DIFFERENTIAL IDENTITIES

by

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Abstract. – We deal here with homogeneous polynomials in many variables and their hypercube representation, introduced in [5]. Associated with this representation there is a norm (Bombieri's norm) and a scalar product. We investigate differential identities connected with this scalar product. As a corollary, we obtain Bombieri's inequality (originally proved in [4]), with significant improvements.

The hypercube representation of a polynomial was elaborated in order to meet the requests of massively parallel computation on the "Connection Machine" at E.T.C.A. ; we see here once again (after [3] and [5]) the theoretical power of the model.

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1. The hypercube representation and Bombieri's norm.

Let

$$P(x_1,\dots,x_N) \;=\; \sum_{|\alpha|=m} a_\alpha\; x_1^{\alpha_1}\cdots x_N^{\alpha_N} \tag{1}$$

be a homogeneous polynomial in N variables $x_1,\dots,x_N$, with complex coefficients and degree m. Here, as usual, we write $\alpha=(\alpha_1,\dots,\alpha_N)$, $|\alpha|=\alpha_1+\cdots+\alpha_N$.

For any $i_1,\dots,i_m$, $1\le i_1\le N,\dots,1\le i_m\le N$, we define, as in [4] :

$$c_{i_1,\dots,i_m} \;=\; \frac{1}{m!}\;\frac{\partial^m P}{\partial x_{i_1}\cdots\partial x_{i_m}}\;, \tag{2}$$

and by Taylor's formula, we have :

$$P(x_1,\dots,x_N) \;=\; \sum_{i_1,\dots,i_m=1}^{N} c_{i_1,\dots,i_m}\; x_{i_1}\cdots x_{i_m}\;, \tag{3}$$

which is called *symmetric form* of the polynomial.

For a polynomial P of degree m, Bombieri's norm is defined as (see [4]) :

$$\left[P\right]_{(m)} \;=\; \left(\sum_{i_1,\dots,i_m=1}^{N} |c_{i_1,\dots,i_m}|^2\right)^{1/2}\;. \tag{4}$$

We usually omit the subscript (m) but it should be clear that the norm depends on the degree of the polynomial.

As explained in [5], both (3) and (4) have a geometric description, by means of the hypercube representation of the polynomial : in the hypercube $[0,1]^m$, we define the N^m points $M_{i_1,\dots,i_m}$ with coordinates $i_1/N,\dots,\ i_m/N$, $(1\le i_1\le N,\dots,1\le i_m\le N)$. We now put each coefficient $c_{i_1,\dots,i_m}$ onto the corresponding point $M_{i_1,\dots,i_m}$: this operation is called representation of the polynomial on the hypercube. Bombieri's norm appears as the canonical l_2-norm associated with this representation.

If we start with any polynomial P given as in (1), it can be written in many ways under the form

$$P(x_1,\dots,x_N) \;=\; \sum_{i_1,\dots,i_m=1}^{N} b_{i_1,\dots,i_m}\; x_{i_1}\cdots x_{i_m}\;, \tag{5}$$

but the symmetric representation (3) has a particular property, as the following Proposition shows (it was communicated to us by Christian Millour) :

Proposition 1. – *Among all representations of P of the form (5), the symmetric one (3) is the one for which the l_2-norm is minimal.*

Proof. – For any $i_1,\dots,i_m$, if σ is a permutation of $\{i_1,\dots,i_m\}$, we have

$$\sum_\sigma b_{\sigma(i_1),\dots,\sigma(i_m)} \;=\; \sum_\sigma c_{\sigma(i_1),\dots,\sigma(i_m)}\;,$$

and $c_{\sigma(i_1),\dots,\sigma(i_m)}=c_{i_1,\dots,i_m}$ for any σ. Therefore, the proposition follows from the observation that, if (a_i) are any complex numbers with fixed sum, $\sum|a_i|^2$ is minimal when all the a_i's are equal.

2. The associated scalar product.

Canonically associated with Bombieri's norm $[\ ]_{(m)}$, there is a scalar product : if P, Q are two homogeneous polynomials with same degree m, written in symmetric form as

$$P(x_1,\dots,x_N) \;=\; \sum_{i_1,\dots,i_m=1}^{N} c_{i_1,\dots,i_m}\; x_{i_1}\cdots x_{i_m}\ ,$$

$$Q(x_1,\dots,x_N) \;=\; \sum_{i_1,\dots,i_m=1}^{N} d_{i_1,\dots,i_m}\; x_{i_1}\cdots x_{i_m}\ ,$$

then we set :

$$\big[P,Q\big]_{(m)} \;=\; \sum_{i_1,\dots,i_m=1}^{N} c_{i_1,\dots,i_m}\; \overline{d}_{i_1,\dots,i_m}.$$

This scalar product appears already in Bruce Reznick [10], where the following result can be found :

Proposition 2. – *Let $b=(b_1,\dots,b_N)$ and define $\delta_b = b_1x_1+\cdots+b_Nx_N$. Then, for any homogeneous polynomial P with degree m :*

$$P(b_1,\dots,b_N) \;=\; \big[P,\delta_{\overline{b}}{}^m\big].$$

Proof. – We just observe that $\delta_{\overline{b}}{}^m$ can be written in symmetric form

$$\delta_{\overline{b}}^m(x_1,\dots,x_N) \;=\; \sum_{i_1,\dots,i_m=1}^{N} \overline{b}_{i_1}\cdots\overline{b}_{i_m}\; x_{i_1}\cdots x_{i_m}, \tag{6}$$

and the result follows. This result justifies the notation "δ_b", since this polynomial behaves as a Dirac measure for this scalar product.

As we did for the norm, we usually omit the subscript (m) in the notation of the scalar product and write simply $\big[P,Q\big]$, but one should remember that P and Q must be of the same degree (or are considered so), and that the scalar product depends on that degree.

We observe that, in order to define the scalar product, only one of the polynomials needs to be written in symmetric form :

Proposition 3. – *Let $P=\sum_{i_1,\dots,i_m} c_{i_1,\dots,i_m}x_{i_1}\cdots x_{i_m}$ be written in symmetric form (3), and*

$$Q \;=\; \sum_{j_1,\dots,j_m} d_{j_1,\dots,j_m}x_{j_1}\cdots x_{j_m}$$

be any homogeneous polynomial of degree m (the d's need not be invariant under permutation of indices). Then :

$$\big[P,Q\big] \;=\; \sum_{i_1,\dots,i_m} c_{i_1,\dots,i_m}\overline{d_{i_1,\dots,i_m}}.$$

Proof. – Let

$$Q \;=\; \sum_{j_1,\dots,j_m} d'_{j_1,\dots,j_m}x_{j_1}\cdots x_{j_m}$$

be the symmetric form of Q. Then

$$\big[P,Q\big] \;=\; \sum_{i_1,\dots,i_m} c_{i_1,\dots,i_m}\overline{d'_{i_1,\dots,i_m}}. \tag{7}$$

But

$$d'_{j_1,\dots,j_m} \;=\; \frac{1}{m!}\sum_{\sigma\in S_m} d_{j_{\sigma(1)},\dots,j_{\sigma(m)}}$$

where S_m is the group of permutations of $\{1,\dots,m\}$. Substituting into (7) and taking into account the symmetry of the c's, we obtain the result.

We now investigate a few special situations :

Proposition 4. – *Let* $P_1, \dots, P_k$ *be homogeneous polynomials in* N *variables* $x_1, \dots, x_N$, *with degrees* $m_1, \dots, m_k$. *Let* $m = m_1 + \cdots + m_k$, *and let also* $q_1, \dots, q_m$ *be homogeneous polynomials of degree* 1. *Then :*

$$\big[P_1 \cdots P_k \ , q_1 \cdots q_m\big] \ = \ \frac{1}{m!} \sum_\sigma \big[P_1, \ q_{\sigma(1)} \cdots q_{\sigma(m_1)}\big] \times \cdots \times \big[P_k, \ q_{\sigma(m-m_k+1)} \cdots q_{\sigma(m)}\big],$$

where σ *runs over all permutations of* $\{1, \dots m\}$.

Proof. – For $i = 1, \dots, m$, we write

$$q_i \ = \ \sum_{j=1}^N \ q_{ij} x_j \ ,$$

and we obtain a symmetric representation of $q_1 \cdots q_m$:

$$q_1 \cdots q_m \ = \ \frac{1}{m!} \sum_{i_1, \dots, i_m = 1}^N \ \sum_\sigma \ q_{1, i_{\sigma(1)}} \cdots q_{m, i_{\sigma(m)}} \ x_{i_1} \cdots x_{i_m}.$$

This can also be written :

$$q_1 \cdots q_m \ = \ \frac{1}{m!} \sum_{i_1, \dots, i_m = 1}^N \ \sum_\sigma \ q_{\sigma(1), i_1} \cdots q_{\sigma(m), i_m} \ x_{i_1} \cdots x_{i_m}.$$

Now, we write each P_j under the form

$$P_j \ = \ \sum_{I_j} \ c^{(j)}_{I_j} \ X_{I_j}$$

where $I_j = \{i_{m_{j-1}+1}, \dots, i_{m_j}\}$, and X_{I_j} stands for $x_{i_{m_{j-1}+1}} \cdots x_{i_{m_j}}$.

Then a non-symmetric representation of $P_1 \cdots P_k$ is given by

$$P_1 \cdots P_k \ = \ \sum_{I_1, \dots, I_k} \ c^{(1)}_{I_1} \cdots c^{(k)}_{I_k} \ X_{I_1} \cdots X_{I_k}.$$

Using Proposition 3, we find :

$$\begin{aligned}
\big[P_1 \cdots P_k, q_1 \cdots q_m\big] \ &= \ \frac{1}{m!} \sum_{i_1, \dots, i_m} \ \sum_\sigma \ c^{(1)}_{I_1} \cdots c^{(k)}_{I_k} \ \overline{q}_{\sigma(1), i_1} \cdots \overline{q}_{\sigma(m), i_m} \\
&= \ \frac{1}{m!} \sum_\sigma \ (\sum_{I_1} \ c^{(1)}_{I_1} \overline{q}_{\sigma(1), i_1} \cdots \overline{q}_{\sigma(m), i_{m_1}}) \times \cdots \\
&\qquad \cdots \times \ (\sum_{I_k} \ c^{(k)}_{I_k} \overline{q}_{\sigma(m-m_k+1), i_{m-m_k+1}} \cdots \overline{q}_{\sigma(m), i_m}) \\
&= \ \frac{1}{m!} \sum_\sigma \ \big[P_1, q_{\sigma(1)} \cdots q_{\sigma(m_1)}\big] \cdots \big[P_k, q_{\sigma(m-m_k+1)} \cdots q_{\sigma(m)}\big],
\end{aligned}$$

which concludes the proof.

This proposition has several corollaries (which of course have direct proofs) :

Corollary 5. – *Let* $p_1, \dots, p_m$, $q_1, \dots, q_m$ *be two sets of homogeneous polynomials of degree* 1, *with variables* $x_1, \dots, x_N$. *Then :*

$$\big[p_1 \cdots p_m, q_1 \cdots q_m\big] \ = \ \frac{1}{m!} \sum_{\sigma \in S_m} \ \big[p_1, q_{\sigma(1)}\big] \cdots \big[p_m, q_{\sigma(m)}\big],$$

where σ *runs over the set* S_m *of all permutations of* $\{1, \dots, m\}$.

In the next corollary, we give an expression of the scalar product of two polynomials in one variable z, with same degree m. This expression uses the zeros of both polynomials :

Corollary 6. – *Let* $P=(z-a_1)\cdots(z-a_m)$, $Q=(z-b_1)\cdots(z-b_m)$. *Then :*

$$\big[P,Q\big] \;=\; \frac{1}{m!}\sum_{\sigma\in S_m}(1+a_1\overline{b}_{\sigma(1)})\cdots(1+a_m\overline{b}_{\sigma(m)}),$$

where σ *runs over the set* S_m *of all permutations of* $\{1,\ldots,m\}$.

This corollary is an obvious consequence of Corollary 5. We identify the one-variable polynomial $z-a$ with the homogeneous two-variable polynomial $z-az'$.

An expression of $\big[P,Q\big]$, using the zeros of P and Q, was already given by Y. Legrandgérard [8]. It differs from this one, and is more complicated.

3. The multi-linear functional associated with a many-variable polynomial.

Let P be a homogeneous polynomial in N variables. Then, Littlewood's theory associates to it a multi-linear form L on $\mathbb{C}^N\times\cdots\times\mathbb{C}^N$, by the formula

$$\begin{aligned} L(Z^{(1)},\ldots,Z^{(m)}) \;=\; &\frac{1}{m!}\,\frac{1}{2^m}\sum_{\varepsilon_1,\ldots,\varepsilon_m=\pm1}\varepsilon_1\cdots\varepsilon_m\;\times \\ &\times\;P(\sum_{j=1}^{m}\varepsilon_j\;Z_1^{(j)},\ldots,\sum_{j=1}^{m}\varepsilon_j\;Z_N^{(j)}), \end{aligned} \tag{8}$$

where each $Z^{(j)}$ stands for $(Z_1^{(j)},\ldots,Z_N^{(j)})$.

That fact that L is indeed multi-linear is not a priori clear ; a proof can be found in S. Dineen [6]. But this fact will become obvious once we establish :

Proposition 7. – *The form* L *coincides with the multi-linear form generated by the hypercube, that is :*

$$L_1(Z^{(1)},\ldots,Z^{(m)}) \;=\; \sum_{i_1,\ldots,i_m=1}^{N} c_{i_1,\ldots,i_m}\;Z_{i_1}^{(1)}\cdots Z_{i_m}^{(m)}\;. \tag{9}$$

Since quite obviously L_1 is linear with respect to each variable, the same will be true for L.

Proof. – If in (8) we write P under symmetric form (3) and substitute, we get :

$$\begin{aligned} L(Z^{(1)},\ldots,Z^{(m)}) \;=\; &\frac{1}{m!\;2^m}\sum_{\varepsilon_1,\ldots,\varepsilon_m=\pm1}\varepsilon_1\cdots\varepsilon_m\sum_{i_1,\ldots,i_m=1}^{N}c_{i_1,\ldots,i_m}\;\times \\ &\times\;(\sum_{j=1}^{m}\varepsilon_j\;Z_{i_1}^{(j)})\cdots(\sum_{j=1}^{m}\varepsilon_j\;Z_{i_m}^{(j)}) \\ =\;\frac{1}{m!\;2^m}\sum_{i_1,\ldots,i_m=1}^{N}c_{i_1,\ldots,i_m}&\sum_{\varepsilon_1,\ldots,\varepsilon_m=\pm1}\varepsilon_1\cdots\varepsilon_m\;(\sum_{j=1}^{m}\varepsilon_j\;Z_{i_1}^{(j)})\cdots(\sum_{j=1}^{m}\varepsilon_j\;Z_{i_m}^{(j)}). \end{aligned}$$

Using Rademacher functions (see for instance [2]), this can be written more simply :

$$L \;=\; \frac{1}{m!}\sum_{i_1,\ldots,i_m=1}^{N}c_{i_1,\ldots,i_m}\int_0^1 r_1(t)\cdots r_m(t)(\sum_{j=1}^{m}r_j(t)Z_{i_1}^{(j)})\cdots(\sum_{j=1}^{m}r_j(t)Z_{i_m}^{(j)})dt.$$

We expand all products and observe that all terms have integral 0, except those which give $r_1^2(t)\cdots r_m^2(t)$: these ones have integral 1. This way, we obtain :

$$L \;=\; \frac{1}{m!}\sum_{u}\sum_{i_1,\ldots,i_m=1}^{N}c_{i_1,\ldots,i_m}\;Z_{i_1}^{(u_1)}\cdots Z_{i_m}^{(u_m)}\;,$$

where $u=(u_1,\ldots,u_m)$ runs through all permutations of $\{1,\ldots,m\}$. We rewrite

$$Z_{i_1}^{(u_1)}\cdots Z_{i_m}^{(u_m)} \;=\; Z_{i_{u^{-1}(1)}}^{(1)}\cdots Z_{i_{u^{-1}(m)}}^{(m)}\;,$$

and use the fact that $c_{i_{u^{-1}(1)},\ldots,i_{u^{-1}(m)}}=c_{i_1,\ldots,i_m}$ and obtain the result.

As an example, the polynomial $P(x_1, x_2) = x_1^3 - 3x_1^2\ x_2 + x_2^3$ gives in symmetric form $x_1^3 - (x_1x_1x_2 + x_1x_2x_1 + x_2x_1x_1) + x_2^3$, and the associated tri-linear form is :

$$L(Z^{(1)}, Z^{(2)}, Z^{(3)}) = Z_1^{(1)} Z_1^{(2)} Z_1^{(3)} \ - \ (Z_1^{(1)} Z_1^{(2)} Z_2^{(3)} + Z_1^{(1)} Z_2^{(2)} Z_1^{(3)} + Z_2^{(1)} Z_1^{(2)} Z_1^{(3)}) + \ Z_2^{(1)} Z_2^{(2)} Z_2^{(3)}.$$

As pointed out to us by Andrew Tonge, the above proposition is known to the specialists of the multi-linear functional, though we could not find a published proof. Moreover, it does not seem to provide any quantitative improvement of the results on the norm of the multi-linear functional, obtained originally by Banach and completed by various authors (R. Aron - J. Globevnik [1], L. Harris [7], Y. Sarantopoulos [11], I. Zalduendo [12]).

4. Differential identities.

Let's come back to scalar products. The basic observation (already made by Bruce Reznick in [10]) is that multiplication by a variable on one side becomes derivation on the other side. Precisely, we have :

Lemma 9. – *If P is of degree $m-1$ and Q of degree m :*

$$\big[x_1 P,\ Q\big]_{(m)} \ = \ \frac{1}{m}\ \big[P,\ \frac{\partial Q}{\partial x_1}\big]_{(m-1)}.$$

Proof. – Of course, it's enough to prove the formula when P, Q are monomials, say $P = x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ (with $|\alpha| = m-1$), $Q = x_1^{\beta_1} \cdots x_N^{\beta_N}$ (with $(|\beta| = m)$. Then

$$\big[x_1 P, Q\big]_{(m)} \ = \ \big[x_1^{\alpha_1+1}\ x_2^{\alpha_2} \cdots x_N^{\alpha_N},\ x_1^{\beta_1} \cdots x_N^{\beta_N}\big]_{(m)}$$

and this is 0, except if $\beta_1 = \alpha_1 + 1$, $\beta_2 = \alpha_2, \ldots, \beta_N = \alpha_N$, in which case the value is

$$\big[x_1^{\beta_1} \cdots x_N^{\beta_N}\big]^2_{(m)} \ = \ \frac{\beta_1! \cdots \beta_N!}{m!}.$$

Similarly,

$$\big[P, \frac{\partial Q}{\partial x_1}\big]_{(m-1)} \ = \ \big[x_1^{\alpha_1} \cdots x_N^{\alpha_N}, \beta_1 x_1^{\beta_1 - 1} x_2^{\beta_2} \cdots x_N^{\beta_N}\big]_{(m-1)}$$

is 0, except if $\alpha_1 = \beta_1 - 1$, $\alpha_2 = \beta_2, \ldots, \alpha_N = \beta_N$, in which case the value is

$$\beta_1 \big[x_1^{\alpha_1} \cdots x_N^{\alpha_N}\big]^2_{(m-1)} \ = \ \beta_1 \frac{\alpha_1! \cdots \alpha_N!}{(m-1)!} \ = \ \frac{\beta_1! \cdots \beta_N!}{(m-1)!} \ ;$$

the result follows.

Corollary 10. – *(Transposition of a linear factor) Let P be of degree $m-1$ and Q be of degree m. Then*

$$\big[(\sum a_j x_j) P, Q\big]_{(m)} \ = \ \frac{1}{m}\ \big[P, \sum \overline{a}_j \frac{\partial Q}{\partial x_j}\big]_{(m-1)} \ .$$

If P and Q are one-variable polynomials (with the identification between $\sum_0^m a_j z^j$ and $\sum_0^m a_j z^j z'^{m-j}$), the polynomial $a\frac{\partial Q}{\partial z} + a'\frac{\partial Q}{\partial z'}$ is the homogeneous version of the polynomial $mQ(z) + (a-z)Q'(z)$, which is called the *polar derivative* of Q at the point a (cf. Marden [9]) and denoted by $Q_1(a, z)$. So, for one variable polynomials, the above formula becomes

$$\big[(az+1)P, Q\big]_{(m)} \ = \ \frac{1}{m}\ \big[P, Q_1(\overline{a}, z)\big]_{(m-1)} \ .$$

From Lemma 9 will follow several differential identities. In order to state them, we recall the following definitions, which are standard in P.D.E.

If $P(x_1, \ldots, x_N) = \sum a_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ is a polynomial, the associated differential operator is

$$P(D_1, \ldots, D_N) = \sum a_\alpha D_1^{\alpha_1} \cdots D_N^{\alpha_N}$$

where D_j stands for $\dfrac{\partial}{\partial x_j}$. This operator is usually written simply $P(D)$, with $D = (D_1, \ldots, D_N)$. We write simply P_i instead of $\dfrac{\partial P}{\partial x_i}$, and more generally $P_{i_1,\ldots,i_k}$ instead of $\dfrac{\partial^k P}{\partial x_{i_1} \cdots \partial x_{i_k}}$.

We also define

$$P^*(x_1,\ldots,x_N) \;=\; \sum_{|\alpha|=m} \overline{a_\alpha}\; x_1^{\alpha_1}\cdots x_N^{\alpha_N}.$$

A simple generalization of Lemma 9, also stated by Bruce Reznick [10], is :

Lemma 11. *– Let P, Q, R be homogeneous polynomials, with $\deg P = p$, $\deg Q = q$, $\deg R = r$, with $r = p+q$. Then :*

$$\big[PQ,\ R\big]_{(r)} \;=\; \frac{q!}{r!}\,\big[Q,\ P^*(D)R\big]_{(q)}\ .$$

We can now state the most general form of the differential identities.

Theorem 12. *– Let P, Q, R, S be four homogeneous polynomials, respectively of degree p, q, r, s, with $p+q = r+s$. Then*

$$\big[PQ,\ RS\big]_{(p+q)} \;=\; \frac{1}{(p+q)!}\sum_{k\geq 0}\frac{(q-r+k)!}{k!}\sum_{i_1,\ldots,i_k=1}^{N}\big[R^*_{i_1,\ldots,i_k}(D)Q,\ P^*_{i_1,\ldots,i_k}(D)S\big]_{(q-r+k)}.$$

We observe that in this sum the terms are 0
– if $k>p$ (since $P^*_{i_1,\ldots,i_k}=0$),
– if $k>r$ (since $R^*_{i_1,\ldots,i_k}=0$),
– if $r-k>q$ (since $R^*_{i_1,\ldots,i_k}(D)Q=0$) and
– if $r-k>s$ (since $P^*_{i_1,\ldots,i_k}(D)S=0$).

We will give two proofs of the theorem : the first one is longer but more transparent, the second one shorter and less transparent.

<u>First proof</u>. We have if, $P=\sum c_{i_1,\ldots,i_p}x_{i_1}\cdots x_{i_p}$, $R=\sum d_{j_1,\ldots,j_r}x_{j_1}\cdots x_{j_r}$:

$$\begin{aligned}\big[PQ,RS\big] \;&=\; \sum_{i_1,\ldots,i_p}\sum_{j_1,\ldots,j_r} c_{i_1,\ldots,i_p}\overline{d}_{j_1,\ldots,j_r}\big[x_{i_1}\cdots x_{i_p}\ Q,\ x_{j_1}\cdots x_{j_r}\ S\big]_{p+q}\\ &=\; \frac{1}{p+q}\sum_{i_1,\ldots,i_p}\sum_{j_1,\ldots,j_r} c_{i_1,\ldots,i_p}\overline{d}_{j_1,\ldots,j_r}\big[x_{i_2}\cdots x_{i_p}Q,\ \frac{\partial}{\partial x_{i_1}}(x_{j_1}\cdots x_{j_r}\ S)\big]_{p+q-1}\ ,\end{aligned}$$

by Lemma 9.

We now need a lemma which describes the exchange between a multiplication and a derivation.

Lemma 13. *– For all homogeneous polynomials R, Q with same degree,*

$$\sum_{i,j} c_{i,j}\big[R,\ \frac{\partial}{\partial x_i}(x_jQ)\big] \;=\; (\sum_j c_{j,j})\ \big[R,Q\big] + \sum_{i,j} c_{i,j}\big[R,\ x_j\frac{\partial Q}{\partial x_i}\big]$$

Proof of Lemma 13. We have simply

$$\begin{aligned}\sum_{i,j} c_{i,j}\big[R,\ \frac{\partial}{\partial x_i}(x_jQ)\big] \;&=\; \sum_j\sum_{i\neq j} c_{i,j}\ \big[R,\frac{\partial}{\partial x_i}(x_jQ)\big] + \sum_j c_{j,j}\big[R,\ \frac{\partial}{\partial x_j}(x_jQ)\big]\\ &=\; \sum_j\sum_{i\neq j} c_{i,j}\ \big[R, x_j\ \frac{\partial Q}{\partial x_i}\big] \;+\; \sum_j c_{j,j}\big[R,\ x_j\frac{\partial Q}{\partial x_j}+Q\big],\end{aligned}$$

and the Lemma follows.

We observe that the statement of this lemma is quite similar to what one gets, computing a derivative in the sense of distributions : a derivative plus a jump. The similarity is quite natural : here also, we have a derivative inside a duality.

Let's now return to the proof of Theorem 12.

Using Lemma 13, we find :

$$\begin{aligned}
\big[PQ, RS\big] &= \frac{1}{p+q} \sum_{i_1,\dots,i_p} \sum_{j_2,\dots,j_r} c_{i_1,\dots,i_p} \overline{d}_{i_1,j_2,\dots,j_r} \big[x_{i_2}\cdots x_{i_p}\ Q,\ x_{j_2}\cdots x_{j_r}\ S\big] \\
&+ \frac{1}{p+q} \sum_{i_1,\dots,i_p} \sum_{j_1,\dots,j_r} c_{i_1,\dots,i_p} \overline{d}_{j_1,\dots,j_r} \big[x_{i_2}\cdots x_{i_p}\ Q,\ x_{j_1} \frac{\partial}{\partial x_{i_1}} (x_{j_2}\cdots x_{j_r}\ S)\big]
\end{aligned}$$

Repeating the process, we get :

$$\begin{aligned}
&= \frac{r}{p+q} \sum_{i_1,\dots,i_p} \sum_{j_2,\dots,j_r} c_{i_1,\dots,i_p} \overline{d}_{i_1,j_2,\dots,j_r} \big[x_{i_2}\cdots x_{i_p}\ Q,\ x_{j_2}\cdots x_{j_r}\ S\big] \\
&+ \frac{1}{p+q} \sum_{i_1\cdots i_p} \sum_{j_1\cdots j_r} c_{i_1\cdots i_p} \overline{d}_{j_1\cdots j_r} \big[x_{i_2}\cdots x_{i_p}\ Q,\ x_{j_1}\cdots x_{j_r} S_{i_1}\big].
\end{aligned}$$

In order to argue by induction, we define

$$\begin{aligned}
L(l,k) &= \sum_{i_1,\dots,i_p} \sum_{j_{l+1},\dots,j_r} c_{i_1,\dots,i_p}\ \overline{d}_{i_1,\dots,i_l,j_{l+1},\dots,j_r} \times \\
&\qquad \times \big[x_{i_{p-k+1}}\cdots x_{i_p} Q,\ x_{j_{l+1}}\cdots x_{j_r}\ S_{i_{l+1},\dots,i_{p-k}}\big]
\end{aligned}$$

In this notation, l stands for the numbers of "links" between the c's and the d's, that is the number of indexes appearing in both, and k stands for the number of variables before Q (from $x_{i_{p-k+1}}$ to x_{i_p}). We observe that $\big[PQ, RS\big] = L(0,p)$.

Then, the same computation as above, using Lemma 13, yields the induction formula :

$$L(l,k) = \frac{r-l}{q+k}\ L(l+1,k-1) + \frac{1}{k+q}\ L(l,k-1).$$

From this formula, we deduce by induction on j that, for all j :

$$\begin{aligned}
L(0,p) = \frac{(p+q-j)!}{(p+q)!} \Big(& r(r-1)\cdots(r-j+1)\ L(j,p-j) \\
&+ \binom{j}{1} r(r-1)\cdots(r-j+2)\ L(j-1,p-j) \\
&\ \vdots \\
&+ \binom{j}{\lambda} r\cdots(r-j+\lambda+1)\ L(j-\lambda,p-j) \\
&\ \vdots \\
&+ \binom{j}{j-1} r\ L(1,p-j) \\
&+ \binom{j}{j}\ L(0,p-j) \Big)
\end{aligned}$$

Taking $j = p$, we get :

$$L(0,p) \;=\; \frac{q!}{(p+q)!}\big(r(r-1)\cdots(r-p+1)\ L(p,0)\ +\cdots+\ \binom{p}{l} r\cdots(r-l+1)\ L(l,0)+ \\ +\cdots+\ \binom{p}{1} r\ L(1,0)+\ L(0,0)\big).$$

But, for every l :

$$L(l,0) \;=\; \sum_{i_1,\dots,i_p}\ \sum_{j_{l+1},\dots,j_r} c_{i_1,\dots,i_p}\ \overline{d}_{i_1,\dots,i_l,j_{l+1},\dots,j_r}\ \big[Q,\ x_{j_{l+1}}\cdots x_{j_r}\ S_{i_{l+1},\dots,i_p}\big]$$

and by Lemma 9 :

$$\begin{aligned}
&= \frac{(q-r+l)!}{q!} \sum_{i_1,\dots,i_p}\ \sum_{j_{l+1},\dots,j_r} c_{i_1,\dots,i_p}\ \overline{d}_{i_1,\dots,i_l,j_{l+1},\dots,j_r}\ \big[Q_{j_{l+1}\cdots j_r},\ S_{i_{l+1},\dots,i_p}\big] \\
&= \frac{(q-r+l)!}{q!} \sum_{i_1,\dots,i_l} \Big[\sum_{j_{l+1},\dots,j_r} \overline{d}_{i_1,\dots,i_l,j_{l+1},\dots,j_r}\ Q_{j_{l+1},\dots,j_r},\ \sum_{i_{l+1},\dots,i_p} \overline{c}_{i_1,\dots,i_p} S_{i_{l+1},\dots,i_p}\Big] \\
&= \frac{(q-r+l)!}{q!}\ \frac{(r-l)!}{r!}\ \frac{(p-l)!}{p!} \sum_{i_1,\dots,i_l} \big[R^*_{i_1,\dots,i_l}\ (D)Q,\ P^*_{i_1,\dots,i_l}\ (D)S\big]
\end{aligned}$$

Substituting each $L(l,0)$ into $L(0,p)$ above yields the announced result.

<u>Second proof</u>. – Using Euler's formula, we can write $P = \frac{1}{p}\sum x_i P_i$, and thus :

$$\big[PQ, RS\big] \;=\; \frac{1}{p}\sum_i \big[x_i P_i Q,\ RS\big] \;=\; \frac{1}{p(r+s)}\sum_i \big[P_i Q,\ (RS)_i\big].$$

Repeating this process on P_i, we get :

$$\big[PQ, RS\big] \;=\; \frac{(r+s-p)!}{p!(r+s)!} \sum_{i_1,\dots,i_p} \big[P_{i_1,\dots,i_p} Q,\ (RS)_{i_1,\dots,i_p}\big].$$

We now study the derivative $(RS)_{i_1,\dots,i_p}$. Let τ_p be the set of partitions of $\{1,\dots,p\}$ into two subsets. If u, v are two such subsets, we let k be the number of elements in u, k' the number of elements in v (of course $k+k'=p$). We also write $u(j)$ for the j-th element of u, $v(j')$ for the j'-th element of v. Then :

$$(RS)_{i_1,\dots,i_p} \;=\; \sum_{(u,v)\in\tau_p} R_{i_{u(1)},\dots,i_{u(k)}}\ S_{i_{v(1)},\dots,i_{v(k')}}.$$

Therefore :

$$\big[PQ, RS\big] \;=\; \frac{q!}{p!(p+q)!} \sum_{i_1,\dots,i_p}\ \sum_{(u,v)\in\tau_p} \big[P_{i_1,\dots,i_p} Q,\ R_{i_{u(1)},\dots,i_{u(k)}}\ S_{i_{v(1)},\dots,i_{v(k')}}\big].$$

Repeating the same argument, we get :

$$\begin{aligned}
\big[PQ, RS\big] \;=\; &\frac{q!}{p!(p+q)!} \sum_{(u,v)\in\tau_p}\ \sum_{i_1,\dots,i_p} \frac{(q-r+k)!}{(r-k)!q!} \times \\
&\times \sum_{j_{k+1},\dots,j_r} \big[(P_{i_1,\dots,i_p}Q)_{j_{k+1},\dots,j_r},\ R_{i_{u(1)},\dots,i_{u(k)} j_{k+1},\dots,j_r}\ S_{i_{v(1)},\dots,i_{v(k')}}\big].
\end{aligned}$$

But $P_{i_p,\ldots,i_p}$ is a scalar, and therefore :

$$\begin{aligned}[PQ,RS] &= \frac{q!}{p!(p+q)!}\sum_{(u,v)\in\tau_p}\sum_{i_1,\ldots,i_p}\frac{(q-r+k)!}{(r-k)!q!}\times \\ &\times \sum_{j_{k+1},\ldots,j_r}\big[P_{i_1,\ldots,i_p}Q_{j_{k+1},\ldots,j_r},\ R_{i_{u(1)},\ldots,i_{u(k)}j_{k+1},\ldots,j_r}\ S_{i_{v(1)},\ldots,i_{v(k')}}\big]\end{aligned}$$

Since both $P_{i_1,\ldots,i_p}$ and $R_{i_{u(1)},\ldots,i_{u(k)},j_{k+1},\ldots,j_r}$ are scalars, this is also

$$\begin{aligned}
&= \frac{1}{p!(p+q)!}\sum_{(u,v)\in\tau_p}\sum_{i_1,\ldots,i_p}\frac{(q-r+k)!}{(r-k)!}\times \\
&\quad\times \sum_{j_{k+1},\ldots,j_r}\big[\overline{R}_{i_{u(1)},\ldots,i_{u(k)},j_{k+1},\ldots,j_r}\ Q_{j_{k+1},\ldots,j_r}\ \overline{P}_{i_1,\ldots,i_p}\ S_{i_{v(1)},\ldots,i_{v(k')}}\big] \\
&= \frac{1}{p!(p+q)!}\sum_{(u,v)\in\tau_p}\frac{(q-r+k)!}{(r-k)!}\sum_{i_{u(1)},\ldots,i_{u(k)}} \\
&\big[\sum_{j_{k+1},\ldots,j_r}\overline{R}_{i_{u(1)},\ldots,i_{u(k)},j_{k+1},\ldots,j_r}\ Q_{j_{k+1},\ldots,j_r},\ \sum_{i_{v(1)},\ldots,i_{v(k')}}\overline{P}_{i_1,\ldots,i_p}\ S_{i_{v(1)},\ldots,i_{v(k')}}\big], \\
&= \frac{1}{p!(p+q)!}\sum_{(u,v)\in\tau_p}\frac{(q-r+k)!}{(r-k)!}\times \\
&\times \sum_{i_{u(1)},\ldots,i_{u(k)}}\big[(r-k)!\ R^*_{i_{u(1)},\ldots,i_{u(k)}}(D)Q,\ (p-k)!\ P^*_{i_{u(1)},\ldots,i_{u(k)}}(D)S\big] \\
&= \frac{1}{p!(p+q)!}\sum_{(u,v)\in\tau_p}(q-r+k)!(p-k)!\sum_{i_1,\ldots,i_k}\big[R^*_{i_1,\ldots,i_k}(D)Q,\ P^*_{i_1,\ldots,i_k}(D)S\big] \\
&= \frac{1}{p!(p+q)!}\sum_{k=0}^{p}\binom{p}{k}(q-r+k)!(p-k)!\sum_{i_1,\ldots,i_k}\big[R^*_{i_1,\ldots,i_k}(D)Q,\ P^*_{i_1,\ldots,i_k}(D)S\big],
\end{aligned}$$

which gives :

$$[PQ,RS] = \frac{1}{(p+q)!}\sum_{k=0}^{p}\frac{(q-r+k)!}{k!}\sum_{i_1,\ldots,i_k}\big[R^*_{i_1,\ldots,i_k}(D)Q,\ P^*_{i_1,\ldots,i_k}(D)S\big]$$

and concludes the second proof.

Taking $P=R,\ Q=S$ (thus $q=s,\ p=r$), we deduce immediately :

Corollary 14. – *For any homogeneous polynomials P, Q of degree p and q respectively :*

$$[PQ]^2 = \frac{1}{(p+q)!}\sum_{k=0}^{p}\frac{(q-p+k)!}{k!}\sum_{i_1,\ldots,i_k}\big[P^*_{i_1,\ldots,i_k}(D)Q\big]^2.$$

Since all terms on the right-hand side are positive, we deduce, taking $k=p$:

$$[PQ]^2 \geq \frac{q!}{(p+q)!p!}\sum_{i_1,\ldots,i_p}\big[P^*_{i_1,\ldots,i_p}(D)Q\big]^2.$$

But $P^*_{i_1,\ldots,i_p} = p!\ \overline{c}_{i_1,\ldots,i_p}$ is just a constant. So :

$$\sum_{i_1,\ldots,i_p}\big[P^*_{i_1,\ldots,i_p}(D)Q\big]^2 = p!^2\sum_{i_1,\ldots,i_p}|c_{i_1,\ldots,i_p}|^2[Q]^2 = p!^2\ [P]^2[Q]^2\ ,$$

and we deduce

$$[PQ]^2 \geq \frac{p!q!}{(p+q)!}\ [P]^2[Q]^2,$$

which is Bombieri's inequality [4].

Using Bombieri's proof in [4], J.L. Frot observed that the quantity $\sum_i \left[P_i Q,\ PQ_i\right]$ was real and positive. We deduce a stronger statement from Theorem 12 :

Theorem 15. – *For all homogeneous polynomials P, Q, with degree m and n respectively, we have :*

$$\sum \left[P_i Q,\ PQ_i\right] \ = \ \frac{1}{(m+n-1)!} \sum_{k=0}^{m} \frac{(n-m+k)!}{k!} \ (m-k) \sum_{i_1,\dots,i_k=1}^{N} \left[P^*_{i_1,\dots,i_k}(D)\ Q\right]^2.$$

Proof. – We set $\varphi(P,Q) = \sum \left[P_i Q,\ PQ_i\right]$. Then an immediate computation shows that

$$\varphi(P,Q) \ = \ m(m+n)\ \left[PQ\right]^2 - \sum_{i=1}^{N} \left[P_i Q\right]^2. \tag{10}$$

Applying Theorem 11, we have

$$\left[PQ\right]^2 \ = \ \frac{1}{(m+n)!} \sum_{k=0}^{m} \frac{(n-m+k)!}{k!} \sum_{i_1,\dots,i_k=1}^{N} \left[P^*_{i_1,\dots,i_k}\ (D)Q\right]^2$$

but

$$(P_i)_{i_1,\dots,i_{k-1}}(D) \ = \ P_{i_1,\dots,i_{k-1},i}(D),$$

and thus

$$\begin{aligned} \varphi(P,Q) \ = \ & \frac{m}{(m+n-1)!} \sum_{k=0}^{m} \frac{(n-m+k)!}{k!} \sum_{i_1,\dots,i_k} \left[P^*_{i_1,\dots,i_k}(D)Q\right]^2 \\ & - \sum_{k=1}^{m} \frac{(n-m+k)!}{(k-1)!} \sum_{i,i_1,\dots,i_{k-1}} \left[P^*_{i_1,\dots,i_{k-1},i}(D)Q\right]^2, \end{aligned}$$

which gives the result.

The previous results are algebraic identities. They have analytic consequences, which are quantitative estimates, obtained by means of inequalities. These estimates will be the object of forthcoming papers.

REFERENCES.

[1] R. Aron - J. Globevnik : Interpolation by analytic functions on c_0. *Math. Proc. Camb. Phil. Soc*, 1988, 104, pp. 295-302.

[2] B. Beauzamy : Introduction to Banach Spaces and their geometry, *North Holland, Mathematics Studies, Second Ed.*, 1985.

[3] B. Beauzamy : Products of many-variable polynomials : Pairs that are maximal in Bombieri's norm. *To appear.*

[4] B. Beauzamy, E. Bombieri, P. Enflo, H. Montgomery : Products of polynomials in many variables. *Journal of Number Theory*, vol. 36, 2, oct. 1990, pp. 219–245.

[5] B. Beauzamy, J.L. Frot, C. Millour : Massively parallel computations on many-variable polynomials : when seconds count. *To appear.*

[6] S. Dineen : Complex Analysis in locally convex spaces. *Mathematics Studies* 83, North Holland, Amsterdam, 1981.

[7] L. Harris : Bounds on the derivative of holomorphic functions. *Colloque d'Analyse*, Rio de Janeiro, 1972. Ed. Hermann, Act. Sci. et Ind. no 1367 (L. Nachbin, editor).

[8] Y. Legrangérard : On some representations of the norm $[.]_2$. *To appear.*

[9] M. Marden : Geometry of Polynomials. *A. M. S. Math. Surveys*, 3rd edition, 1985.

[10] B. Reznick : An Inequality for products of polynomials. *Proceedings A.M.S.*, vol. 117, 4, april 1993, pp. 1063–1073.

[11] Y. Sarantopoulos : Extremal multilinear forms on Banach Spaces. *Proc. A.M.S.*, vol. 99, 2, 1987, pp. 340–346.

[12] I. Zalduendo : An estimate for multilinear forms on l_p-spaces. *To appear.*