Extremal products in Bombieri's norm

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Abstract. – Let P be a homogeneous polynomial in any number of variables, of any degree, with complex coefficients. We give a complete description of the Q's, of all degrees, such that $[Q] = 1$ and $[PQ]$ is either maximal or minimal, where $[\ .\ ]$ is Bombieri's norm. For this, we introduce a matrix, built with partial derivatives of P ; the quantity $[PQ]$ appears as the l_2 norm of the product of this matrix by a vector column associated with Q : thus products of polynomials are replaced by the product of a matrix by a vector, a much simpler feature. The extreme values of $[PQ]$ are eigenvalues of this matrix. As an application, we give an exact estimate of the norm of the differential operator $P(D)$.

January 1995.

0. Introduction

Let $P(x_1, \ldots, x_N) = \sum_{|\alpha|=m} a_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ be a homogeneous polynomial of degree m in N variables ; we write $\alpha = (\alpha_1, \ldots, \alpha_N)$, and $|\alpha| = \alpha_1 + \cdots + \alpha_N$.

A non-homogeneous polynomial can be transformed into a homogeneous one by adding one more variable ; so, for instance, an ordinary one-variable polynomial $P(z) = \sum_0^m a_j z^j$ becomes homogeneous in two variables : $P(x_1, x_2) = \sum_{j=0}^m a_j x_1^j x_2^{m-j}$. We will use freely this identification in the sequel, so what we say applies a fortiori to one-variable ordinary polynomials.

Let $\| \, . \, \|$ be a norm on the space of polynomials, and let P be a given polynomial. The problem : To find

$$I(P) \;=\; \inf\{\|PQ\| \; ; \; \|Q\| \;=\; 1\} \tag{1}$$

has been extensively studied over the years, as well as the problem

$$S(P) \;=\; \sup\{\|PQ\| \; ; \; \|Q\| \;=\; 1\}. \tag{2}$$

Indeed, they correspond to a very natural question : P represents a fixed single system, and we want to act on it ; only multiplicative actions are allowed. We want to find the best action, that is the Q which optimizes some norm, depending on the problem (for instance : the lowest energy).

Mathematically speaking, such problems heavily depend on the norm which is taken. For instance, if the norm is the l_1-norm in one variable ($\|P\| = \sum |a_j|$), problem (1) is connected with spectral synthesis (see J.-P. Kahane [13]).

Both problems are trivial for the L_∞ norm :

$$\|P\|_\infty \;=\; \sup\{|P(e^{i\theta_1}, \ldots, e^{i\theta_N})| \; ; \; \theta_1, \ldots, \theta_N \in \Pi\},$$

and, for other norms, they may be trivial for special P's. But except in these cases, none of them has been solved, for any norm.

They have finite dimensional versions :

$$I_n(P) \;=\; \inf\{\|PQ\| \; ; \; \|Q\| \;=\; 1, \; \deg Q = n\}, \tag{3}$$

$$S_n(P) \;=\; \sup\{\|PQ\| \; ; \; \|Q\| \;=\; 1, \; \deg Q = n\}. \tag{4}$$

For one-variable polynomials, the problems are easier with another normalization, namely $Q(0) = 1$, and this is why they have been mostly studied with this normalization. For instance,

$$\inf\{\|PQ\|_2 \; ; \; Q(0) = 1\} \;=\; M(P) \tag{5}$$

where $\|P\|_2 = (\sum |a_j|^2)^{1/2}$, $M(P) = \exp \int_0^{2\pi} \log |P(e^{i\theta})| \frac{d\theta}{2\pi}$ is Mahler's measure. The result (5) is Szegö's Theorem (see for instance [2], p. 195).

The same way, Bombieri and Vaaler [8] studied $\inf |PQ|_\infty$, where $|P|_\infty = \max |a_j|$ is the *height* of the polynomial, in the case where both P and Q have integer coefficients (which implies $|Q(0)| \geq 1$) ; they showed that this infimum is again $M(P)$.

Among many other pieces of work, one should quote Donaldson and Rahman [12], for the $\|.\|_2$ norm, and Q of degree 1.

Here, we will solve completely problems (3) and (4) for Bombieri's norm $[P]$. The problem of the size of factors of polynomials under Bombieri's norm, which is related to the present study, was already considered by Beauzamy-Bombieri-Enflo-Montgomery [4] and by Boyd [9], [10], [11]. We recall that, with our present notation,

$$[P] = \left(\sum_{|\alpha|=m} |a_\alpha|^2 \frac{\alpha!}{m!} \right)^{1/2},$$

where $\alpha! = \alpha_1! \ldots \alpha_N!$.

We also recall the two inequalities

$$\sqrt{\frac{m! n!}{(m+n)!}} \, [P][Q] \leq [PQ] \leq [P][Q] \, ; \qquad (6)$$

the left-hand side being due to Bombieri [4], the right-hand side to the author [3]. Initially, they suggested the problem of finding

$$\inf\{[PQ] \, ; \, [P] = 1, \, [Q] = 1\}$$

and

$$\sup\{[PQ] \, ; \, [P] = 1, \, [Q] = 1\},$$

that is to find the extremal pairs (P, Q). These pairs were determined in Beauzamy-Frot-Millour [6] and [7], Beauzamy [3], Reznick [14]. But here the problem is different : P is fixed, and we can play only with Q. We now turn to a description of our results.

1. Description of the results

The key idea will be to build from P a sequence of polynomials (π_j) (basically, they will just be partial derivatives of P, multiplied by some extra variables), and then build a matrix, made of cross scalar products of these polynomials. Then, for any Q, the quantity $[PQ]^2$ appears to be just $\sum |[\pi_j, Q]|^2$, and optimization problems can be handled very easily on the associated matrix.

Let n be fixed (the degree of Q in (3) or (4)). For any $i = 1, \ldots, N$, we write $P_i = \frac{\partial P}{\partial x_i}$. For any $k = 0, \ldots, \min(m, n)$, any indexes $i_{k+1}, \ldots, i_m$, $j_{k+1}, \ldots, j_n$, between 1 and N, we define the polynomial

$$\Lambda_{\substack{i_{k+1}, \ldots, i_m \\ j_{k+1}, \ldots, j_n}} (P) = x_{j_{k+1}} \cdots x_{j_n} \, P_{i_{k+1}, \ldots, i_m}. \qquad (7)$$

All of them are polynomials of degree n. For a given k, there are $N^{n-k} \cdot N^{m-k} = N^{m+n-2k}$ such polynomials. Altogether, there are

$$\sum_{k=0}^{\min(m,n)} N^{m+n-2k} = N^{m+n} \frac{1 - N^{-2(\min(m,n)+1)}}{1 - N^{-2}} = \nu$$

such polynomials. Let $\pi_1, \ldots, \pi_\nu$ be an enumeration of the polynomials

$$\frac{n!}{\sqrt{(m+n)!(m-k)!(n-k)!}} \, \Lambda_{\substack{i_{k+1}, \ldots, i_m \\ j_{k+1}, \ldots, j_n}} (P) \qquad (8)$$

(the order of the enumeration does not matter).

We observe that all these polynomials are well-defined and very easy to compute : they are just derivatives of P, multiplied by some variables. Then we have :

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Proposition 1. – *Let P be a homogeneous polynomial of degree m, in any number of variables. For any n, any homogeneous polynomial Q of degree n, one has the identity :*

$$[PQ]^2 = \frac{n!^2}{(m+n)!} \sum_{k=0}^{\min(m,n)} \frac{1}{(m-k)!(n-k)!} \times$$

$$\times \sum_{\substack{i_{k+1},\dots,i_m \\ j_{k+1},\dots,j_n}} \left| \left[x_{j_{k+1}} \cdots x_{j_n} \ P_{i_{k+1},\dots,i_m} \, , \, Q \right] \right|^2 ,$$

where $[.\,,\,.]$ denotes the scalar product associated with Bombieri's norm. Using the (π_j)'s, this identity can be written :

$$[PQ]^2 = \sum_{j=1}^{\nu} \left| [\pi_i \, , \, Q] \right|^2 .$$

We can now state the solution to the problems (3) and (4).

Theorem 2. – *Let P be a homogeneous polynomial of degree m, in any number of variables. For a given n, the solution to problem*

$$\inf\{[PQ] \ ; \ Q \text{ homogeneous polynomial of degree } n, \ [Q] = 1\}$$

is the square root of the smallest non-zero eigenvalue of the self-adjoint matrix

$$G = \left([\pi_j \, , \, \pi_{j'}] \right)_{1 \le j, j' \le \nu} .$$

Similarly, the solution to the problem

$$\sup\{[PQ] \ ; \ Q \text{ homogeneous polynomial of degree } n, \ [Q] = 1\}$$

is the square root of the largest eigenvalue of the same matrix.

The matrix G is quite large : roughly speaking it has size N^{m+n}. We can reduce it to a smaller matrix, and get at the same time a description of the Q's for which the infimum [resp : supremum] is attained.

Let $e_1, \dots, e_\mu$ be any orthonormal basis (for the above scalar product) of the space of polynomials in N variables of degree n (recall that $\mu = \binom{N+n-1}{n}$) ; we can take for this basis simply monomials, with proper normalization.

Let $\alpha_{i,j} = [\pi_i \, , \, e_j]$, $i = 1, \dots, \nu$, $j = 1, \dots, \mu$ and $A = (\alpha_{i,j})$. This is a $\nu \times \mu$ matrix. Note that $\mu = \binom{N+n-1}{n}$ does not depend on m.

In the statements which follow, we identify a polynomial Q with the column-vector X of its expansion in the basis $(e_j)_{j \le \mu}$.

We can now state the second description of our results :

Proposition 3. – *Let P, Q be as above ; we have the identity*

$$[PQ]^2 = \|A\overline{X}\|_2^2 ,$$

where $\| \cdot \|_2$ is the usual euclidean norm.

Let H be the matrix A^*A : this is a $\mu \times \mu$ self-adjoint matrix (thus smaller than G). We get :

Theorem 4. – *The solution of problems (3) and (4) is also the square root of the smallest (resp. largest) non-zero eigenvalue of H. The corresponding Q is, after conjugation, the corresponding eigenvector of this matrix.*

We now turn to the proofs of the theorems.

2. The proofs

We assume the reader to be familiar with the contents of Beauzamy-Bombieri-Enflo- Montgomery [4] and Beauzamy-Dégot [5]. We just briefly recall some notation.

Rather than under the reduced form (1), the polynomial P is written under the symmetric form

$$P(x_1,\ldots,x_N) \;=\; \sum_{i_1,\ldots,i_m=1}^{N} c_{i_1,\ldots,i_m}\, x_{i_1}\cdots x_{i_m}, \tag{9}$$

where

$$c_{i_1,\ldots,i_m} \;=\; \frac{1}{m!}\,\frac{\partial^m P}{\partial x_{i_1}\cdots \partial x_{i_m}}\ .$$

Both formulas are linked by the relation

$$c_{i_1,\ldots,i_m} \;=\; \frac{\alpha!}{m!}\, a_\alpha\ ,$$

and Bombieri's norm is then just :

$$[P] \;=\; \left(\sum_{i_1,\ldots,i_m=1}^{N} |c_{i_1,\ldots,i_m}|^2\right)^{1/2}.$$

Now let $u_1,\ldots,u_m$, $v_1,\ldots,v_n$ be integers between 1 and $m+n$. Let

$$U = (u_1,\ldots,u_m)\ ,\qquad V = (v_1,\ldots,v_n).$$

We say that (U,V) forms a *shuffle* of type (m,n) if $u_1 < \cdots < u_m$, $v_1 < \cdots < v_n$, and the set $\{u_1,\ldots,u_m,\ v_1,\ldots,v_n\}$ is a permutation of $\{1,\ldots,m+n\}$. We write $\mathrm{sh}\,(m,n)$ for the set of shuffles of type (m,n). We recall from [4] the formula (see also [14] for another presentation)

$$[PQ]_2^2 \;=\; \left(\frac{m!\,n!}{(m+n)!}\right)^2 \sum_{(U,V),(U',V')} \sum_{l_{U\cap U'}} \sum_{l_{V\cap V'}} \left| \sum_{l_W} c_{l_{U\cap U'},l_W}\, \overline{d}_{l_W,l_{V\cap V'}} \right|^2 \tag{10}$$

where l_U stands for $l_{u_1},\ldots,l_{u_m}$, the same with $l_{U'}$, l_V, $l_{V'}$, l_W ; the c's are the coefficients of P and the d's are the coefficients of Q :

$$Q(x_1,\ldots,x_N) \;=\; \sum_{j_1,\ldots,j_n}^{N} d_{j_1,\ldots,j_n}\, x_{j_1}\cdots x_{j_n} \tag{11}$$

written under symmetric form.

First, we observe that the cardinality of W is determined by that of $U \cap U'$ (or that of $V \cap V'$). Its value is

$$|W| \;=\; m - |U \cap U'| \;=\; n - |V \cap V'|.$$

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Next, we observe (as G. Bacquet already did in [1]) that the value of any sum

$$\sum_{l_W} c_{l_{U \cap U'}, l_W} \; \overline{d}_{l_W, l_{V \cap V'}}$$

depends only on the cardinality of W, since all the c's and all the d's are invariant under permutation of the indexes.

Let ν_k be the number of terms, in (10), for which $|W| = k$ $(k = 0, 1, \ldots, \min(m, n))$. Then easy combinatorics show that

$$\nu_k \;=\; \binom{m+n}{m} \binom{m}{k} \binom{n}{k}. \tag{12}$$

For $k = 0, 1, \ldots, \min(m, n)$, we define

$$\alpha_k \;=\; \alpha_k(P, Q) \;=\; \sum_{\substack{i_{k+1}, \ldots, i_m \\ j_{k+1}, \ldots, j_n}} \Big| \sum_{i_1, \ldots, i_k} c_{i_1, \ldots, i_m} \; \overline{d}_{i_1, \ldots, i_k, j_{k+1}, \ldots, j_n} \Big|^2. \tag{13}$$

Then, using (12) and (13), formula (10) becomes :

$$\begin{aligned}
[PQ]^2 &= \left(\frac{m! n!}{(m+n)!} \right)^2 \sum_{k=0}^{\min(m,n)} \nu_k \; \alpha_k \\
&= \sum_{k=0}^{\min(m,n)} \frac{\binom{m}{k} \binom{n}{k}}{\binom{m+n}{m}} \; \alpha_k \; .
\end{aligned} \tag{14}$$

We will now introduce partial derivatives and consider (13) as a scalar product. Recall from [1], [5], [14] that

$$\begin{aligned}
P_{i_{k+1}, \ldots, i_m}(x_1, \ldots, x_N) &= \frac{m!}{k!} \sum_{i_1, \ldots, i_k} c_{i_1, \ldots, i_m} \; x_{i_1} \cdots x_{i_k} \\
Q_{j_{k+1}, \ldots, j_n}(x_1, \ldots, x_N) &= \frac{n!}{k!} \sum_{j_1, \ldots, j_k} d_{j_1, \ldots, j_n} \; x_{j_1} \cdots x_{j_k}
\end{aligned} \tag{15}$$

which are both polynomials of degree k.

If we set (as was done in [1]) :

$$A_k \;=\; A_k(P, Q) \;=\; \sum_{\substack{i_{k+1}, \ldots, i_m \\ j_{k+1}, \ldots, j_n}} \Big| \big[P_{i_{k+1}, \ldots, i_m}, \; Q_{j_{k+1}, \ldots, j_n} \big] \Big|^2, \tag{16}$$

we find

$$A_k \;=\; \frac{m!^2 n!^2}{k!^4} \; \alpha_k \; ,$$

and (14) becomes :

$$[PQ]^2 \;=\; \frac{1}{(m+n)!} \sum_{k=0}^{\min(m,n)} \frac{k!^2}{(m-k)!(n-k)!} \; A_k \; . \tag{17}$$

Now, in the expression of A_k, we will put all partial derivatives on P. This is done using the transposition formulas from [5] :

$$\Big[P_{i_{k+1}, \ldots, i_m}, \; \frac{\partial^{m-k} Q}{\partial x_{j_{k+1}} \cdots \partial x_{j_n}} \Big] \;=\; \frac{n!}{k!} \big[x_{j_{k+1}} \cdots x_{j_n} P_{i_{k+1}, \ldots, i_m} \, , \; Q \big]. \tag{18}$$

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We obtain the identity

$$[PQ]^2 = \frac{n!^2}{(m+n)!} \sum_{k=0}^{\min(m,n)} \frac{1}{(m-k)!(n-k)!} \times$$
$$\times \sum_{\substack{i_{k+1},\ldots,i_m \\ j_{k+1},\ldots,j_n}} \left| \left[x_{j_{k+1}} \cdots x_{j_n} \, P_{i_{k+1},\ldots,i_m}, \, Q \right] \right|^2. \tag{19}$$

With the notation introduced in section 1, this becomes

$$[PQ]^2 = \sum_{j=1}^{\nu} \left| [\pi_i , Q] \right|^2, \tag{20}$$

which proves Proposition 1.

To find the minimum or the maximum of (20) when $[Q] = 1$ becomes a simple problem.

Let, as we already said, $e_1, \ldots, e_\mu$ be an orthonormal basis of the space of polynomials of degree n, homogeneous in N variables. Let

$$Q = \sum_{k=1}^{\mu} \gamma_k e_k,$$

$$\pi_i = \sum_{j=1}^{\mu} \alpha_{i,j} e_j, \qquad i = 1, \ldots, \nu,$$

be the orthogonal decompositions of Q and the π_i's respectively on this basis. Then :

$$[PQ]^2 = \sum_{i=1}^{\nu} |\sum_{k=1}^{\mu} \alpha_{i,k} \overline{\gamma}_k|^2. \tag{21}$$

If A is the matrix $(\alpha_{i,j})_{\substack{i=1,\ldots,\nu \\ j=1,\ldots,\mu}}$ and $X = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_\mu \end{pmatrix}$, then (21) becomes :

$$[PQ]^2 = \|A\overline{X}\|_2^2 = {}^t X A^* A \overline{X}, \tag{22}$$

and Proposition 3 is proved.

Since X and $\overline{X}$ have the same norm, the infimum in (22) is obtained for the smallest eigenvalue of A^*A ($\overline{X}$ being the corresponding eigenvector), and the supremum for the largest eigenvalue. We observe that the eigenvalues are real positive, and that none of them can be zero. This proves Theorem 4. In order to prove Theorem 2, we just observe that the matrix G is simply AA^*, so the non-zero eigenvalues are the same for G and for H.

The statement of Proposition 1, that is formula (19), may be given with P under contracted form :

Proposition 5. – *Let P, Q be homogeneous polynomials of degrees m and n respectively. Then :*

$$[PQ]^2 = \frac{n!^2}{(m+n)!} \sum_{k=0}^{\min(m,n)} \sum_{\substack{|\alpha|=n-k \\ |\beta|=m-k}} \frac{1}{\alpha!\beta!} \left| [x^\alpha D_\beta P, Q] \right|^2,$$

where

$$x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N} , \qquad D_\beta = \frac{\partial^{m-k}}{\partial x_1^{\beta_1} \cdots \partial x_N^{\beta_N}} .$$

Indeed, this follows immediately from (19) : for given $\alpha_1, \ldots, \alpha_N$, there are $\frac{(n-k)!}{\alpha!}$ terms $x_{j_{k+1}} \cdots x_{j_n}$ equal to x^α, and for given $\beta_1, \ldots, \beta_N$, there are $(m-k)!/\beta!$ terms $P_{i_{k+1}, \ldots, i_m}$ equal to $D_\beta P$.

Using the contracted form of P, we can, instead of the enumeration (8), consider the enumeration of the polynomials

$$\frac{n!}{\sqrt{(m+n)!\alpha!\beta!}} \, x^\alpha D_\beta P, \tag{23}$$

for $k = 0, \ldots, \min(m,n)$, $|\alpha| = n - k$, $|\beta| = m - k$, and take for G the matrix, say G_1, made of scalar products of these polynomials. This is a self-adjoint matrix, of size

$$\nu_1 \;=\; \sum_{k=0}^{\min(m,n)} \binom{N+n-k-1}{n-k} \binom{N+m-k-1}{m-k}$$

and $\nu_1 < \nu$. The general term of G_1 is

$$\frac{n!^2}{(m+n)!\sqrt{\alpha!\beta!\alpha'!\beta'!}} \, \big[x^\alpha D_\beta P, x^{\alpha'} D_{\beta'} P \big]. \tag{24}$$

Using the identity given in Beauzamy-Dégot [5], one can transform the above expression, in order to get scalar products involving only derivatives of P (and no x^α's anymore). Details of the computation are left to the reader ; (24) is transformed into :

$$\frac{n!^2}{(m+n)!} \sqrt{\frac{\alpha!\alpha'!}{\beta!\beta'!}} \sum_{\substack{\nu \geq 0 \\ |\gamma| = \nu}} \frac{(k+k'+\nu-n)!}{\gamma!^3} \big[D_{\alpha'-\gamma+\beta} P, D_{\alpha-\gamma+\beta'} P \big]. \tag{25}$$

Since Bombieri's norm depends on the degree (in fact, there is a different norm for each degree), problem (3) was stated for a fixed degree of Q. One can now wonder : what is the solution of problem (1) that is, what do we get if we let the degree of Q vary also ?

3. Limit problems

For the minimization problem, we obtain :

Proposition 6. – If $\deg P > 0$, $I(P) = 0$.

Proof. We will show that $I_n(P) \to 0$ when $n \to +\infty$.

In order to do this, we use Theorem 1, with the matrix $G_1 = G_1(n)$ defined in (24).

Let λ_n be the smallest non-zero eigenvalue of $G_1(n)$. We want to show that $\lambda_n \to 0$. Since $\lambda_n = 1/\|G_n^{-1}\|$, all we have to do is to find vectors X_n, with $\|X_n\| = 1$, such that $G_1(n)X_n \to 0$. Here the norms are l_2-norms, but this won't matter, as we will see.

For X_n, we simply take $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, so $G_1(n)X_n$ is just the first column of $G_1(n)$. We now identify this first column, using (24). We have $k = 0$, $\alpha = (n, 0, \ldots, 0)$, $\beta = (m, 0, \ldots, 0)$, and $D_\beta P$ is just $m!$ times the coefficient of x_1^m in P, which we simply call c. So we get from (24) :

$$\frac{n!^{3/2}}{(m+n)!} \sqrt{\frac{m!}{\alpha'!\beta'!}} \, c \, \big[x_1^n, x^{\alpha'} D_{\beta'} P \big]. \tag{26}$$

This scalar product is 0, unless $\alpha' = (n - k', 0, \ldots, 0)$. Then we get

$$\frac{n!^{3/2}}{(m+n)!} \sqrt{\frac{m!}{(n-k')!\beta'!}} \, c \, \big[x^{k'}, D_{\beta'} P \big], \tag{27}$$

where $|\beta'| = m - k'$, $k' \leq m$.

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So we see that the number of no-zero terms in (27) is independent of n, though the size of $G_1(n)$ increases with n.

Let

$$A = \frac{n!^{3/2}}{(m+n)!} \sqrt{\frac{m!}{(n-k')!\beta'!}}.$$

Then

$$A \leq \frac{n!^{3/2}}{(m+n)!} \sqrt{\frac{m!}{(n-m)!}}.$$

So

$$A^2 \leq \frac{n!^2}{(n+m)!^2} \frac{n!}{(n-m)!} m!$$

and, if $m > 0$, this can be written

$$A^2 \leq \frac{n}{(n+m)^2} \cdots \frac{n-m+1}{(n+1)^2} m!$$

and each of the fractions tends to zero when $n \to +\infty$. This proves our claim.

Corollary 7. – *For any non-constant homogeneous polynomial P, there is a sequence of homogeneous polynomials Q_n (of increasing degrees), such that $[Q_n] = 1$ and $[PQ_n] \to 0$.*

The Q_n's can be explicitly determined from Theorem 4. In some cases, they are immediate : if $P = (1+z)^m$, then $Q_n = (1-z)^n/2^{n/2}$. Indeed, since the pair $((1+z)^m, (1-z)^n)$ is extremal in Bombieri's inequality, we have

$$[PQ_n] = \sqrt{\frac{m!n!}{(m+n)!}} \, [P][Q_n] = \sqrt{\frac{m!n!}{(m+n)!}} \, 2^{m/2}.$$

We now turn to problem (2), dealing with the supremum. First, we observe that, trivially,

$$\sup\{[PQ] \; ; \; Q \text{ homogeneous of any degree and } [Q] = 1\} = [P],$$

since $[PQ] \leq [P]$ by [3] and since the supremum is attained for $Q = 1$.

If we now look at the finite dimensional problem $S_n(P)$, things are not so simple. For polynomials of the form $P = (\sum a_i x_i)^m$, one has $S_n(P) = [P]$ for all n, but there are cases for which $S(P) < [P]$:

Proposition 8. – *The polynomial $P = x_1 x_2$ satisfies $[P] = 1/\sqrt{2}$, $\lim S_n(P) = 1/2$.*

Proof. We will build the matrix G_1 defined in (24), for Q of fixed degree n. So first we have to build the list (23), where $m = 2$, that is the enumeration of the polynomials

$$\frac{n!}{\sqrt{(n+2)!\alpha!\beta!}} x_1^{\alpha_1} x_2^{\alpha_2} D_\beta P,$$

with $|\alpha| = n - k$, $|\beta| = 2 - k$, $k = 0, 1, 2$.

For $k = 0$, this leads to a first list :

$$L_1 : \quad \frac{n!}{\sqrt{(n+2)!(n-j)!j!}} x_1^{n-j} x_2^j, \qquad j = 0, \dots, n.$$

For $k = 1$, $\beta = (1, 0)$, we get

$$L_2 : \quad \frac{n!}{\sqrt{(n+2)!(n-j-1)!j!}} x_1^{n-j} x_2^j, \qquad j = 1, \dots, n.$$

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For $k = 1$, $\beta = (0, 1)$, we get

$$L_3 \; : \qquad \frac{n!}{\sqrt{(n+2)!(n-j-1)!j!}} \; x_1^{n-j} x_2^j \;, \qquad j = 0, \ldots, n-1.$$

For $k = 2$,

$$L_4 \; : \qquad \frac{n!}{\sqrt{(n+2)!(n-j-1)!(j-1)!}} \; x_1^{n-j} x_2^j \;, \qquad j = 1, \ldots, n-1.$$

The matrix G_1 is built from (24), using the cross-scalar products of all elements, so it has 16 blocks $A_{1,1}, \ldots, A_{4,4}$, where $A_{i,j}$ is the block made of the scalar products of the list L_i with the list L_j. All these blocks are square, except, for some of them, a first or a last row of zeros, and all of them are diagonal. The diagonal terms are very easy to compute : with $c = 1/(n+1)(n+2)$, we find for $A_{1,1} \; : c$; for $A_{1,2} \; : c\sqrt{n-j}$; for $A_{1,3} \; : c\sqrt{j}$; for $A_{1,4} \; : c\sqrt{(n-j)j}$; for $A_{2,2} \; : cj$; for $A_{2,3} \; : c\sqrt{(n-j)j}$; for $A_{2,4} \; : cj\sqrt{n-j}$; for $A_{3,3} \; : c(n-j)$; for $A_{3,4} \; : c(n-j)\sqrt{j}$; for $A_{4,4} \; : cj(n-j)$.

All these blocks, except $A_{4,4}$, have an operator norm tending to zero when $n \to \infty$ (recall that, for a diagonal matrix with positive entries, the operator norm is just the maximum of the terms on the diagonal). For $A_{4,4}$, the maximum is attained for $j = n/2$ and its value is $\frac{n^2}{4(n+1)(n+2)} \to 1/4$ when $n \to \infty$. So $S_n(P) \to 1/2$, and this proves our claim.

As an application of the previous concepts, we now compute the norm of a differential operator $P(D)$.

4. The norm of a differential operator

Let P be as before a homogeneous polynomial in N variables. We now call p its degree, since m and n will be used for other purposes. If the polynomial is written as in (1), the associated differential operator is

$$P\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_N}\right) \; = \; \sum_{|\alpha|=p} a_\alpha \; \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}},$$

which is written $P(D_1, \ldots, D_n)$, or more simply $P(D)$.

Let $\mathcal{P}_n$ be the space of homogeneous polynomials in N variables, of degree n, endowed with the norm $[\,.\,]_{(n)}$. Then the differential operator $P(D)$ acts on $\mathcal{P}_n$ and has its value in $\mathcal{P}_m$, $m = n - p$ ($m = 0$ if $p \geq n$). We may therefore define its operator norm, from $\mathcal{P}_n$ into $\mathcal{P}_m$, by the standard formula :

$$[P(D)]_{op(n,m)} \; = \; \sup\{[P(D)Q]_{(m)} \; ; \; Q \in \mathcal{P}_n, \; [Q]_{(n)} \leq 1\}.$$

We can now state our Theorem :

Theorem 9. – *Let P be a homogeneous polynomial in N variables $x_1, \ldots, x_N$, with (total) degree p. Let $P(D)$ be the associated differential operator. Then :*

$$[P(D)]_{op(n,m)} \; = \; \frac{n!}{m!} \, S_n(P),$$

where $S_n(P)$ is defined in formula (4), for Bombieri's norm.

Proof. Indeed, we have, using the transposition formulas of [5] or [14],

$$
\begin{aligned}
[P(D)]_{op(n,m)} \; &= \; \sup\{[P(D)Q]_{(m)} \; ; \; [Q]_{(n)} \leq 1\} \\
&= \; \sup\{|[P(D)Q, R]| \; ; \; [Q]_{(n)} \leq 1, \; [R]_{(m)} \leq 1\} \\
&= \; \frac{n!}{m!} \sup\{|[Q, \overline{P}R]| \; ; \; [Q]_{(n)} \leq 1, \; [R]_{(m)} \leq 1\} \\
&= \; \frac{n!}{m!} \sup\{[\overline{P}R]_{(n)} \; ; \; [R]_{(m)} \leq 1\} \\
&= \; \frac{n!}{m!} \sup\{[\overline{PR}]_{(n)} \; ; \; [\overline{R}]_{(m)} \leq 1\} \\
&= \; \frac{n!}{m!} S_m(P),
\end{aligned}
$$

as we announced. This proves the Theorem.

We observe that, since

$$[PR] \geq \sqrt{\frac{p!m!}{n!}} \, [P][R]$$

by Bombieri's inequality, we find a fortiori

$$[P(D)]_{op(n,m)} \geq \frac{n!}{m!} \sqrt{\frac{p!m!}{n!}} \, [P] = \sqrt{\frac{n!p!}{m!}} \, [P] = \sqrt{\frac{n!(n-m)!}{m!}} \, [P].$$

5. Other problems

The identities given in Proposition 1 and 3 also allow us to deal with other optimization problems, involving other normalizations. We have considered here the normalization $[Q] = 1$ because it is the hardest, but the corresponding problems with other constraints are immediately handled. Let us consider for instance the problem (in one variable) :

$$\inf\{[PQ] \; ; \; Q(0) = 1, \; deg(Q) = n\}.$$

This means that the homogeneous associated Q finishes with the term x_2^n. So the vector column X is of the form

$$X = \begin{pmatrix} \gamma_0 \\ \vdots \\ \gamma_{n-1} \\ 1 \end{pmatrix}$$

If we denote by A_1, A_2, ... A_{n+1} the column-vectors of A, we have

$$A\overline{X} = \overline{\gamma}_0 A_1 + \cdots \overline{\gamma}_{n-1} A_n + A_{n+1} \tag{28}$$

and the minimum of $A\overline{X}$ is just the distance between A_{n+1} and the vector space spanned by A_1, ..., A_n, that is the distance between A_{n+1} and its projection onto this subspace.

But A_{n+1} is made of the constant coefficients of each π_j, and the last π_j in the enumeration is just $\dfrac{n!}{\sqrt{(n+m)!(n-m)!}} x_2^{n-m} P$, whose constant term (that is the coefficient of x_2^n) is at most $|P(0)|$. The other terms in the enumeration have constant terms at most $Cn!/\sqrt{(n+m)!(n-m+1)!} \leq C/\sqrt{n-m+1} \to 0$. Since conversely $[PQ] \geq |P(0)|$, we get the formula

$$\inf\{[PQ] \; ; \; Q(0) = 1\} = |P(0)|.$$

This formula itself is obvious (taking simply $Q = x_2^n$ gives it), so the interest of (28) lies rather in the finite-dimensional geometric description, than in the asymptotics when $n \to \infty$.

References

[1] Bacquet, G. : Produits de polynômes en plusieurs variables : expression du reste dans une inégalité de Bombieri. *Note C.R.A.S.* Paris, oct. 1993.

[2] Beauzamy, B. : Introduction to Operator Theory and Invariant Spaces. *North Holland Mathematical Library*, 1988.

[3] Beauzamy, B. : Products of polynomials and a priori estimates for coefficients in polynomial decompositions : a sharp result. *Journal of Symbolic Computation*, vol. 13, 1992, pp. 463–472.

[4] Beauzamy, B. – Bombieri, E. – Enflo, P. – Montgomery, H. : Products of polynomials in many variables. *Journal of Number Theory*, vol. 36, 2, oct. 1990, pp. 219–245.

[5] Beauzamy, B. – Dégot, J. : Differential Identities. To appear in the *Trans. A.M.S.*

[6] Beauzamy, B. – Frot, J.-L. – Millour, C. : Massively parallel computations on many-variable polynomials : When seconds count. Special volume "Maths and Computer Science", *Annals of Maths and IA*, M. Nivat and S. Grigorieff, editors, 1994.

[7] Beauzamy, B. : Product of many-variable polynomials : pairs that are maximal in Bombieri's norm. To appear in the *J. of Number Theory*.

[8] Bombieri, E. – Vaaler, J. : On Siegel's Lemma. *Invent. Math.*, 73, 1983, pp. 11–32.

[9] Boyd, D. W. : Bounds for the height of a factor of a polynomial in terms of Bombieri's norms : I. The largest factor. *Journal of Symbolic Computation*, 16, 1993, pp. 115–130.

[10] Boyd, D. W. : Bounds for the height of a factor of a polynomial in terms of Bombieri's norms : II. The smallest factor. *Journal of Symbolic Computation*, 16, 1993, pp. 131–145.

[11] Boyd, D. W. : Two sharp inequalities for the norm of a factor of a polynomial. *Mathematika*, 39, 1992, pp. 341–349.

[12] Donaldson, J. D.– Rahman, Q. I. : Inequalities for polynomials with a prescribed zero. *Pacific J. of Maths* , 41, 1972, pp. 375–378.

[13] Kahane, J.-P. : Séries de Fourier absolument convergentes. *Springer Verlag*, 1970.

[14] Reznick, Bruce : An inequality for products of polynomials. *Proceedings A.M.S.*, vol. 117, 4, 1993, pp. 1063–1073.