ON THE ZEROS OF POLYNOMIALS WITH CONCENTRATION AT LOW DEGREES

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Abstract. – Let P be a polynomial with concentration d at degree k, with zeros written in increasing order of moduli : $0 \leq |z_1| \leq |z_2| \leq \cdots$.

We show that the quantity $\left|\sum_{j>k} \frac{1}{z_j}\right|$ is bounded from above by a number depending only on d and k, for which we give numerical estimates. More precise ones are obtained for Hurwitz polynomials. We finally show that the theory built for Hurwitz polynomials, can be extended to a class of entire functions.

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Recall that a polynomial $P = \sum_0^n a_j z^j$, with complex coefficients, is said to have *concentration* d ($0 < d \leq 1$) at degree k if

$$\sum_0^k |a_j| \;\geq\; d \sum_0^n |a_j|. \tag{0.1}$$

This concept, introduced by Beauzamy-Enflo in [1], has proved to be useful in order to obtain quantitative estimates, independent of the degree : for instance, Jensen's Inequality (Beauzamy [2], Rigler-Trimble-Varga [12]), products of polynomials (Beauzamy-Enflo [1], Beauzamy-Bombieri-Enflo-Montgomery [4]), zeros of H^2 functions (Beauzamy [5]).

Let's write the zeros of P in increasing order of moduli :

$$0 \;\leq\; |z_1| \;\leq\; |z_2| \;\leq\; \cdots \tag{0.2}$$

In [6], S. Chou showed that, under assumption (0.1), the $k+1$-st zero of P satisfies

$$|z_{k+1}| \;\geq\; R(d,k),$$

with $R(d,k) > 0$ depending only on d and k. Precise estimates were given for this number, which was computed exactly for the class of Hurwitz polynomials (see [6]).

The present paper may be regarded as a continuation of [6], since we study here the distribution of all zeros after the $k+1$-st. It is organized as follows :

-In the first section, we give a general theory for the zeros of polynomials satisfying (0.1) ; we derive several consequences, among them a radius of inclusion for the smallest zero. The methods used rely heavily on the theory of analytic functions in the disk.

In the second section, we restrict ourselves to Hurwitz polynomials, for which, using direct proofs, more precise numerical estimates can be given. We derive a generalization of Bernstein's inequality, which does not involve the degree anymore, but only the concentration data d and k.

The third section deals with a generalization of the second one to a class of entire functions ; it is a part of the second author's thesis ([7], chap. IV).

1. General Theory.

The main result of this section is :

Theorem 1.1. – *If P is a polynomial with concentration d at degree k, with zeros ordered as in (0.2), then*

$$\left|\sum_{j>k} \frac{1}{z_j}\right| \;\leq\; C(d,k) \;;\quad \prod_{j>k} |z_j| \;\geq\; \frac{1}{C(d,k)}\;,$$

with $C(d,k) = \dfrac{9^{k+3}}{d^3}\left(\dfrac{2(1+d)}{d}\right)^k$.

Before turning to the proof, we observe that estimates starting with $j > 0$ (when $k \geq 1$) cannot be true : the polynomial $z^k + z^{k+1}$ has concentration $1/2$ at degree k and has k zeros at the origin. We observe also that estimates of $\sum_{j>k} 1/|z_j|$ cannot hold with just our assumptions : the polynomials $1+z^n$, $n \geq 1$, all have concentration $1/2$ at degree 0, but $\sum_{j>0} 1/|z_j| = n$. However,as we will see in § 2, such estimates will be given for Hurwitz polynomials.

In order to prove Theorem 1.1, we will argue by induction on k. The case $k = 0$ is fairly simple.

Proposition 1.2. – *Let P be a polynomial with concentration d at degree 0. Then*

$$\left|\sum_{j=1}^{n} \frac{1}{z_j}\right| \le \frac{1}{d} \quad ; \quad \prod_{j=1}^{n} |z_j| \ge d. \tag{1.1}$$

Proof of Proposition 1.2. – The assumption $|a_0| \ge d\sum_1^n |a_j|$ implies $|a_0| \ge d|a_1|$ and $|a_0| \ge d|a_n|$. The first one gives $|\prod_1^n z_j| \ge d|\sum_{j=1}^n \prod_{i\ne j} z_i|$, the second one $|\prod_1^n z_j| \ge d$.

In order to argue by induction, that is to pass from $k-1$ to k, we would like to have a tool saying that, if α is a zero of P and if P has concentration d at degree k, then $Q = P/(\alpha - z)$ has some concentration $\lambda(d,k) > 0$ at degree $k-1$. But this is not true : $P = (1+z)z = z + z^2$ has concentration $1/2$ at degree 1, but if we remove the factor $1+z$, $Q = z$ has no concentration at degree 0. Here, the root removed is the largest, but if we take $1 - z^n$ and remove $1 - z$, what remains does not have a fixed concentration at degree 0.

So the true statement is more complex, and is as follows : either the root z_1 is small enough, and then indeed $Q = P/(z_1 - z)$ has concentration $\lambda(d,k) > 0$ at degree $k-1$, or all the zeros are rather large, and in this case P already has some concentration at degree 0. The last part is studied in the following Theorem.

Theorem 1.3. – *Let P be a polynomial with concentration d at degree k. Assume that all zeros satisfy $|z_j| \ge \lambda$ ($0 < \lambda \le 1$). Then P has concentration d' at degree 0, with*

$$d' = \frac{d^3\lambda^k}{8e^2 9^{k+1}} .$$

Proof of Theorem 1.3. – It will again be by induction on k. For $k = 0$, it is obvious. Assume it holds for $k-1$. Assume P has concentration d at degree k. Then a first (trivial) case can be excluded :

– if $\sum_0^{k-1} |a_j| \ge (d/2)\sum_0^n |a_j|$, P has concentration $d/2$ at degree $k-1$, and thus by the induction hypothesis, concentration $d' = d^3\lambda^{k-1}/2^6e^2 9^k \ge d^3\lambda^k/8e^2 9^{k+1}$ at degree 0.

– so we are left with the case

$$|a_k| \ge \frac{d}{2}\sum_0^n |a_j|. \tag{1.2}$$

Define $P_\lambda(z) = P(\lambda z)$, $0 < \lambda \le 1$. All zeros of P_λ have modulus ≥ 1, so P_λ, considered as a function in the space H^2, is outer (see for instance Duren [8]). We will compute its concentration at degree k, using l_2-norms :

Lemma 1.4. – *Let P be a polynomial with coefficients satisfying (1.2), and with zeros satisfying $|z_j| \ge \lambda$ ($0 < \lambda \le 1$). Then*

$$\left(\sum_0^k |\lambda^j a_j|^2\right)^{1/2} \ge \frac{d}{2}\left(\sum_0^n |\lambda^j a_j|^2\right)^{1/2} .$$

Proof of Lemma 1.4. – By (1.2),

$$|a_k| > \frac{d}{2}(\sum_0^n |a_j|^2)^{1/2} \ge \frac{d}{2}(\sum_0^k |a_j|^2)^{1/2} . \tag{1.3}$$

Consider the function

$$f(\lambda) = \frac{(\lambda^k|a_k|)^2}{\sum_{j=k}^n \lambda^{2j}|a_j|^2} = \frac{|a_k|^2}{\sum_{j=k}^n \lambda^{2(j-k)}|a_j|^2} .$$

It is a decreasing function of λ, $0 < \lambda \leq 1$. So it attains its minimum for $\lambda = 1$, which gives by (1.3) :

$$(\lambda^k |a_k|)^2 \ \geq \ \frac{d^2}{4} \sum_{j=k}^{n} \lambda^{2j} |a_j|^2 \ ,$$

or

$$\sum_{j=k}^{n} \lambda^{2j} |a_j|^2 \ \leq \ \frac{4}{d^2} |\lambda^k a_k|^2 \ .$$

Using a trivial estimate for $j = 0, \cdots, k-1$, we get

$$\sum_{j=0}^{n} \lambda^{2j} |a_j|^2 \ \leq \ \frac{4}{d^2} \sum_{0}^{k} |\lambda^j a_j|^2 \ ,$$

which proves our Lemma.

We now use a result of [5], Prop. 1.6 : If F is an outer function in H^2, with concentration d at degree k, measured in l_2-norm, that is, if

$$(\sum_{0}^{k} |a_j|^2)^{1/2} \ \geq \ d\,(\sum_{0}^{n} |a_j|^2)^{1/2} \ ,$$

then it has concentration $d' = d^2/e^2 9^{k+1}$ at degree 0, in l_2-norm :

$$|a_0| \ \geq \ \frac{d^2}{e^2 9^{k+1}} (\sum_{0}^{n} |a_j|^2)^{1/2} \ .$$

Applying this to P_λ, we get, by Lemma 1.4 :

$$\begin{aligned} |a_0| \ &\geq \frac{d^2}{4e^2 9^{k+1}} (\sum_{0}^{n} |\lambda^j a_j|^2)^{1/2} \\ &\geq \frac{d^2}{4e^2 9^{k+1}} |\lambda^k a_k| \\ &\geq \frac{d^3}{8e^2 9^{k+1}} \lambda^k \sum_{0}^{n} |a_j| \end{aligned}$$

using (1.2) again, and Theorem 1.3 is proved in this case also.

Remark. – As pointed out by the referee, the dichotomy we use in the proof of Theorem 1.3 (either $\sum_0^{k-1} |a_j| \geq (d/2) \sum_0^n |a_j|$, or $|a_k| \geq (d/2) \sum_0^n |a_j|$) is not optimal. If we consider instead $\sum_0^{k-1} |a_j| \geq (1-a) d \sum_0^n |a_j|$, or $|a_k| \geq a d \sum_0^n |a_j|$, the same proof works up to $a = 1 - (1/9)^{1/3} \sim 0.51925$, and the 8 in the denominator of d' can be replaced by a^{-3}.

If we restrict our attention to a_k only, the proof of Theorem 1.3 gives

Theorem 1.5. – *If* $|a_k| \geq d(\sum_0^n |a_j|^2)^{1/2}$, *and if all zeros satisfy* $|z_j| \geq \lambda$, *then*

$$|a_0| \ \geq \ \frac{d^2 \lambda^k}{e^2 9^{k+1}} |a_k| \ ,$$

a result which should be compared to [5], Prop. 1.6.

Before returning to the proof of Theorem 1.1, we observe that we can deduce from Theorem 1.5 an estimate for the smallest zero of P.

A *radius of inclusion*, for the smallest zero, is the radius of a circle centered at the origin, containing z_1. An estimate for such a radius can be found in Marden [11] (exercise 1, p. 126), but it depends on the degree, whereas ours depends only on the concentration data d and k.

Corollary 1.6. – *(Radius of inclusion for the smallest zero of a polynomial). Let P be any polynomial with complex coefficients. Let*

$$R \;=\; \min_k \left(e^2 9^{k+1} \frac{|a_0| \sum_0^n |a_j|^2}{|a_k|^3} \right)^{1/k} .$$

Then the disk $D(0,R)$ contains at least one zero of P.

Proof of Corollary 1.6. – Choose a k for which the minimum is attained. Set

$$d \;=\; d_k \;=\; |a_k| \,/ (\sum_0^n |a_j|^2)^{1/2} .$$

Then $|a_k| \geq d (\sum_0^n |a_j|^2)^{1/2}$, and by Theorem 1.5, if all zeros were $> R$, we would have

$$|a_0| \;>\; \frac{d^2 R^k |a_k|}{e^2 9^{k+1}} .$$

But $|a_0| = d^2 R^k |a_k| / e^2 9^{k+1}$. Thus the disk of radius R contains at least a zero.

This criterion is useful when one of the coefficients is large. For instance, consider $P = 1 + 1000z + 3z^n$, for any $n \geq 2$. Then $R \leq e^2 9^2 / 1000 \sim 0.6$.

We come back to the proof of Theorem 1.1. We need one more lemma, dealing with concentrations. We define $\mathrm{cf}_k(P) = \sum_0^k |a_j| / \sum_0^n |a_j|$, the concentration factor of P at degree k.

Lemma 1.7. – *Let $P = (\alpha - z) Q$, with $|\alpha| < 1$. Then*

$$\mathrm{cf}_k(P) \;\leq\; \frac{1}{1 - |\alpha|} \,\mathrm{cf}_{k-1}(Q) + \frac{|\alpha|}{1 - |\alpha|} \,\mathrm{cf}_k(Q).$$

Proof of Lemma 1.7. – Writing $P = \sum_0^n a_j z^j$, $Q = \sum_0^{n-1} b_l z^l$, we get

$$\begin{aligned} \mathrm{cf}_k(P) \;&=\; \frac{|\alpha b_0| + \sum_{j=1}^k |-b_{j-1} + \alpha b_j|}{|\alpha b_0| + \sum_{j=1}^{n-1} |-b_{j-1} + \alpha b_j| + |b_{n-1}|} \\ &\leq\; \frac{\sum_{j=0}^{k-1} |b_j| + |\alpha| \sum_0^k |b_j|}{(1 + |\alpha|)|b_0| + (1 - |\alpha|) \sum_{j=1}^{n-1} |b_j|} \\ &\leq\; \frac{\sum_{j=0}^{k-1} |b_j| + |\alpha| \sum_0^k |b_j|}{(1 - |\alpha|) \sum_{j=0}^{n-1} |b_j|} , \end{aligned}$$

which proves the result.

Lemma 1.8. – *Let P and Q be as in Lemma 1.7. If $\mathrm{cf}_k(P) \geq d$ and $|\alpha| \leq \dfrac{d}{2(1+d)}$, then $\mathrm{cf}_{k-1}(Q) \geq d/2$.*

Proof of Lemma 1.8. – This follows immediately from Lemma 1.7, since $\mathrm{cf}_{k-1}(Q) \geq d - |\alpha|(1 + d)$.

We may now prove Theorem 1.1. Assuming it holds for $k-1$, we wish to prove it for k. Let P satisfy (0.1).

– If the first zero, z_1, satisfies $|z_1| \le d/2(1+d)$, then $Q = P/(z_1 - z)$ has concentration $d/2$ at degree $k-1$, by Lemma 1.8. Applying the induction hypothesis to Q yields

$$\left|\sum_{j>k} \frac{1}{z_j}\right| \;\le\; C(\frac{d}{2}, k-1)\ , \quad \prod_{j>k} |z_j| \;\ge\; C^{-1}(\frac{d}{2}, k-1).$$

– If z_1, and therefore all other zeros satisfy

$$|z_j| \;\ge\; \frac{d}{2(1+d)}\ , \tag{1.4}$$

Theorem 1.3, with $\lambda = \dfrac{d}{2(1+d)}$, shows that, at degree 0, P has concentration

$$d' \;=\; \frac{d^3}{8e^2 9^{k+1}} \left(\frac{d}{2(1+d)}\right)^k .$$

Therefore, by Proposition 1.2,

$$|\sum_1^n \frac{1}{z_j}| \;\le\; \frac{1}{d'}, \tag{1.5}$$

$$\prod_1^n |z_j| \;\ge\; d'. \tag{1.6}$$

But, by (1.4), for all j, $1/|z_j| \le 2(1+d)/d$, so

$$|\sum_{j>k} \frac{1}{z_j}| \;\le\; \frac{2k(1+d)}{d} + \frac{1}{d'} \;\le\; \frac{9^{k+3}}{d^3} \left(\frac{2(1+d)}{d}\right)^k .$$

We take

$$C(d,k) \;=\; \frac{9^{k+3}}{d^3} \left(\frac{2(1+d)}{d}\right)^k ,$$

and check that $C(d/2, k-1) \le C(d,k)$. Therefore, $C(d,k)$ is a suitable bound for both cases.

To study $\prod_{k+1}^n |z_j|$, let p be the index (if it exists) such that $|z_p| \le 1 < |z_{p+1}|$.

– If $p \le k$, then $|z_{k+1}|, \ldots, |z_n| \ge 1$, and $\prod_{k+1}^n |z_j| \ge 1$,

– If $p > k$, then $|z_1|, \ldots, |z_k| < 1$, and $\prod_{k+1}^n |z_j| \ge \prod_1^n |z_j|$, and Theorem 1.1 is proved.

Examples. – We can mention for instance :

- the family of polynomials $p_n(z) = 1/n + 2z + (-1)^n z^n$. If $z_{j,n}$ is the set of zeros of p_n, then, for any n,

$$|\sum_{j=2}^n \frac{1}{z_{j,n}}| \;\le\; 3^{11} \cdot 5/8,$$

since all these polynomials have concentration 2/3 at degree 1.

- the partial sums $s_n(z)$ of the exponential function. All these partial sums have concentration $1/e$ at degree 0, and therefore, if $z_{j,n}$ are the zeros of s_n,

$$|\sum_{j=1}^n \frac{1}{z_{j,n}}| \;\le\; 9^3 \cdot e^3.$$

We now turn to Hurwitz polynomials, for which the general theory will become more precise.

2. Hurwitz polynomials.

Recall that a polynomial P is a *Hurwitz* polynomial if its coefficients are real and positive and its roots z_j satisfy $\Re z_j \le 0$ (here $\Re z$ is the real part of z). Such a polynomial may be written

$$P(z) \;=\; \prod_l (z+\alpha_l) \prod_{l'} (z+\beta_{l'})(z+\bar\beta_{l'}), \tag{2.1}$$

where the α_l's are real ≥ 0 and the $\beta_{l'}$'s satisfy $\Re\beta_{l'} \ge 0$.

Hurwitz polynomials play a special rôle in the context of dynamic stability (see Marden [11], chap. IX, § 36) ; their study, in the frame of concentration at low degrees, was initiated by Rigler-Trimble-Varga [12].

The main theorem of this section is :

Theorem 2.1. – *Let P be a Hurwitz polynomial, with concentration d at degree k. Then*

$$\sum_1^n \frac{1}{1-z_j} \;\le\; C_H(d,k),$$

with $C_H(d,k) = 9\log(1/d) + (11k+9)\log 2$.

The general idea behind the proof is the same as before : either one root has small modulus ; we remove it and get a polynomial with some concentration at degree $k-1$, or all the roots are large, and then P has some concentration at degree 0. However, the steps of the proof are technically quite different. The next theorem is the analogue of Theorem 1.3. We define

$$\mathrm{tf}_k(P) \;=\; \frac{a_k}{\sum_0^n a_j} \;=\; \frac{a_k}{P(1)}\ ,$$

which might be called the *true* concentration factor of P at degree k, for it indicates the importance of the coefficient a_k among all coefficients.

Theorem 2.2. – *Let P be a Hurwitz polynomial with $\mathrm{tf}_k(P) \ge d$. Assume that all zeros satisfy $|z_j| \ge \lambda$. Then :*

$$a_0 \;\ge\; \frac{d a_k}{(4+\frac{2}{\lambda})^k}\ .$$

Proof. – Put $\zeta_j = -1/z_j$, $j = 1,\ldots,n$. We can write

$$\frac{P}{a_0} \;=\; (z\zeta_1+1)\cdots(z\zeta_n+1),$$

and therefore

$$\frac{a_k}{a_0} \;=\; \sum_{l_1<\cdots<l_k} \Re(\zeta_{l_1}\cdots\zeta_{l_k}),$$

which gives

$$\left(\frac{a_k}{a_0}\right)^2 \;=\; \sum_{\substack{l_1<\cdots<l_k\\ l'_1<\cdots<l'_k}} \Re(\zeta_{l_1}\cdots\zeta_{l_k}\cdot\zeta_{l'_1}\cdots\zeta_{l'_k})\ .$$

In a product $\zeta_{l_1}\cdots\zeta_{l_k}\cdot\zeta_{l'_1}\cdots\zeta_{l'_k}$, the number of distinct indexes is between k and $2k$; we write it as $k+p$ $(0 \le p \le k)$. If $l_1 = l'_1$ (for instance), we just replace $\zeta_{l_1}^2$ by $\frac{1}{\lambda}\zeta_{l_1}$, using the fact that all the ζ_j have moduli $\le 1/\lambda$. We also observe that products of length $k+p$ occur in the expansion of a_{k+p}/a_0. Precisely :

$$\frac{a_{k+p}}{a_0} \;=\; \sum_{j_1<\cdots<j_{k+p}} \zeta_{j_1}\cdots\zeta_{j_{k+p}}\ .$$

Combining suitable terms, we obtain this way

$$\left(\frac{a_k}{a_0}\right)^2 \;\le\; \sum_{p=0}^{k} \binom{k+p}{k}\binom{k}{p}\left(\frac{1}{\lambda}\right)^{k-p} \frac{a_{k+p}}{a_0}\;. \tag{2.2}$$

The assumption $\mathrm{tf}_k(P) \ge d$ implies, for $p = 1, \cdots, k$,

$$a_{k+p} \;\le\; \frac{a_k}{d}\;.$$

Therefore,

$$\left(\frac{a_k}{a_0}\right)^2 \;\le\; \frac{a_k}{da_0}\sum_{p=0}^{k} \binom{k+p}{k}\binom{k}{p}\left(\frac{1}{\lambda}\right)^{k-p}\;. \tag{2.3}$$

Let $A_k = \sum_{p=0}^{k} \binom{k+p}{k}\binom{k}{p}\left(\frac{1}{\lambda}\right)^{k-p}$. A rough estimate is :

Lemma 2.3. – *We have* $A_k \le 2^k(2+\frac{1}{\lambda})^k$.

Proof of Lemma 2.3. – We write :

$$\binom{k+p}{k} \;=\; \sum_{m=0}^{p}\binom{p}{m}\binom{k}{k-m},$$

thus

$$\begin{aligned}
A_k &= \sum_{p=0}^{k}\sum_{m=0}^{p}\binom{k}{p}\binom{p}{m}\binom{k}{k-m}\left(\frac{1}{\lambda}\right)^{k-p}\\
&= \sum_{m=0}^{k}\binom{k}{m}\sum_{p=m}^{k}\binom{k}{p}\binom{p}{m}\left(\frac{1}{\lambda}\right)^{k-p}\\
&= \sum_{m=0}^{k}\binom{k}{m}\sum_{p=m}^{k}\binom{k}{m}\binom{k-m}{p-m}\left(\frac{1}{\lambda}\right)^{k-p}\\
&= \sum_{m=0}^{k}\binom{k}{m}^2\sum_{p=m}^{k}\binom{k-m}{p-m}\left(\frac{1}{\lambda}\right)^{k-p}\\
&= \sum_{m=0}^{k}\binom{k}{m}^2\left(1+\frac{1}{\lambda}\right)^{k-m}\\
&\le 2^k\sum_{m=0}^{k}\binom{k}{m}\left(1+\frac{1}{\lambda}\right)^{k-m}\\
&\le 2^k\left(2+\frac{1}{\lambda}\right)^{k},
\end{aligned}$$

which proves the Lemma. Coming back to (2.3), we get :

$$\left(\frac{a_k}{a_0}\right)^2 \;\le\; \frac{2^k a_k}{da_0}\,(2+\frac{1}{\lambda})^k,$$

which proves Theorem 2.2.

The next two lemmas indicate what happens to the concentration when a real root is removed, or when two complex conjugate roots are removed.

Lemma 2.4. – *If $-\alpha$ is a real root of P and $Q = P/(\alpha + z)$, then*

$$\frac{a_k}{P(1)} \;=\; \frac{\alpha}{1+\alpha}\,\frac{b_k}{Q(1)} + \frac{1}{1+\alpha}\,\frac{b_{k-1}}{Q(1)}\ ,$$

and therefore

$$\mathrm{tf}_k(P) \;\le\; \max\{\mathrm{tf}_k(Q), \mathrm{tf}_{k-1}(Q)\}.$$

Lemma 2.5. – *If $-\beta$, $-\bar\beta$ are non-real roots of P and $Q = P/(\beta + z)(\bar\beta + z)$,*

$$\frac{a_k}{P(1)} \;=\; \frac{|\beta|^2}{|\beta|^2 + 2\Re\beta + 1}\,\frac{b_k}{Q(1)} + \frac{2\Re\beta}{|\beta|^2 + 2\Re\beta + 1}\,\frac{b_{k-1}}{Q(1)} + \frac{1}{|\beta|^2 + 2\Re\beta + 1}\,\frac{b_{k-2}}{Q(1)}\ ,$$

and therefore

$$\mathrm{tf}_k(P) \;\le\; \max\{\mathrm{tf}_k(Q), \mathrm{tf}_{k-1}(Q), \mathrm{tf}_{k-2}(Q)\}.$$

The proofs are elementary, and are left to the reader. We now turn to the proof of Theorem 2.1, which will again be made by induction on k. We observe that the quantity $\sum_1^n 1/(1 - z_j)$ is always real and positive.

For $k = 0$, the assumption is

$$|\prod_1^n z_j| \ge d \prod_1^n |1 - z_j|, \tag{2.4}$$

and the result in the case $k = 0$ is given by :

Proposition 2.6. – *If P is a Hurwitz polynomial with concentration d at degree 0, then*

$$\sum_1^n \frac{1}{1 - z_j} \;\le\; 2 \log \frac{1}{d}\ .$$

Proof of Proposition 2.6. – We write

$$\begin{aligned}
\sum_1^n \frac{1}{1 - z_j} &= n + \sum_1^n \frac{z_j}{1 - z_j} \\
&\le n - \sum_1^n \left|\frac{z_j}{1 - z_j}\right|^2 \\
&\le n\left(1 - \left(\prod_1^n \left|\frac{z_j}{1 - z_j}\right|^2\right)^{1/n}\right) \\
&\le n(1 - d^{2/n}) \\
&\le 2 \log 1/d\ .
\end{aligned}$$

The estimate is best possible : the polynomials

$$P_n \;=\; \left(z^2 + \frac{d^{1/n}}{1 - d^{1/n}}\right)^n$$

all have concentration d at degree 0, and

$$\sum_1^{2n} \frac{1}{1 - z_j} \;=\; 2n(1 - d^{1/n}) \;\to\; 2 \log 1/d,$$

when $n \to \infty$.

Assume now Theorem 2.1 holds for $k-1$, we will prove it for k. Let P be a Hurwitz polynomial with concentration d at degree k.

– If $\sum_0^{k-1} a_j \geq d/2 \sum_0^n a_j$, P has concentration $d/2$ at degree $k-1$, and by the induction hypothesis,

$$\sum_1^n \frac{1}{1-z_j} \ \leq \ C_H(d/2, k-1). \tag{2.5}$$

– Otherwise,

$$\mathrm{tf}_k(P) \ \geq \ \frac{d}{2}\ ; \tag{2.6}$$

we now consider this case. Let $\lambda = 1/2$. Let m be the last index (if it exists) such that $|z_m| \leq \lambda$. Let $Q = P/(z-z_1)\cdots(z-z_m)$: this is a Hurwitz polynomial.

By Lemmas 2.4 and 2.5, for some k', $0 \leq k' \leq k$, we have $\mathrm{tf}_{k'}(Q) \geq d/2$. By Theorem 2.2, writing $Q = \sum_0^{n-m} b_j z^j$, we get :

$$b_0 \ \geq \ \frac{d}{2\cdot 8^{k'}}\, b_{k'} \ \geq \ \frac{d^2}{4\cdot 8^{k'}} \sum_0^{n-m} b_j\ ,$$

using (2.6) again.

By Proposition 2.6 applied to Q,

$$\sum_{m+1}^n \frac{1}{1-z_j} \ \leq \ 2\log\frac{4\cdot 8^{k'}}{d^2} \ \leq \ 2\log\frac{4\cdot 8^k}{d^2}\ . \tag{2.7}$$

Since $a_k \geq (d/2)\sum_0^n a_j$, we also have

$$a_k \ \geq \ \frac{d}{2}\,(\sum_0^n a_j^2)^{1/2}\ ,$$

and [5], Theorem 2.1, shows that the number m of zeros of P in the disk $D(O, 1/2)$ is at most

$$N \ = \ \frac{\log(2/d) + k\log 2}{\log(5/4)}\ .$$

So, since $\Re z_j \leq 0$, we get

$$\sum_1^m \frac{1}{1-z_j} \ \leq \ N, \tag{2.8}$$

and we finally deduce from (2.7) and (2.8) the estimate :

$$\sum_1^n \frac{1}{1-z_j} \ \leq \ 9\log(1/d) + (11k+9)\log 2,$$

which proves Theorem 2.1, since $C_H(d/2, k-1) \leq C_H(d,k)$.

We observe that, for fixed d, $C_H(d,k)$ is proportional to k. This order of magnitude is best possible. Indeed, $P = (z^2+1)^{k+1}$ has concentration $1/2$ at degree k, and $\sum_1^{2(k+1)} 1/(1-z_j) = k+1$.

For fixed k, $C_H(d,k)$ is proportional to $\log 1/d$, and this order of magnitude is also best possible, as we already mentioned (Proposition 2.6).

We now deduce an interesting corollary :

Corollary 2.7. *A Generalized Bernstein Inequality. – Let P be a Hurwitz polynomial, with concentration d at degree k. Define $\|P\|_\infty = \max_\theta |P(e^{i\theta})|$. Then*

$$\|P'\|_\infty \ \le\ C_H(d,k)\,\|P\|_\infty \ ,$$

where, as before, $C_H(d,k) = 9\log(1/d) + (11k+9)\log 2$.

Proof of Corollary 2.7. – It follows immediately from Theorem 2.1, since $\|P\|_\infty = P(1)$, $\|P'\|_\infty = P'(1)$, and

$$\frac{P'(1)}{P(1)} \ =\ \sum_1^n \frac{1}{1-z_j} \ .$$

Classical Bernstein's inequality is valid for any polynomial, but involves the degree : if the degree of P is n,

$$\|P'\|_\infty \ \le\ n\,\|P\|_\infty \ .$$

Our extension does not involve the degree (it uses only d and k), but it is valid only for Hurwitz polynomials. It cannot be valid in general : $P_n = 1 - z^n$ all have concentration $1/2$ at degree 0, but $\|P'\|_\infty / \|P\|_\infty = n$.

From Theorem 2.1, we can easily deduce estimates for the quantities we considered in § 1 :

Corollary 2.8. *– Let P be a Hurwitz polynomial with concentration d at degree k. Then*

$$|\sum_{j>k} \frac{1}{z_j}| \ \le\ (1+\frac{1}{R_H})^2 (C_H + 1) \ ,$$

where $R_H = 1/(1-d)^{1/(k+1)} - 1$ is the lower bound for $|z_{k+1}|$ obtained in [6].

Proof. – We observe that $\sum_{j>k} 1/z_j$ may not be real : it depends on whether z_k and z_{k+1} are conjugate or not. However, in all cases,

$$\left|\sum_{j>k} \frac{1}{1-z_j}\right| \ \le\ \sum_1^n \frac{1}{1-z_j} + 1.$$

Now, if α is real, $\alpha > R_H$,

$$\frac{1}{\alpha} \ \le\ (1+\frac{1}{R_H})\frac{1}{1+\alpha} \ , \tag{2.9}$$

and if $\Re\beta > 0$, $|\beta| > R_H$,

$$\frac{1}{\beta} + \frac{1}{\bar\beta} \ \le\ (1+\frac{1}{R_H})^2 \Big(\frac{1}{1+\beta} + \frac{1}{1+\bar\beta}\Big) \ ; \tag{2.10}$$

the Corollary follows.

Corollary 2.9. *– Let P be as above. Then*

$$\prod_{k+1}^n |z_j| \ \ge\ e^{-(1+\frac{1}{R_H})^2 (1+C_H)} \ .$$

Proof. – We have :

$$\sum_{k+1}^n \log\frac{1}{|z_j|^2} \ \le\ \sum_{k+1}^n \frac{1}{|z_j|^2} \ .$$

If z_j is real, $z_j \le 0$, we use (2.9) and obtain

$$\frac{1}{|z_j|} \ \le\ (1+\frac{1}{R_H})\,\frac{1}{1-z_j} \ ,$$

$$\frac{1}{|z_j|^2} \;\le\; (1+\frac{1}{R_H})^2\,\frac{1}{(1-z_j)^2} \;\le\; (1+\frac{1}{R_H})^2\,\frac{1}{1-z_j} \;.$$

If z_j is not real but $\Re z_j \le 0$ and $|z_j| \ge R_H$:

$$\begin{aligned}\frac{1}{|z_j|^2} &\le (1+\frac{1}{R_H^2})\,(\frac{1}{1-z_j}+\frac{1}{1-\bar z_j})\\ &\le (1+\frac{1}{R_H})^2\,(\frac{1}{1-z_j}+\frac{1}{1-\bar z_j})\;,\end{aligned}$$

Summing up, we obtain

$$\begin{aligned}\sum_{k+1}^{n}\frac{1}{|z_j|^2} &\le (1+\frac{1}{R_H})^2\left(\sum_{\substack{z_j\ \text{real}\\ j>k}}(\frac{1}{1-z_j}+\sum_{\substack{z_j\ \text{not real}\\ j>k}}(\frac{1}{1-z_j}+\frac{1}{1-\bar z_j})\right)\\ &\le 2(1+\frac{1}{R_H})^2\sum_{1}^{n}\frac{1}{1-z_j}\\ &\le 2(1+\frac{1}{R_H})^2\,C_H\;,\end{aligned}$$

which gives the result.

Within the framework of Hurwitz polynomials, the process of removing one (or two conjugate) roots leads to a polynomial with concentration at degree $k-1$ (we saw in § 1 that this was not the case in general) :

Proposition 2.10. – *Let P be a Hurwitz polynomial with $\mathrm{tf}_k(P) \ge d$. With z_1 as before, define $Q = P/(z_1 - z)$ if z_1 is real, $Q = P/(z_1 - z)(\bar z_1 - z)$ if z_1 is not real. Then Q is a Hurwitz polynomial, with concentration $d^2/2(4+d)^k$ at degree $k-1$.*

Proof. – If $|z_1| < d/2$, this is clear from Lemmas 2.4 and 2.5. If $|z_1| \ge d/2$, Theorem 2.2 shows that P has concentration $d' = d^2/2(4+d)^k$ at degree 0. But $\mathrm{cf}_0(Q) \ge \mathrm{cf}_0(P)$, so Q itself has concentration d' at degree 0, so at degree $k-1$.

Remark. – If we define $Q = P/(z_1 - z)$, no matter whether z_1 is real or not, then Q has concentration $\dfrac{4+2\sqrt2}{4+3\sqrt2}\,d'$, at degree $k-1$ (but of course Q is not Hurwitz if z_1 is not real).

Indeed, the case $|z_1| < d/2$ is handled as before. Assume z_1 is not real and $|z_1| > d/2$. Set $z_1 = -\beta$, $R = P/(\beta+z)(\bar\beta+z) = \sum_0^{n-2} c_j z^j$, $Q = (z+\bar\beta)R$. We have

$$d' \;=\; \frac{P(0)}{P(1)} \;=\; \frac{c_0|\beta|^2}{R(1)|1+\beta|^2}\;, \tag{2.11}$$

and

$$\mathrm{cf}_0(Q) \;=\; \frac{|\beta|c_0}{|R(z)(\bar\beta+z)|_1}\;,$$

with

$$\begin{aligned}|R(z)(\bar\beta+z)|_1 &= |c_0\bar\beta| + |c_0 + c_1\bar\beta| + \cdots + |c_{k-1} + c_k\bar\beta| + \cdots\\ &\le (\sum c_j)(1+|\beta|),\end{aligned}$$

since all c_j's are ≥ 0. So we get

$$\mathrm{cf}_0(Q) \;\ge\; \frac{c_0|\beta|}{R(1)(1+|\beta|)} \;=\; d'\,\frac{|1+\beta|^2}{|\beta|(1+|\beta|)}\;,$$

by (2.11). Writing $|1+\beta|^2 \geq 1+|\beta|^2$, since $Re(\beta) \geq 0$, we obtain

$$\mathrm{cf}_0(Q) \;\geq\; \frac{d'(1+|\beta|^2)}{|\beta|(1+|\beta|)} \;.$$

The function $(1+x^2)/(x+x^2)$, for $x \geq 0$, takes its minimum at $x = 1+\sqrt{2}$, and its minimal value is $(4+2\sqrt{2})/(4+3\sqrt{2})$. So we find

$$\mathrm{cf}_0(Q) \;\geq\; \frac{4+2\sqrt{2}}{4+3\sqrt{2}}\, d',$$

which proves our claim.

We now give a converse to Corollary 2.7 :

Theorem 2.11. – *Let $C > 0$. Let P be a Hurwitz polynomial, satisfying*

$$\frac{\|P'\|_\infty}{\|P\|_\infty} \;\leq\; C. \tag{2.12}$$

Then, for every $k \geq C$, the concentration of P at degree k is $\exp\{-C(1+2\rho)(1+\rho)/2\rho^2\}$, *where*

$$\rho \;=\; \max\left\{\frac{k-C+1}{C}, \sqrt{\frac{k-C+1}{C}}\right\} \;. \tag{2.13}$$

We need two simple lemmas :

Lemma 2.12. – *Let $z \in \mathbb{C}$, with $\Re z \leq 0$. Then*

$$\Re\frac{1}{1-z} \;\geq\; \frac{1}{1+\max(|z|,|z|^2)} \;.$$

Lemma 2.13. – *Let $z \in \mathbb{C}$, with $\Re z \leq 0$, and $|z| > \rho > 0$. Then*

$$|1-\frac{1}{z}|^2 \;\leq\; 1 + a(\rho)\,\Re\frac{1}{1-z} \;,$$

where

$$a(\rho) \;=\; \frac{(1+2\rho)(1+\rho)}{\rho^2} \;.$$

The proofs are left to the reader.

We now prove Theorem 2.11. First, take any $\rho > 0$. Under assumptions (2.12), the number N of zeros of P in the disk $D(O,\rho)$ is bounded by a number depending only on C, ρ. Indeed, writing $P = \prod_1^n (z-z_j)$, let m be the last index such that $|z_j| \leq \rho$. We get

$$\sum_1^m \frac{1}{1-z_j} \;\leq\; \sum_1^n \frac{1}{1-z_j} \;=\; \frac{P'(1)}{P(1)} \;\leq\; C \;;$$

from which follows, by Lemma 2.12,

$$N \;\leq\; C\left(1+\max\{\rho,\rho^2\}\right). \tag{2.14}$$

We now set $Q = \prod_1^m (z-z_j)$, $R = \prod_{m+1}^n (z-z_j)$. We have found a bound for the degree of Q, and we now show that R has a concentration at degree 0, depending only on ρ and C. Let $\delta = \mathrm{cf}_0(R)$. Then :

$$\delta \;=\; \prod_{m+1}^n \left|\frac{z_j}{1-z_j}\right|,$$

$$\begin{aligned}
\frac{1}{\delta^2} &= \prod_{m+1}^{n} \left|1 - \frac{1}{z_j}\right|^2 \\
&\le \left(\frac{1}{m-n} \sum_{m+1}^{n} \left|1 - \frac{1}{z_j}\right|^2 \right)^{n-m} \\
&\le \left(1 + \frac{a(\rho)}{n-m} \sum_{m+1}^{n} \frac{1}{1-z_j} \right)^{n-m} \\
&\le \left(1 + \frac{a(\rho)C}{n-m} \right)^{n-m} \\
&\le e^{a(\rho)C},
\end{aligned}$$

and therefore

$$\delta \ \ge \ e^{-a(\rho)C/2} \ . \qquad (2.15)$$

Since $P = Q \cdot R$, we deduce from (2.14) and (2.15) that the polynomial P has concentration $\exp\{-a(\rho)C/2\}$ at degree $k = [C(1 + \max\{\rho, \rho^2\})]$. The choice of ρ indicated in (2.13) gives the result.

We now give an extension to a class of entire functions.

3. Extension to a class of entire functions.

As we already explained, the results presented in the previous paragraphs are independent of the degrees of the polynomials involved, and depend only on the concentration data (d, k). Therefore, they will extend naturally to a class of entire functions, when the proper framework is defined.

We first consider the space of functions with absolutely convergent Fourier series :

$$\mathcal{A}(\Pi) \ = \ \{f = \sum_{-\infty}^{\infty} c_j e^{ij\theta} \ ; \ \sum_{-\infty}^{\infty} |c_j| < \infty\},$$

and inside this space the subspace of one-sided series :

$$\mathcal{A}_+(\Pi) \ = \ \{f = \sum_{0}^{\infty} c_j e^{ij\theta} \ ; \ \sum_{0}^{\infty} |c_j| < \infty\},$$

equipped with the norm $\|f\|_{\mathcal{A}} = \sum_0^\infty |c_j|$.

We refer the reader to the book by J.-P. Kahane [9] for a detailed study of these spaces.

The space $\mathcal{A}_+$ is obviously isometric to the space $l_1(\mathbb{N})$, in the isometry $f \to (c_j)_{j \ge 0}$, so we will write $|f|_1$ instead of $\|f\|_{\mathcal{A}}$.

We also observe that if f is in $\mathcal{A}_+$, the function $f(z) = \sum_0^\infty c_j z^j$ is analytic in the unit disk.

We define the partial sums of f by $s_k(f) = \sum_0^k c_j z^j$, and the concentration at degree k by

$$\mathrm{cf}_k(f) \ = \ \frac{|s_k(f)|_1}{|f|_1} \ .$$

We denote by $\mathcal{A}_+(d, k)$ the set of functions in $\mathcal{A}_+$ with concentration d at degree k.

We also define the Taylor-Hurwitz functions (generalizing Hurwitz polynomials) : these are functions of the form :

$$f(z) \ = \ a z^m \prod_{1}^{\infty} (1 + \frac{z}{\alpha_j}) \qquad (3.1)$$

where :

- either α_j is real positive, or α_j is not real, but satisfies $\Re \alpha_j \ge 0$ and the term with $\bar{\alpha}_j$ also exists,
- the sequence of (α_j) satisfies $\sum_1^\infty 1/|\alpha_j| < \infty$.

We denote by $\mathcal{A}_H$ the space of Taylor-Hurwitz functions (equipped with the same norm), and by $\mathcal{A}_H(d,k)$ the subspace of functions having concentration d at degree k.

Any Taylor-Hurwitz function has genus 0 and order at most 1 (see for instance Levin [10] for definitions). It may have order 1 : this is the case of the function

$$f(z) \;=\; \prod_2^\infty (1+\frac{z}{n(\ln n)^2}).$$

The theory and results of § 2 extend naturally, and we get :

Theorem 3.1. – *Let f be a function in $\mathcal{A}_H(d,k)$. Then*

$$\frac{|f'|_1}{|f|_1} \;\le\; C_H(d,k),$$

where $C_H(d,k)$ is defined in Theorem 2.1.

Since for such a function the Taylor coefficients are real and positive, we have $|f|_1 = f(1) = \|f\|_\infty$, and we obtain again a generalization of Bernstein's inequality.

The converse also holds :

Theorem 3.2. – *Let $C>0$ and f in $\mathcal{A}_+$, with*

$$\frac{|f'|_1}{|f|_1} \;\le\; C.$$

Then, for every $k \ge C$, f has concentration $\exp\{-C(1+2\rho)(1+\rho)/2\rho^2\}$ at degree k ; the number ρ is defined as in (2.13).

The statement of Theorem 3.1 would be false in the framework of functions in $\mathcal{A}_+$, assuming only the coefficients to be positive. Indeed, the set of functions

$$f_n(z) \;=\; e^{(z^n)} \;=\; 1 + z^n + \frac{z^{2n}}{2!} + \cdots$$

all have concentration $1/e$ at degree 0, but $f_n'(1)/f_n(1) = n$.

So the fact that the order of the function is at most 1 plays an essential rôle in Theorem 3.1. This theorem, however, can be extended to functions of higher order, but the bound on $|f'|_1/|f|_1$ then depends on d, k, and on the order.

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