta\right)^{1/2} \\
&\le \sqrt{\pi}\ .
\end{aligned}
$$

This gives :

$$
\int\!\!\int_{|\phi|>\delta} \log|\phi(x,y)|\, dxdy \ \le\ \frac{\pi}{2}\ . \tag{3.8}
$$

From (3.7) and (3.8) we now deduce :

$$
\begin{aligned}
\int\!\!\int_{|\phi|\le\delta} \log|\phi(x,y)|\, dxdy &\ge \pi\, \log \frac{d^2}{e^{k+1}3^{2k}} - \frac{\pi}{2} \\
&\ge \pi\, \log \frac{d^2}{e^{k+3/2}3^{2k}} \\
&\ge \pi\, \log \frac{d^2}{3^{3k+2}}\ ,
\end{aligned}
$$

which gives :

$$
\int\!\!\int_{|\phi|\le\delta} \log \frac{1}{|\phi(x,y)|}\, dxdy \ \le\ \pi\, \log \frac{3^{3k+2}}{d^2}\ ,
$$

and finally :

$$
\big(\log \frac{1}{\delta}\big)\, M\{|\phi| \le \delta\} \ \le\ \pi\, \log \frac{3^{3k+2}}{d^2}\ ,
$$

which proves the Theorem.

Of course, outside the unit disk, the statement does not make sense since the function $f$ is not defined in general. But if we take a mere polynomial, the question may be asked, since Cartan's theorem was valid in the whole plane. Here, the result remains true, for a polynomial, inside a disk of given radius (with data depending on the radius ; same proof as above), but an inequality such as :

$$
M\{z \in \mathbb{C}\ ;\ |P(z)| \le \delta\} \ \le\ \varepsilon_{d,k}(\delta),
$$

with $\varepsilon_{d,k}(\delta) \to 0$ when $\delta \to 0$, cannot be true. Indeed, take any polynomial $P$ and a root $z_0$. There is a small disk around $z_0$, of radius $\alpha$, where $|P(z)| < \delta$. Let's consider $Q(z) = P(z/n)$, for $n \in \mathbb{N}$. We have $|Q(z)| < \delta$ in a disk centered at $nz_0$, with radius $n\alpha$. Moreover, $Q$ has concentration $d$ at degree $k$, if $P$ had concentration $d$ at degree $k$.

Finally, we observe that, in this paragraph, we do not get Cartan's result in its full strength (beside the exact values of the constants) : we control the measure of the set, but cannot say that it is contained in a reunion of disks, with a control upon the sum of radii (for instance, a line of length $l$ has measure 0, but if we cover it by disks, the sum of radii will be at least $l/2$). This question will be solved in the next paragraph.

A result of a similar nature, with much stronger assumptions, can be found in Levin [7], th. 11 p. 21 :

**Theorem.** – *If $f$ is holomorphic in the disk $\{|z| \le 2e\}$ and satisfies $|f(0)| = 1$, then, for every $\eta$, $0 < \eta < 3e/2$, outside a reunion of disks with sum of radii $< 4\eta$, one has, for every $z$, $|z| \le 1$,*

$$
|f(z)| \ \ge\ K^{-(2+\log(3e/2\eta))}\ ,
$$

*where*

$$
K = \max\{|f(z)|\ ;\ |z| = 2e\}.
$$

IV. The shape of the set where an $H^2$ function is small.

In this section, we are going to extend Cartan's result in its full strength, to $H^2$ functions with concentration at low degrees. Namely, we prove :

**Theorem 4.1.** – *Let $d$, $0 < d \leq 1$ and $k \in \mathbb{N}$. For any function $f$ in $H^2$, with concentration $d$ at degree $k$, the set $\{z \in D\ ;\ |f(z)| < \varepsilon\, \|f\|_2\}$ can be covered by a countable union of disks $D_i$, with radius $r_i$, satisfying $\sum r_i \leq \phi_{d,k}(\varepsilon)$, with :*

$$\phi_{d,k}(\varepsilon) \;=\; C\,\frac{\log\log 1/\sqrt{\varepsilon}}{\log 1/\sqrt{\varepsilon}}\,\log\frac{2^k\sqrt{k+1}}{d}\;,$$

*and $C = 3\cdot 2^{10}\,\pi^2$.*

In an earlier version of this paper, the conclusion was established only for the sets $\{z \in D(r)\ ;\ |f(z)| < \varepsilon\,\|f\|_2\}$ (which is much easier to obtain). We are greatly indebted to Tom Wolff to drawing our attention to Carleson's theorems 3.1 and 3.2, section VIII in Garnett [6], and explaining the connection with the present situation. So the proof we present now is essentially T. Wolff's adaptation of these theorems (valid for the upper half-plane). We have also been informed by A. Ancona that a proof can be given using some arguments from Potential Theory.

**Proof**. – Let $f$ be an $H^2$ function with concentration $d$ at degree $k$ and $\|f\|_2 = 1$. We write the canonical factorization $f = B\cdot S\cdot F$.

First, we reduce the problem to an $H^\infty$ function. We write the outer part :

$$F(z) \;=\; \exp\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log h(\theta)\,\frac{d\theta}{2\pi},$$

and we define $h_1 = h$ if $h \geq 1$, $h_1 = 1$ otherwise, and $h_2 = 1$ if $h \geq 1$, $h_2 = h$ otherwise, so $h = h_1\cdot h_2$. If $F_1$ and $F_2$ are the corresponding outer functions, we get $F = F_1\cdot F_2$.

Moreover,

$$\begin{aligned}\int h_1^2\,\frac{d\theta}{2\pi} \;&=\; \int_{\{h_1\leq 1\}} h_1^2\,\frac{d\theta}{2\pi} + \int_{\{h_1>1\}} h_1^2\,\frac{d\theta}{2\pi}\\ &=\; m\{h_1\leq 1\} + \int_{\{h>1\}} h^2\,\frac{d\theta}{2\pi} \;\leq\; 2,\end{aligned}$$

which shows that $\|F_1\|_2 \leq \sqrt{2}$.

Since $|F_1(e^{i\theta})| = h_1(\theta)$ a.e., we have $|F_1(e^{i\theta})| \geq 1$ a.e., and the same way $|F_2(e^{i\theta})| \leq 1$. If we set $f_1 = B\cdot S\cdot F_2$, we have $\|f_1\|_\infty \leq 1$, and

$$\{|f(z)| < \varepsilon\} \;\subset\; \{|f_1(z)| < \varepsilon\}.$$

Finally, Proposition 1.1 gives

$$d \;\leq\; \|s_k(f)\|_2 \;\leq\; \|s_k(f_1)F_1\|_2 \;\leq\; \|s_k(f_1)\|_\infty\|F_1\|_2\;,$$

which gives

$$\|s_k(f_1)\|_\infty \;\geq\; \frac{d}{\sqrt{2}}\;,$$

and

$$\|s_k(f_1)\|_2 \;\geq\; \frac{d}{\sqrt{2(k+1)}}\;. \qquad (4.1)$$

So the problem reduces to an $H^\infty$ function, $f_1$, satisfying $\|f_1\|_\infty \le 1$ and (4.1).

For the Blaschke factor $B$, Corollary 1.3 gives

$$\|s_k(B)\|_2 \ \geq \ \frac{d}{\sqrt{k+1}} \ , \tag{4.2}$$

and the same for the inner part $B \cdot S$ :

$$\|s_k(B \cdot S)\|_2 \ \geq \ \frac{d}{\sqrt{k+1}} \tag{4.3}$$

and for $F_2$, we deduce from (4.1), using Proposition 1.1 :

$$\|s_k(F_2)\|_2 \ \geq \ \frac{d}{\sqrt{2(k+1)}} \ . \tag{4.4}$$

The set $\{|f| < \varepsilon\}$ is contained in the union

$$\{|B \cdot S| < \sqrt{\varepsilon}\} \cup \{|F_2| < \sqrt{\varepsilon}\},$$

so we will prove separately Theorem 1.1 for $B \cdot S$ and for $F_2$. We start with $B$, and consider first the case where it has concentration $d$ at degree 0.

**Theorem 4.2.** – *For any Blaschke product $B(z)$ satisfying $|B(0)| \ge d$, for every $\varepsilon > 0$, the set $\{z \in D \ ; \ |B(z)| < \varepsilon\}$ can be covered by a countable union of disks, with sum of radii $\le \phi_d(\varepsilon)$, with*

$$\phi_d(\varepsilon) \ = \ c \, \frac{\log\log 1/\varepsilon}{\log 1/\varepsilon} \, \log 1/d \ ,$$

*and $c = 2^{10}\,\pi^2$.*

**Proof** of Theorem 4.2. – We write

$$|B(z)| \ = \ \prod_1^\infty \frac{|z-\alpha_j|}{|1-\bar\alpha_j z|} \ .$$

We fix $\eta > 0$, and we define :

$$E_\eta \ = \ \{z \in D \ ; \ |z-\alpha_j| \ > \ \eta\,(1-|\alpha_j|) \quad \forall j\}.$$

Then we have

**Proposition 4.3.** – *If $z \in E_\eta$,*

$$\log \frac{1}{|B(z)|} \ \le \ A_\eta\,(1-|z|^2) \sum_j \frac{1-|\alpha_j|^2}{|1-\bar\alpha_j z|^2} \ ,$$

*with*

$$A_\eta \ = \ \frac{1}{2}(1+\log\frac{3}{\eta}).$$

**Proof**. – We first need a lemma.

**Lemma 4.4.** – *The condition* $|z-\alpha_j| \geq \eta(1-|\alpha_j|)$ *implies*

$$|z-\alpha_j| \;\geq\; \frac{\eta}{3}|1-\bar{\alpha}_j z|.$$

**Proof** of Lemma 4.4. – We have

$$\begin{aligned} |1-\bar{\alpha}_j z| &\leq |1-\bar{\alpha}_j\alpha_j| + |\bar{\alpha}_j\alpha_j - \bar{\alpha}_j z| \\ &\leq 1-|\alpha_j|^2 + |\alpha_j||\alpha_j - z| \\ &\leq \big(\frac{2}{\eta}+1\big)|z-\alpha_j|, \end{aligned}$$

from which the Lemma follows obviously.

Now, we observe that, if $0 < a^2 \leq t < 1$,

$$\begin{aligned} \frac{-\log t}{1-t} &\leq \frac{-2\log a}{1-a^2} \\ &\leq 1 + 2\log\frac{1}{a} \end{aligned}$$

and thus

$$-\log t \;\leq\; (1+2\log\frac{1}{a})(1-t). \tag{4.4}$$

Using this inequality, we deduce from Lemma 4.4 that, for $z \in E_\eta$,

$$\begin{aligned} -\log|B(z)|^2 &= -\sum \log\frac{|z-\alpha_j|^2}{|1-\bar{\alpha}_j z|^2} \\ &\leq (1+2\log\frac{3}{\eta})\sum\big(1-\frac{|z-\alpha_j|^2}{|1-\bar{\alpha}_j z|^2}\big). \end{aligned}$$

But

$$1-\frac{|z-\alpha_j|^2}{|1-\bar{\alpha}_j z|^2} \;=\; \frac{(1-|z|^2)(1-|\alpha_j|^2)}{|1-\bar{\alpha}_j z|^2}\ ,$$

which gives the Proposition.

From this estimate, we see that no point $z$, with $|z| \leq 1/2$ and $|B(z)| < \varepsilon$, can be in $E_\eta$, if $A_\eta$ is not too large, namely :

$$A_\eta \;\leq\; \frac{1}{6}\frac{\log 1/\varepsilon}{\log 1/d}\ . \tag{4.5}$$

Indeed, assume $|z| \leq 1/2$, $|B(z)| < \varepsilon$, and $z \in E_\eta$. Proposition 4.3 gives

$$\log 1/\varepsilon \;\leq\; A_\eta\,(1-|z|^2)\sum_j \frac{1-|\alpha_j|^2}{|1-\bar{\alpha}_j z|^2}\ .$$

But $|1-\bar{\alpha}_j z| \geq 1-|z|$, and $\dfrac{1-|z|^2}{(1-|z|)^2} = \dfrac{1+|z|}{1-|z|} \leq 3$, so :

$$\log 1/\varepsilon \;\leq\; 3A_\eta\sum 1-|\alpha_j|^2 \;\leq\; 6A_\eta\sum 1-|\alpha_j|.$$

But, since $|B(0)| \geq d$, we have :

$$\sum 1-|\alpha_j| \;\leq\; -\sum\log|\alpha_j| \;\leq\; \log 1/d\ , \tag{4.6}$$

which proves our claim. The estimate (4.5) will be satisfied at the end. From now on, we go on investigating $E_\eta$, keeping in mind $|z| \geq 1/2$.

We now consider a measure on the unit circle :

$$\mu \;=\; \sum_j (1-|\alpha_j|^2)\delta_{\alpha_j/|\alpha_j|} \;,$$

where $\delta_w$ is the Dirac measure at the point $w$. We also define the function :

$$\begin{aligned} P_\mu(z) \;&=\; \int_{\mathcal{C}} \frac{1-|z|^2}{|1-\bar{z}\zeta|^2}\, d\mu(\zeta) \\ &=\; \int P_r(\theta - t) d\mu(t) \;, \quad \text{if } z = re^{i\theta} \;. \end{aligned}$$

Then we have :

**Proposition 4.5.** – *If* $z \in E_\eta$,

$$\log \frac{1}{|B(z)|} \;\leq\; 4A_\eta\, P_\mu(z).$$

**Proof** of Proposition 4.5. – From the definition of $\mu$ follows :

$$P_\mu(z) \;=\; (1-|z|^2)\sum_j \frac{1-|\alpha_j|^2}{|1-\bar{z}\frac{\alpha_j}{|\alpha_j|}|^2} \;.$$

But

$$\begin{aligned} |1-\bar{z}\frac{\alpha_j}{|\alpha_j|}| \;&=\; |1-\frac{\bar{\alpha}_j}{|\alpha_j|}z| \\ &\leq\; |1-\bar{\alpha}_j z| + \left|\bar{\alpha}_j z - z\frac{\bar{\alpha}_j}{|\alpha_j|}\right| \\ &\leq\; |1-\bar{\alpha}_j z| + |z|(1-|\alpha_j|) \\ &\leq\; 2|1-\bar{\alpha}_j z|. \end{aligned}$$

This gives

$$P_\mu(z) \;\geq\; \frac{1}{4}(1-|z|^2)\sum_j \frac{1-|\alpha_j|^2}{|1-\bar{\alpha}_j z|^2} \;,$$

and the result follows from Proposition 4.3.

We now define a subset $\Gamma_0$ of the unit disk, by the polar equation of its boundary :
– if $|\theta| \leq \pi/3,\ |\theta| \;=\; \pi\,\dfrac{1-r}{1+r}$,
– if $|\theta| > \pi/3,\ r = 1/2$.

Let $A = 1$ and $M = e^{i\theta}$ in the complex plane. Let $l(AM)$ be the length of the arc $AM$. Let $P$ be the intersection of $OM$ with $\partial\Gamma_0$, boundary of $\Gamma_0$.

**Lemma 4.6.** – *The circle centered at* $M$, *with radius* $l(AM)$, *contains the sector* $APM$.

**Proof**. – Assume $\theta > 0$. The length of the arc $AM$ is $\theta$. We first prove that the circle contains $P$. The length of $MP$ is $1-r$.

– if $\theta \leq \pi/3$, $1-r \ = \ \theta\,\dfrac{1+r}{\pi} \ \leq \ 2\theta/\pi$,

– if $\theta > \pi/3$, $1-r = 1/2 < 2\theta/\pi$.

Assume first $\theta < \pi/3$. Let $M'$ be a point on the unit circle, between $A$ and $M$, and $P'$ the corresponding point on $\partial\Gamma_0$. The circle centered at $M'$ is contained in the circle centered at $M$, since $MM' + \theta' \ < \ \theta - \theta' + \theta' \ = \ \theta$. So $P'$ belongs to the circle centered at $M$. This proves the Lemma when $\theta < \pi/3$.

If $\theta > \pi/3$, the circle contains also the point $A' = 1/2$, thus the segments $AA'$ and $A'P$, and so it contains the curve, and the Lemma is proved.

For $\varepsilon > 0$, $\eta > 0$, we define

$$Y_{\varepsilon,\eta} \ = \ \{z \in E_\eta \ ; \ |B(z)| < \varepsilon\}.$$

For any $x \in \mathcal{C}$, we define $\Gamma_x$ as the set $\Gamma_0$ after a rotation of angle $\arg(x)$. This set intersects $\mathcal{C}$ at the point $x$ only. We set :

$$Y^*_{\varepsilon,\eta} \ = \ \{x \in \mathcal{C} \ ; \ \Gamma_x \text{ intersects } Y_{\varepsilon,\eta}\}.$$

This is an open set, so it is a countable union of open intervals $I_k$ on the unit circle. Let $l(I_k)$ be the length of $I_k$ (on the circle), and $M_k$ be le middle of $I_k$ (also on the circle). Then :

**Lemma 4.7.** – *The reunion of the disks* $D_k$ *centered at* $M_k$ *and with radius* $l(I_k)/2$ *covers* $Y_{\varepsilon,\eta}$.

**Proof** of Lemma 4.7. – Let $I$ be any of the $I_k$, and let $E$, $E'$ be its endpoints, $\Gamma$, $\Gamma'$ the corresponding sets. The curves $\partial\Gamma$, $\partial\Gamma'$ intersect at a point $P$. Let $M \in \mathcal{C}$ be the middle of $EE'$ ; it is on the segment $OM$. By Lemma 4.6, the circle centered at $M$, with radius $l(I)/2$, contains the sector between $\Gamma$ and $\Gamma'$. But by definition, the sets $\Gamma$, $\Gamma'$ do not intersect $Y_{\varepsilon,\eta}$, so $Y_{\varepsilon,\eta}$ is contained in the union of these sectors. This proves the Lemma.

We denote by $r(D_k)$ the radius of the disk $D_k$. We have, by Lemma 4.7 :

$$\sum_k r(D_k) \ = \ \frac{1}{2}\sum l(I_k) \ = \ \pi\, m(Y^*_{\varepsilon,\eta}), \tag{4.5}$$

where $m$ is, as before, the normalized Lebesgue measure on $\mathcal{C}$ : that is, $m(\mathcal{C}) = 1$.

We now define, for $x \in \mathcal{C}$,

$$P^*_\mu(x) \ = \ \sup_{\Gamma_x \ni z} P_\mu(z),$$

which is Hardy-Littlewood's maximal function associated to the cones $\Gamma_x$ (see Garnett [6], p. 28). We also put

$$N_\mu(x) \ = \ \sup_{I \ni x} \frac{\mu(I)}{m(I)} \ ,$$

the maximal function associated to the measure $\mu$. Then :

**Lemma 4.8.** – *For every* $x \in \mathcal{C}$, $P^*_\mu(x) \leq 2N_\mu(x)$.

**Proof** of Lemma 4.8. – We follow Garnett, th. 4.2, chap. I (his proof is given for the upper half-plane, and the exact value of the constant – which we need here– is not specified).

It's enough to prove the result for $x = 0$, that is

$$\sup_{z\in\Gamma_0} P_\mu(z) \;\le\; 2\sup_{I\ni 0}\frac{1}{m(I)}\,\mu(I),$$

with

$$\mu(I) \;=\; \sum_{\frac{\alpha_i}{|\alpha_i|}\in I} 1-|\alpha_i|^2\ .$$

First we take $z$ real, $z = r$. Then $P_r(t)$ is an even function of $t$, decreasing for $t \ge 0$. So we can find an increasing sequence of positive functions, $h_n(t)$, each of them being a sum of characteristic functions ($1_E$ is the function which takes the value 1 on the set $E$, 0 outside) :

$$h_n \;=\; \sum_l a_l^{(n)} 1_{J_l}\ ,\qquad a_l^{(n)} \ge 0\ ,\qquad J_l = [-x_l, x_l],$$

with $h_n(t) \le P_r(t)$, for every $t$ ; the sequence $h_n(t)$ is increasing and $\lim_n h_n(t) = P_r(t)$. So

$$\int_0^{2\pi} h_n(t)\frac{dt}{2\pi} \;=\; \sum_l a_l^{(n)} m(J_l) \;\le\; 1.$$

For any $n$,

$$\begin{aligned}
\int h_n(t)d\mu(t) \;&=\; \sum_l a_l^{(n)} \int 1_{J_l} d\mu(t)\\
&=\; \sum_l a_l^{(n)} \sum_{\frac{\alpha_i}{|\alpha_i|}\in J_l} 1-|\alpha_i|^2\\
&=\; \sum_l a_l^{(n)} m(J_l)\frac{1}{m(J_l)} \sum_{\frac{\alpha_i}{|\alpha_i|}\in J_l} 1-|\alpha_i|^2\\
&\le\; N_\mu(0),
\end{aligned}$$

and $\int P_r(t)d\mu(t) \le N_\mu(0)$ follows if we let $n \to \infty$.

Let now $z = re^{i\theta} \in \Gamma_0$, with $r \ge 1/2$, and $0 \le \theta < \pi$. For $s > 0$, we define :

$$\Psi(s) \;=\; \sup\{P_r(\theta - t)\ ;\ |t| > s\}.$$

So $\Psi(s) = P_r(\theta - s)$ if $s \ge \theta$, and $\Psi(s) = P_r(0)$ if $s \le \theta$. We also put $\Psi(-s) = \Psi(s)$. We have a positive, even function, decreasing for $s \ge 0$, which majorizes $P_r(\theta - t)$. Moreover :

$$\int_{-\pi}^{\pi} \Psi(s)\frac{ds}{2\pi} \;\le\; \frac{\theta}{\pi}\,P_r(0) + 1 \;=\; \frac{\theta}{\pi}\frac{1+r}{1-r} + 1.$$

The definition of $\Gamma_0$ gives $\dfrac{\theta}{\pi}\dfrac{1+r}{1-r} = 1$, and so

$$\int_{-\pi}^{\pi} \Psi(s)\frac{ds}{2\pi} \;\le\; 2.$$

Applying the above reasoning to $\Psi(s)$ instead of $P_r(s)$, we obtain the Lemma.

If $x \in Y^*_{\varepsilon,\eta}$, there exists a point $z \in E_\eta$, with $\Gamma_x \ni z$, and $|B(z)| < \varepsilon$. By Proposition 4.5,

$$P^*_\mu(x) \ \geq \ \frac{1}{4A_\eta} \ \log\frac{1}{\varepsilon} \ ,$$

which implies

$$N_\mu(x) \ \geq \ \frac{1}{8A_\eta} \ \log\frac{1}{\varepsilon} \ .$$

Set $a_\eta = 1/(8A_\eta)$. We have

$$Y^*_{\varepsilon,\eta} \ \subset \ \{x \in \mathcal{C} \ ; \ N_\mu(x) \ \geq \ a_\eta \log\frac{1}{\varepsilon}\},$$

and thus

$$m(Y^*_{\varepsilon,\eta}) \ \leq \ m\big(\{x \in \mathcal{C} \ ; \ N_\mu(x) \ \geq \ a_\eta \log\frac{1}{\varepsilon}\}\big).$$

By the weak-type inequalities for the maximal function (Garnett, chap. I, th. 4.3 and th. 5.1) :

$$\begin{aligned} m\big(\{x \in \mathcal{C} \ ; \ N_\mu(x) \ \geq \ a_\eta \log\frac{1}{\varepsilon}\}\big) \ &\leq \ \frac{2}{a_\eta \log 1/\varepsilon} \int d\mu \\ &\leq \ \frac{2}{a_\eta \log 1/\varepsilon} \sum 1 - |\alpha_j|^2. \end{aligned}$$

But since $|B(0)| \geq d$, we have

$$-\log d \ \geq \ -\sum \log|\alpha_j| \ \geq \ \sum 1 - |\alpha_j|,$$

which implies

$$\sum 1 - |\alpha_j|^2 \ \leq \ 2\log 1/d.$$

Finally we obtain :

$$\sum r(D_k) \ \leq \ 32\pi\, A_\eta \, \frac{\log 1/d}{\log 1/\varepsilon}. \tag{4.6}$$

This is the required estimate for the part of $\{|B(z)| < \varepsilon\}$ which is in $E_\eta$. For the part which is not in $E_\eta$, we use trivial estimates. We let

$$\Delta_j \ = \ \{z \in D \ ; \ |z - \alpha_j| \ \leq \ \eta\,(1 - |\alpha_j|)\},$$

and

$$R_j \ = \ \{z \in \Delta_j \ ; \ |B(z)| < \varepsilon\}.$$

Then

$$\sum r(\Delta_j) \ \leq \ \eta \sum 1 - |\alpha_j| \ \leq \ \eta \, \log 1/d.$$

We now prove Theorem 4.2. With $c = 32\pi$, we see that, for every $\eta > 0$, the set $|B(z)| < \delta$ can be covered by the union of the disks $D_k$ and the disks $\Delta_j$, with

$$\sum r(D_k) + \sum r(\Delta_j) \ \leq \ c\,(\frac{1}{2} + \log\frac{3}{\eta})\frac{\log 1/d}{\log 1/\varepsilon} + \eta \, \log 1/d \ .$$

Set

$$f(\eta) \ = \ c\,(\frac{1}{2} + \log\frac{3}{\eta})\frac{1}{\log 1/\varepsilon} + \eta \ .$$

It takes its minimum at $\eta = c/(\log 1/\varepsilon)$, and using this value for $\eta$, we find

$$\sum r(D_k) + \sum r(\Delta_j) \ \leq \ c^2 \, \frac{\log\log 1/\varepsilon}{\log 1/\varepsilon} \, \log 1/d \ ,$$

and we observe that if this number is $\leq 1$, condition (4.5) will hold.

We now prove Theorem 4.2 for the Blaschke term, when the concentration is at degree $k$ (and not only at degree 0). This will be done by means of a Möbius transformation. For $w \in D$, $z_0 \in D$, we set

$$\tilde{f}(w) \;=\; f\left(\frac{w - z_0}{1 - \bar{z}_0 w}\right).$$

**Lemma 4.9.** – *If $f \in H^\infty$ has concentration $d$ at degree $k$, for every $R$, $0 < R < 1$, we can find $z_0$, with $|z_0| = R$, such that the function $\tilde{f}$ has concentration*

$$d' \;=\; dR^k$$

*at degree 0.*

**Proof** of Lemma 4.9. – We assume $\|f\|_\infty = 1$, so $\|\tilde{f}\|_\infty = 1$. We have, if $f = \sum a_j z^j$,

$$\tilde{f}(w) \;=\; \sum a_j \left(\frac{w - z_0}{1 - \bar{z}_0 w}\right)^j ,$$

so, if we set $-z_0 = Re^{i\theta}$,

$$\tilde{f}(0) \;=\; \sum a_j R^j e^{ij\theta} .$$

But

$$\begin{aligned}\left(\int_0^{2\pi} \Big|\sum_0^\infty a_j R^j e^{ij\theta}\Big|^2 \frac{d\theta}{2\pi}\right)^{1/2} \;&=\; \left(\sum_0^\infty |a_j|^2 R^{2j}\right)^{1/2} \\ &\geq\; \left(\sum_0^k |a_j|^2 R^{2j}\right)^{1/2} \\ &\geq\; dR^k.\end{aligned}$$

So there is a $\theta$ such that

$$\Big|\sum_0^\infty a_j R^j e^{ij\theta}\Big| \;\geq\; dR^k,$$

which proves our claim (a similar argument was already used in the proof of Lemma 3.2, in the $H^2$ context).

If $f$ is a Blaschke product $B$ with zeros $\alpha_i$, the function $\tilde{f}$ will also be a Blaschke product $B'$, with zeros $\alpha'_i$. By Theorem 4.2, we can cover the set $|B'(w)| < \varepsilon$ by disks with centers $w_i$. Take $z_0$, with $|z_0| = 1/2$, given by Lemma 4.9, and define $z_i$ by

$$w_i \;=\; \frac{z_i - z_0}{1 - \bar{z}_0 z_i} .$$

Then the disk $\{|w - w_i| < r_i\}$ is also the set

$$\left|\frac{z - z_0}{1 - \bar{z}_0 z} - \frac{z_i - z_0}{1 - \bar{z}_0 z_i}\right| \;<\; r_i . \tag{4.7}$$

But

$$(z - z_0)(1 - \bar{z}_0 z_i) - (z_i - z_0)(1 - \bar{z}_0 z) \;=\; (1 - |z_0|^2)(z - z_i),$$

and since

$$|1 - \bar{z}_0 z||1 - \bar{z}_0 z_i| \;\leq\; (1 + |z_0|)^2,$$

the set (4.7) is contained in the disk

$$|z - z_i| \;\leq\; \frac{1 + |z_0|}{1 - |z_0|} r_i \;=\; 3r_i ,$$

and we obtain :

**Theorem 4.10.** – *Let $B$ be a Blaschke product with concentration $d$ at degree $k$. For every $\varepsilon > 0$, the set $\{z \in D\ ;\ |B(z)| < \varepsilon\}$ can be covered by a countable union of disks, with sum of radii $\leq \phi_{d,k}(\varepsilon)$, with*

$$\phi_{d,k}(\varepsilon) \;=\; 3 \cdot 2^{10} \pi^2 \, \frac{\log\log 1/\varepsilon}{\log 1/\varepsilon} \log 2^k/d \;.$$

Since Blaschke products are uniformly dense in the set of inner functions (see for instance Garnett [6], Corollary 6.5, p. 80), th. 4.10 holds with no change for any inner function.

We now turn to the study of the outer functions. It uses the same ingredients, but is much simpler, so the proofs will be left to the reader.

**Proposition 4.11.** – *Let $F$ be an outer function, with $\|F\|_\infty \leq 1$ and $|F(0)| \geq d$. Then, for every $\varepsilon > 0$, the set $\{z \in D\ ;\ |F(z)| < \varepsilon\}$ can be covered by a reunion of disks, with sum of radii $\phi'_d(\varepsilon)$,*

$$\phi'_d(\varepsilon) \;=\; c \, \frac{\log 1/d}{\log 1/\varepsilon} \;,$$

*with $c = 4\pi$.*

**Proof**. – We write

$$F(z) \;=\; \exp \int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} u(\theta) \, \frac{d\theta}{2\pi},$$

and $u(\theta) = \log h(\theta)$, $h(\theta) = |F(e^{i\theta})|$ a.e.

Set $v = -u$. Then

$$v \geq 0 \;, \qquad \int v(\theta) \frac{d\theta}{2\pi} \;\leq\; \log 1/d \;. \tag{4.8}$$

For every $\varepsilon > 0$, we define $Y_\varepsilon = \{z \in D\ ;\ |F(z)| < \varepsilon\}$. Then

$$Y_\varepsilon \;=\; \{z \in D\ ;\ \big|\log|F(z)|\big| \;>\; \log 1/\varepsilon\}.$$

Set

$$Y_\varepsilon^* \;=\; \{x \in \mathcal{C}\ ;\ \Gamma_x \text{ intersects } Y_\varepsilon\},$$

where $\Gamma_x$ is defined as above. Then, as in Lemma 4.7, $Y_\varepsilon$ is covered by a reunion of disks $D_k$, with $\sum r(D_k) \leq \pi\, m(Y_\varepsilon^*)$.

Set now

$$G^*(x) \;=\; \sup_{z \in \Gamma_x} \big|\log|F(z)|\big|,$$

then $Y_\varepsilon^* \subset \{x\ ;\ G^*(x) > \log 1/\varepsilon\}$.

Let $v^*$ be the maximal function associated with $v$ :

$$v^*(x) \;=\; \sup_{I \ni x} \frac{1}{m(I)} \int_I \big|\log|F(e^{i\theta})|\big| \, \frac{d\theta}{2\pi} \;.$$

Then, as in Lemma 4.8, for every $x \in \mathcal{C}$,

$$G^*(x) \;\leq\; 2v^*(x).$$

So

$$Y_\varepsilon^* \;\subset\; \{x\ ;\ v^*(x) > \frac{1}{2}\log 1/\varepsilon\}.$$

By the weak-$L_1$ inequalities for the maximal function $v^*$ :

$$\begin{aligned} m\{x\ ;\ v^*(x) > \frac{1}{2}\log 1/\varepsilon\} \;&\leq\; \frac{4}{\log 1/\varepsilon} \int v(t) \, \frac{dt}{2\pi} \\ &\leq 4 \, \frac{\log 1/d}{\log 1/\varepsilon} \;, \quad \text{by (4.8)}, \end{aligned}$$

and the Proposition is proved.

**Remark**. – One can also prove this result using an analogue of Th. 3.1, chap. VIII in Garnett, but the present approach gives a better estimate.

If we now consider outer functions with concentration $d$ at degree $k$, using Lemma 4.9, we get :

**Theorem 4.12.** – *Let $F$ be an outer function with $\|F\|_\infty \leq 1$ and $|s_k(F)|_2 \geq d$. Then, for every $\varepsilon > 0$, the set $\{z \in D \ ; \ |F(z)| < \varepsilon\}$ can be covered by a reunion of disks, with sum of radii $\phi'_{d,k}(\varepsilon)$,*

$$\phi'_{d,k}(\varepsilon) \ = \ 12\pi \, \frac{\log(2^k/d)}{\log 1/\varepsilon}.$$

Let's now prove Theorem 4.1. Assume that $f$ has concentration $d$ at degree $k$. So the Blaschke factor satisfies $|s_k(B)|_2 \geq d/\sqrt{k+1}$ by (4.2), and by (4.9), the set $\{|B(z)| < \sqrt{\varepsilon}\}$ can be covered by disks, with

$$\sum r_i \leq 3 \cdot 2^{10} \pi^2 \, \frac{\log\log 1/\sqrt{\varepsilon}}{\log 1/\sqrt{\varepsilon}} \, \log \frac{2^k\sqrt{k+1}}{d} \ .$$

The same is true for the inner factor.

The outer factor $F_2$ satisfies $\|s_k(F_2)\|_2 \geq d/\sqrt{2(k+1)}$ by (4.3), and by Theorem 4.12, the set $\{|F_2(z)| < \sqrt{\varepsilon}\}$ can be covered by disks, with

$$\sum r_i \leq 12\pi \, \frac{1}{\log 1/\sqrt{\varepsilon}} \, \log \frac{2^k\sqrt{2(k+1)}}{d} \ .$$

The set $\{|f(z)| < \varepsilon\}$ is contained in the union of both sets, and the result follows.

The estimate we have obtained is numerically worse than that of § 3. Besides the numerical constant in th. 4.1, which is too big, the order of magnitude is close to best possible. Indeed, we now show that there are examples for which $\sum r_i \sim \dfrac{1}{\log 1/\varepsilon}$.

Let's take a sequence of $(\alpha_n)$ which are real, positive, strictly increasing to 1. We consider

$$B(z) \ = \ \prod_1^\infty \frac{\alpha_n - z}{1 - \alpha_n z} \ .$$

Then we have $|B(z)| < \varepsilon$ if and only if

$$\sum \log \alpha_n + \sum \log \left| 1 - \frac{z}{\alpha_n} \frac{1-\alpha_n^2}{1-\alpha_n z} \right| \ < \ \log \varepsilon.$$

So the set

$$A_\varepsilon \ = \ \{z \ ; \ \sum \log \left| 1 - \frac{z}{\alpha_n} \frac{1-\alpha_n^2}{1-\alpha_n z} \right| \ < \ \log \varepsilon\}$$

is contained in $\{|B(z)| < \varepsilon\}$.

We take $z \in \mathbb{R}$. Then $z \in A_\varepsilon$ as soon as

$$\sum \frac{z}{\alpha_n} \frac{1-\alpha_n^2}{1-\alpha_n z} \ > \ \log 1/\varepsilon \ . \tag{4.9}$$

If $\alpha_n \leq z$, $\dfrac{1-\alpha_n^2}{1-\alpha_n z} \ \geq \ 1$, and so, for every $z \in [0,1]$,

$$\sum \frac{z}{\alpha_n} \frac{1-\alpha_n^2}{1-\alpha_n z} \ \geq \ \text{card}\ \{\alpha_n \ ; \ \alpha_n \leq z\}.$$

Let $f(x)$ be a positive, decreasing, integrable function on $[1,+\infty[$. Set $\alpha_n = 1 - f(n)$. The Blaschke product $B$ will have concentration $1/2$ at degree 0 if

$$\int_1^\infty f(t)dt \ < \ \log 2. \tag{4.10}$$

Then

$$\text{card}\,\{\alpha_n \ ; \ \alpha_n \leq z\} \ = \ f^{-1}(1-z),$$

and (4.9) will hold if

$$f^{-1}(1-z) \ \geq \ \log 1/\varepsilon,$$

or

$$z \ \geq \ 1 - f(\log 1/\varepsilon).$$

So $|B(z)| < \varepsilon$ on the segment $[1 - f(\log 1/\varepsilon), 1]$ which has length $f(\log 1/\varepsilon)$.

For the function $f$, one can choose $f(x) = 1/x^\alpha$ $(\alpha > 1)$, or $\dfrac{1}{x(\log x)^\alpha}$, with proper truncation to ensure (4.10), that is $f(x) = f(x_0)$ if $x \leq x_0$. This way, one gets examples in which the length of the segment where $|B(z)| < \varepsilon$ is of the order $\dfrac{1}{(\log 1/\varepsilon)^\alpha}$, or $\dfrac{1}{\log 1/\varepsilon(\log 1/\varepsilon)^\alpha}$.

To cover such a segment by a reunion of disks requires of course that the sum of radii should be of the same order of magnitude.

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